SLE-decorated LQG as an infinitely divisible law on metric spaces

Jinwoo Sung (Joint with Yuyang Feng)

University of Chicago

Random Conformal Geometry and Related Fields in Jeju

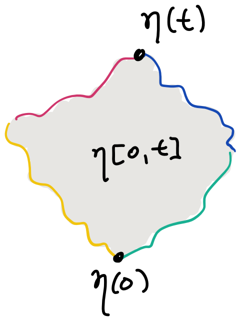
Mating of trees

- Sample the following independently.
  - $(\mathbb{C}, h, 0)$: γ-quantum cone
  - η: Whole-plane space-filling $\mathrm{SLE}_{16/\gamma^2}$ curve from ∞ to ∞
- Then parameterize η by γ-LQG measure μ_h.
  - $\mu_h(\eta[s,t]) = t - s$, $\eta(0) = 0$.
- ν_h: γ-LQG boundary length on SLE_{γ^2}-type curves.

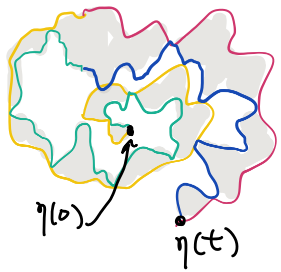
Theorem (Duplantier–Miller–Sheffield, '14)

The process (L_t, R_t) is a planar Brownian motion with correlation $-\cos(\pi\gamma^2/4)$ between the two coordinates. Moreover, (L_t, R_t) a.s. determines h and η up to rotation and scaling of $\mathbb{C}$.

$\gamma \in (0, \sqrt{2}]$:



$\gamma \in (\sqrt{2}, 2)$:



$L_t = \nu_h(\text{red}) - \nu_h(\text{yellow})$

$R_t = \nu_h(\text{blue}) - \nu_h(\text{green})$

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  - $(\mathbb{C}, h, 0)$: γ-quantum cone
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- The γ-LQG surface $\mathcal{C}_{[s,t]} := (\eta[s,t], h|_{\eta[s,t]}, \eta|_{[s,t]})/\sim_\gamma$
  - Independent over disjoint intervals.
  - The law depends only on $t - s$.

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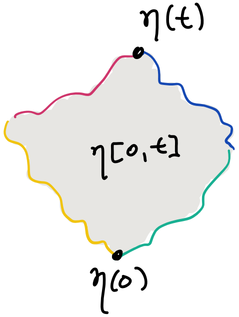

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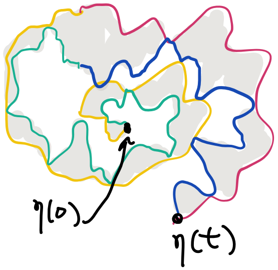

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LQG metric

- For a GFF h on $\mathbb{C}$, the γ-LQG metric D_h is the rigorous construction of

$$\text{“}D_h(z, w) := \inf_{P:z \to w} \int_0^1 e^{\frac{\gamma}{d_\gamma} h(P(t))} |P'(t)| dt\text{”}$$

  where d_γ is the Hausdorff dimension of D_h. Constructed in a series of works by Ding, Dubédat, Falconet, Gwynne, Miller, Pfeffer, Sun, Zeitouni, and Zhang.

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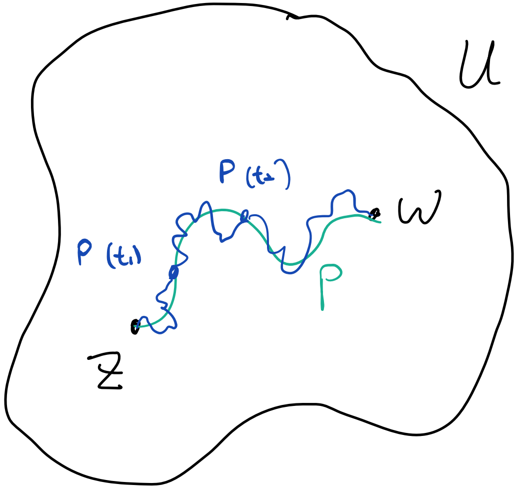
- For a continuous curve $P : [a,b] \to \mathbb{C}$, define

$$\mathrm{len}(P; D_h) := \sup_T \sum_{i=1}^{\#T} D_h(P(t_{i-1}), P(t_i))$$

where the supremum is over all partitions $T : a = t_0 < t_1 < \cdots < t_{\#T} = b$.



- For $U \subset \mathbb{C}$, the **internal metric** of D_h on U is defined by

$$D_h^U(z,w) := \inf_{U \supset P: z\to w} \mathrm{len}(P; D_h).$$

LQG metric axioms

Theorem (Gwynne–Miller, '19)

The γ-LQG metric D_h is the unique continuous metric satisfying the following axioms up to a deterministic multiplicative constant.

1. **Length space**: $\forall z, w \in \mathbb{C}$ and $\varepsilon > 0$, there is a path $P : z \to w$ such that $\mathrm{len}(P; D_h) < D_h(z, w) + \varepsilon$.
2. **Locality:** For a deterministic open set $U \subset \mathbb{C}$, the internal metric D_h^U is a.s. determined by $h|_U$.
3. **Weyl scaling**
4. **Coordinate change for translation and scaling**

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$\Rightarrow$ Since $D_h^{\eta[s,t]}$ is a.s. determined by $h|_{\eta[s,t]}$, which is determined by $(L, R)|_{[s,t]}$, we would like to say that $(\eta[s, t], D_h^{\eta[s,t]})$ has "independent and stationary increments."

Main result

Proposition

For each $s < t$, almost surely,

$$\operatorname{diam}(\eta[s,t]; D_h) := \sup_{z,w\in\eta[s,t]} D_h^{\eta[s,t]}(z,w) \quad \begin{cases} < \infty & \text{if } \gamma \in (0, \sqrt{8/3}), \\ = \infty & \text{if } \gamma \in [\sqrt{8/3}, 2). \end{cases}$$

Theorem (Feng–S., '23+)

The curve-decorated metric measure space $\mathcal{S}_{[s,t]} := (\eta[s,t], D_h^{\eta[s,t]}, \mu_h|_{\eta[s,t]}, \eta|_{[s,t]})$ is measurable under the Gromov–Hausdorff–Prokhorov–uniform topology for $\gamma \in (0, \sqrt{8/3})$. Moreover, it has an infinitely divisible law:

1. For $s_1 < t_1 \leq \cdots \leq s_k < t_k$, the surfaces $\mathcal{S}_{[s_1,t_1]}, \ldots, \mathcal{S}_{[s_k,t_k]}$ are independent.
2. The law is stationary under translations in time: i.e., $\mathcal{S}_{[s,t]} \overset{d}{=} \mathcal{S}_{[0,t-s]}$.

Background: Minkowski content measure for the LQG metric

- $N_\varepsilon(A; D_h) :=$ the smallest number of D_h-balls of radii ε required to cover A.

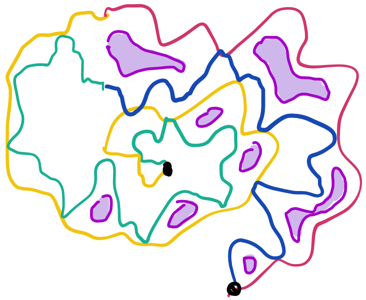
Theorem (Gwynne–S., '22)

For each $\gamma \in (0,2)$, let $\mathfrak{b}_\varepsilon = \mathbb{E}[N_\varepsilon(\eta[0,1]; D_h)]$. Then $\mathfrak{b}_\varepsilon \asymp \varepsilon^{-d_\gamma}$, and for every bounded Borel set $A \subset \mathbb{C}$ with $\mu_h(\partial A) = 0$ a.s., we have

$$\mu_h(A) = \lim_{\varepsilon \to 0} \mathfrak{b}_\varepsilon^{-1} N_\varepsilon(A; D_h) \quad \text{in probability.}$$

Consequently, the pointed metric space $(\mathbb{C}, D_h, 0)$ (measurable w.r.t. the Gromov–Hausdorff topology) a.s. determines the field h up to rotation and scaling of $\mathbb{C}$.

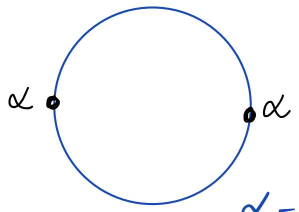
- Define $\widetilde{N}_\varepsilon(A) = N_\varepsilon(\{z \in A : D_h(z, \partial A) \geq \varepsilon\}; D_h)$. This is a.s. determined by D_h^A.
- Key lemma: $(\widetilde{N}_\varepsilon(\eta[s,t]))_{\varepsilon > 0}$ is independent over disjoint intervals and its law is stationary under translations in time.



Proof overview

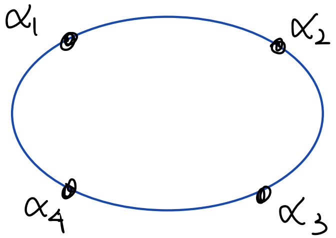
1. Identify the law of $h|_{\eta[s,t]}$.
   - We use the LCFT description for conformal welding by Ang, Holden, Yu, Remy, and Sun.

$\mathcal{M}_2^{\text{disk}}(W) =$



$\alpha = \gamma + \frac{2-W}{\gamma}$

$\mathrm{Quad}(W_1, \cdots, W_4 ; m) =$

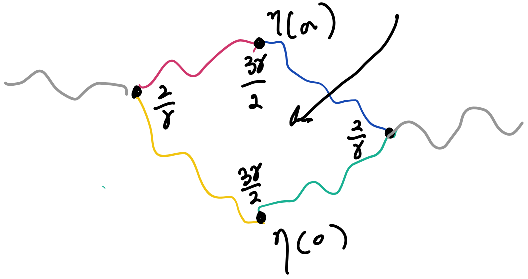


m = modulus

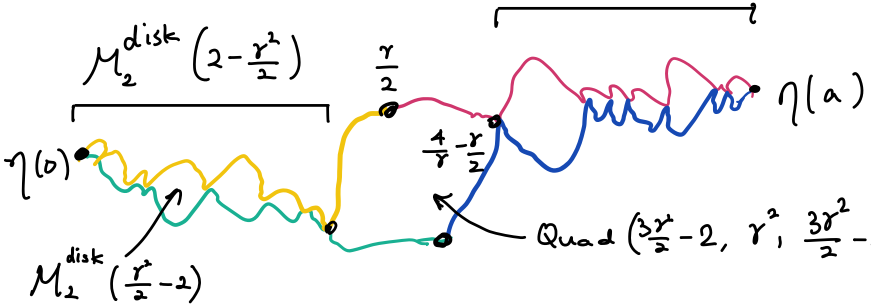
   - Let P_a be the law of the LQG cell $\mathcal{C}_{[0,a]} = (\eta[0,a], h|_{\eta[0,a]}, \eta|_{[0,a]})/\sim_\gamma$.
   - For each γ, there exists an unspecified probability measure ρ on $(0,\infty)$ such that $P_a \times 1_{a>0}\, da$ is proportional to:

$\boxed{\gamma \in (0, \sqrt{2}]}$

$\mathrm{Quad}(2-\frac{\gamma^2}{2}, \gamma^2, 2-\frac{\gamma^2}{2}, \gamma^2 ; m) \times \rho(dm)$



$\boxed{\gamma \in (\sqrt{2}, 2)}$



$\mathrm{Quad}(\frac{3\gamma^2}{2}-2, \gamma^2, \frac{3\gamma^2}{2}-2, \gamma^2 ; m) \times \rho(dm)$

Proof overview

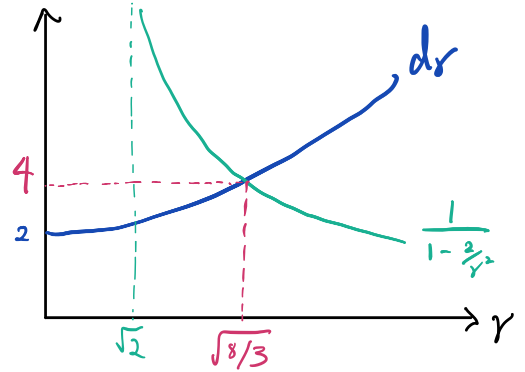
2. Bound the diameter of each quantum disk/quadrangle.
   - Hughes–Miller ('22): A.s., D_h^U extends continuously to ∂U when h is a free-boundary GFF on U.
   - This result extends to fields with boundary insertions of order $\alpha \leq Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.
   - For a thick quantum disk $(\mathcal{D}, h, x_1, x_2)$ sampled from $\mathcal{M}_2^{\mathrm{disk}}(W)$ with $W > \gamma^2/2$, $D_h(x_1, x_2) \leq C\mu_h(\mathcal{D})^{1/d_\gamma}$ with superpolynomially high probability as $C \to \infty$.

Proof overview

3. Concatenate small beads in the thin quantum disk.

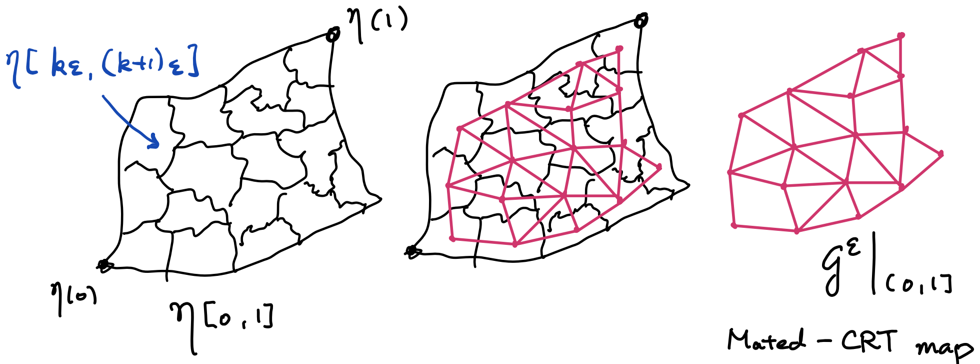
- The thin quantum disk $\mathcal{M}_2^{\text{disk}}(W)$ with $W < \gamma^2/2$ can be sampled by:
  - Sample T from $(1 - \frac{2}{\gamma^2} W)^{-2} 1_{T>0}\, dT$.
  - Sample a Poisson point process $\{(u, \mathcal{D}_u)\}$ from the intensity measure $1_{t\in[0,T]} \times \mathcal{M}_2^{\text{disk}}(\gamma^2 - W)$.
  - Concatenate the disks $\{\mathcal{D}_u\}$ chronogically.
- $\{(u, \mu_h(\mathcal{D}_u))\}$ is a p.p.p. with intensity measure $dt \times a^{(2/\gamma^2-1)-1}\, da$ on $[0, T] \times (0, \infty)$.
- $\{(u, D_h^{\mathcal{D}_u}(x_1, x_2))\}$ is a p.p.p. with intensity measure $dt \times s^{(2/\gamma^2-1)d_\gamma-1}\, ds$ on $[0, T] \times (0, \infty)$.
- The distances between the marked points of the beads are summable if and only if

$$\int_0^1 s^{(\frac{2}{\gamma^2}-2)d_\gamma}\, ds < \infty \quad \Leftrightarrow \quad \left(\frac{2}{\gamma^2} - 1\right) d_\gamma > -1 \quad \Leftrightarrow \quad d_\gamma < \frac{1}{1 - \frac{2}{\gamma^2}}$$



- $\gamma \mapsto d_\gamma$ is strictly increasing and $d_{\sqrt{8/3}} = 4$.

Application: The distance exponent for the mated-CRT map



- Gwynne–Holden–Sun ('16):

$$\chi(\gamma) := \lim_{\varepsilon \to 0} \frac{\log \mathbb{E}[\mathrm{diam}(\mathcal{G}^\varepsilon|_{(0,1]})]}{\log \varepsilon^{-1}}$$

  exists and $\chi(\gamma) \leq 1 - 2/\gamma^2$.

- Conjecture: There exists $\gamma_* \in (\sqrt{2}, \sqrt{8/3}]$ such that for $\gamma \in (0, \gamma_*]$, we have $\chi(\gamma) = 1/d_\gamma$.

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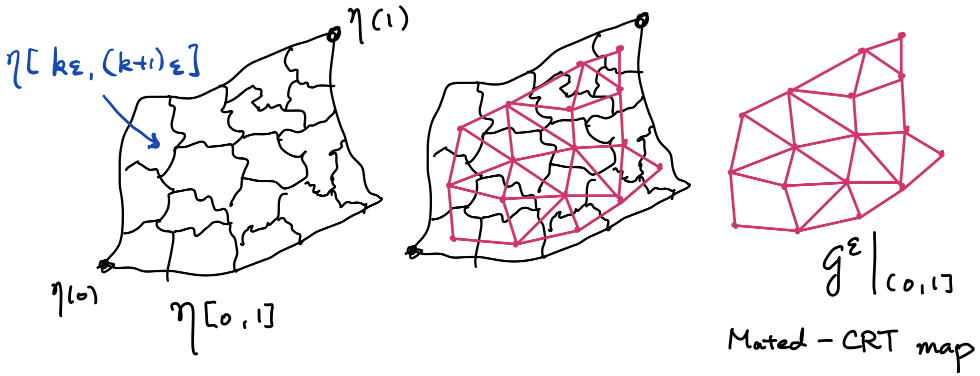

Theorem (Feng–S., '23+)

The conjecture holds with $\gamma_* = \sqrt{8/3}$.
Moreover, $\chi(\gamma) = (1/d_\gamma) \vee (1 - 2/\gamma^2)$.

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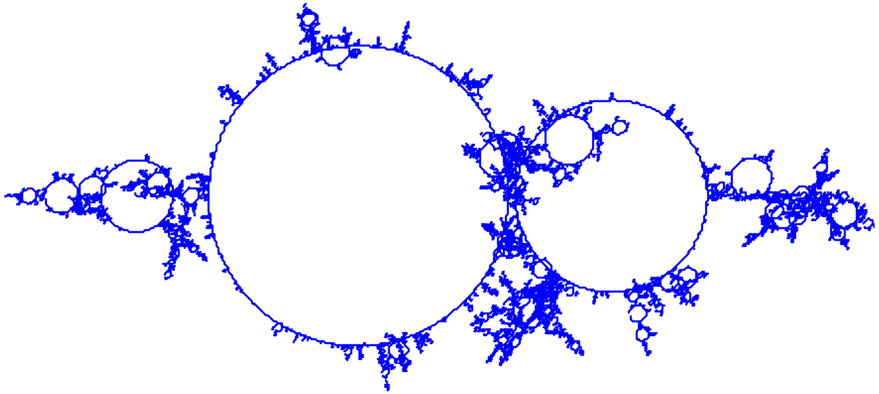
- Proved for $\gamma = \sqrt{2}$ by Gwynne–Pfeffer ('19).
- The proof consists of the following two main inputs:
  - Ang–Falconet–Sun ('20): Almost surely, $\mu_h(\mathcal{B}_r(z; D_h)) = r^{d_\gamma + o(1)}$ as $r \to 0$ uniformly for z on compact sets.
  - $\mathbb{E}[(\mathrm{diam}(\eta[0,1], D_h^{\eta[0,1]}))^p] < \infty$ for some $p > 0$ (in fact, for all $p > 0$).

Questions

- There are metric spaces constructed from branching structures that correspond to $\gamma \geq \sqrt{8/3}$. Are the two thresholds related?
  - $\gamma = \sqrt{8/3}$: Brownian sphere, constructed from a continuum random tree. (Le Gall)

    Corresponds to a $\sqrt{8/3}$-LQG sphere. (Miller–Sheffield)
  - $\gamma \in (\sqrt{8/3}, 2)$: Stable maps, constructed from a stable looptree (Curien–Miermont–Riera, '22+).

    Conjectured to correspond to a CLE_{γ^2} carpet on a γ-LQG sphere.



Stable looptree
(simulation by Curien and Kortchemski)

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- Is there a different space-filling curve that gives an infinitely divisible structure for the γ-LQG metric space, especially when $\gamma \geq \sqrt{8/3}$?
  - Kaavadias–Miller ('22+): For $\gamma \in (\sqrt{4/3}, 2)$, $\mathrm{SLE}_{\gamma^2}(\gamma^2 - 4)$ on γ-LQG can be viewed as a Peano curve between a pair of stable looptrees, which is the scaling limit of bipolar oriented random planar maps with large faces.

감 사 합 니 다

kam sa ham ni da