Random surfaces and Yang-Mills

Scott Sheffield

Includes joint work with Sky Cao (IAS), Minjae Park (Chicago), Joshua Pfeffer (Columbia), Pu Yu (MIT)

June 6, 2023

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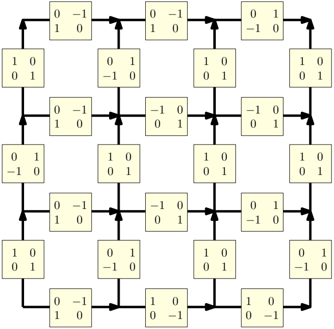

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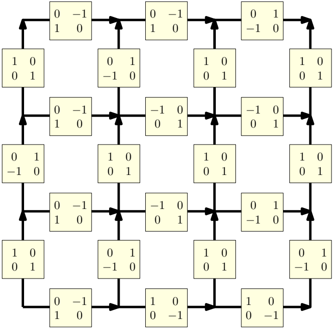

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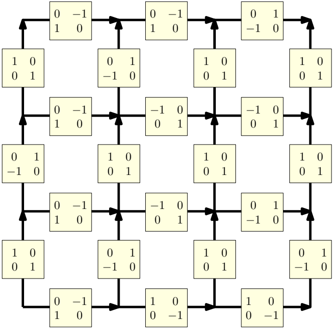

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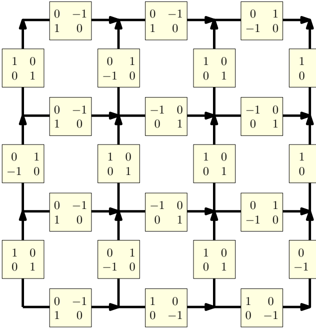

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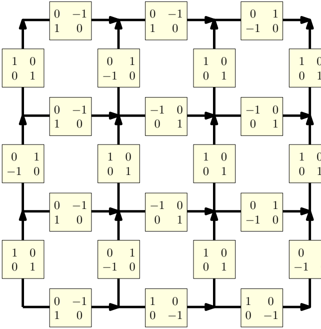

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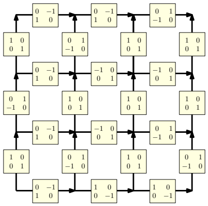
- **Wilson loop:** multiply matrices around directed cycle, find expected trace.
- **Yang Mills problem (roughly):** construct/understand basics of continuum version. Famous prize problem. Important for standard model.

BIG PUZZLE: APPROACHES



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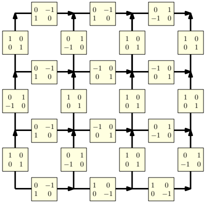

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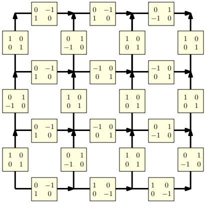

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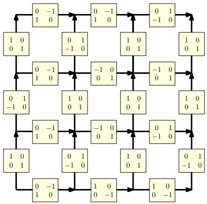

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- **NON-DYNAMIC CONTINUUM APPROACH:** start with Gaussian connection and modify field in some other way (to make it roughly correct at some scales) and take limit as approximation improves.
- **LATTICE APPROACH:** Explore the lattice model, possibly in terms of random surfaces, to gain insight into continuum theory.

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- arXiv paper develops techniques to treat matrix trace problem in continuum 2D Yang-Mills. Forthcoming work applies these techniques to lattice Yang-Mills in higher dimensions.

Wilson loop expectations as sums over surfaces on the plane

Minjae Park
University of Chicago

Joshua Pfeffer
Columbia University

Scott Sheffield
MIT

Pu Yu
MIT

May 4, 2023

Abstract

Although lattice Yang-Mills theory on finite subgraphs of $\mathbb{Z}^d$ is easy to rigorously define, the construction of a satisfactory continuum theory on $\mathbb{R}^d$ is a major open problem when $d \geq 3$. Such a theory should in some sense assign a Wilson loop expectation to each suitable finite collection $\mathcal{L}$ of loops in $\mathbb{R}^d$. One classical approach is to try to represent this expectation as a sum over surfaces with boundary $\mathcal{L}$. There are some formal/heuristic ways to make sense of this notion, but they typically yield an ill-defined difference of infinities.

In this paper, we show how to make sense of Yang-Mills integrals as surface sums for $d = 2$, where the continuum theory is more accessible. Applications include several new explicit calculations, a new combinatorial interpretation of the master field, and a new probabilistic proof of the Makeenko-Migdal equation.

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- **AD:** If t small, matrices$\approx$identity. And $(I + M_1)(I + M_2) \approx I + M_1 + M_2$, so products$\approx$sums. See Narayanan-S for matrix sum spectral analysis.

Large deviations for random hives and the spectrum of the sum of two random matrices

Hariharan Narayanan[1], **and Scott Sheffield**[2]

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Abstract:
Suppose α, β are Lipschitz strongly concave functions from $[0,1]$ to $\mathbb{R}$ and γ is a concave function from $[0,1]$ to $\mathbb{R}$, such that $\alpha(0) = \gamma(0) = 0$, and $\alpha(1) = \beta(0) = 0$ and $\beta(1) = \gamma(1) = 0$. For an $n \times n$ Hermitian matrix W, let $\mathrm{spec}(W)$ denote the vector in $\mathbb{R}^n$ whose coordinates are the eigenvalues of W listed in non-increasing order. Let $\lambda = \partial^- \alpha$, $\mu = \partial^- \beta$ on $(0,1]$ and $\nu = \partial^- \gamma$, at all points of $(0,1]$, where ∂^- is the left derivative. Let $\lambda_n(i) := n^2(\alpha(\frac{i}{n}) - \alpha(\frac{i-1}{n}))$, for $i \in [n]$, and similarly, $\mu_n(i) := n^2(\beta(\frac{i}{n}) - \beta(\frac{i-1}{n}))$, and $\nu_n(i) := n^2(\gamma(\frac{i}{n}) - \gamma(\frac{i-1}{n}))$.

Let X_n, Y_n be independent random Hermitian matrices from unitarily invariant distributions with spectra λ_n, μ_n respectively. We define norm $\|\cdot\|_{\mathcal{I}}$ to correspond in a certain way to the sup norm of an antiderivative. We prove that the following limit exists.

$$\lim_{n\to\infty} \frac{\ln \mathbb{P}\left[\|\mathrm{spec}(X_n + Y_n) - \nu_n\|_{\mathcal{I}} < \epsilon\right]}{n^2}.$$

We interpret this limit in terms of the surface tension σ of continuum limits of the discrete hives defined by Knutson and Tao.

We provide matching large deviation upper and lower bounds for the spectrum of the sum of two random matrices X_n and Y_n, in terms of the surface tension σ mentioned above.

We also prove large deviation principles for random hives with α and β that are C^2, where the rate function can be interpreted in terms of the maximizer of a functional that is the sum of a term related to the free energy of hives associated with α, β and γ and a quantity related to logarithms of Vandermonde determinants associated with γ.

MSC2020 subject classifications: Primary 60F10, 60B20; secondary 82B41.
Keywords and phrases: Large deviations, Random matrices, Random surfaces.

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- **VISUALIZATION PUZZLE:** Suppose boundary values of ϕ map boundary of T to boundary of large planar domain. What do you expect a random instance of T to look like when embedded in 3D via ϕ?

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- Consider a uniformly random planar triangulation T with a fixed number of edges/vertices and a fixed number of boundary edges/vertices.
- Now *weight* the law of T by the partition function for a 3D Gaussian free field ϕ on T (subject to some fixed boundary conditions on ∂T).
- **VISUALIZATION PUZZLE:** Suppose boundary values of ϕ map boundary of T to boundary of large planar domain. What do you expect a random instance of T to look like when embedded in 3D via ϕ?
- **FOLLOW UP:** How would you imagine a measure-preserving Glauber dynamics (where you can resample ϕ at individual points, or remove edge to make quadrilateral, then glue it back along other diagonal)?

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- Did you imagine...
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- **AD:** To visualize random planar maps using **Smith embeddings** see just-posted arXiv paper: Bertacco, Gwynne, S.

Scaling limits of planar maps under the Smith embedding

Federico Bertacco
Imperial College London

Ewain Gwynne
University of Chicago

Scott Sheffield
MIT

Abstract

The Smith embedding of a finite planar map with two marked vertices, possibly with conductances on the edges, is a way of representing the map as a tiling of a finite cylinder by rectangles. In this embedding, each edge of the planar map corresponds to a rectangle, and each vertex corresponds to a horizontal segment. Given a sequence of finite planar maps embedded in an infinite cylinder, such that the random walk on both the map and its planar dual converges to Brownian motion modulo time change, we prove that the a priori embedding is close to an affine transformation of the Smith embedding at large scales. By applying this result, we prove that the Smith embeddings of mated-CRT maps with the sphere topology converge to γ-Liouville quantum gravity (γ-LQG).

Contents

PUZZLE VARIANT: EDGE-PLAQUETTE EMBEDDINGS

- Consider a pair $(\mathcal{M}, \psi)$ where $\mathcal{M}$ is a planar (or higher genus) map and $\psi : \mathcal{M} \to \mathcal{L}$ is a graph homomorphism. We assume that:
  1. The dual graph of $\mathcal{M}$ is bipartite. Faces of $\mathcal{M}$ can be colored blue and yellow.
  2. ϕ maps each yellow face of $\mathcal{M}$ isometrically *onto* a face of $\mathcal{L}$.
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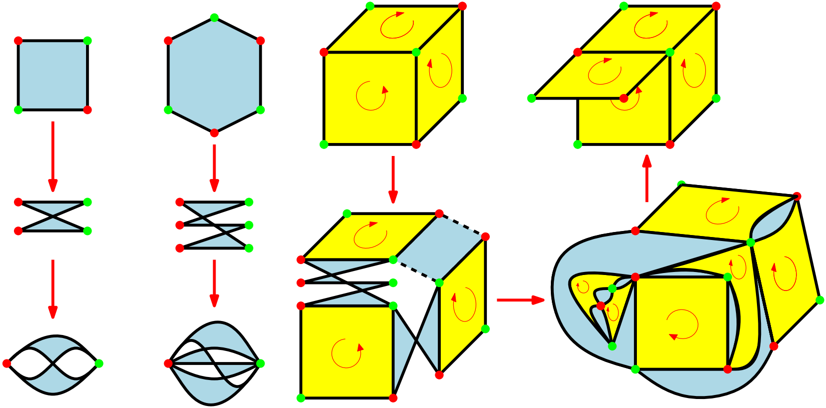
- Call this an **edge-plaquette embedding** (EPE) because each blue face maps onto a single edge, and each yellow face maps onto a single plaquette.



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- Take σ_N to be Haar measure on group like $U(N)$, $SO(N)$, etc.
- Can replace exp with real polynomial that is large near 1 and small (but positive) elsewhere in $[-1, 1]$. For example $x \to x^{100}(1 + x)$.

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$$\mathcal{W}(\sigma) = \frac{1}{(L!)} \sum_{\lambda} \Big[\chi_\lambda(e) \chi_\lambda(\sigma) \prod_{(i,j)\in\lambda} (n + j + i) \Big]$$

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- To simplify $\mathcal{W}$ expression we need nice way to express $\chi_\lambda(\sigma)$.

Yuval Roichman
Massachusetts Institute of Technology

Abstract. Recent developments in the theory of Young tableaux imply good bounds on the characters of the symmetric groups. The applications of these estimates vary from classical problems in representation theory and the theory of permutation groups to random walks and graph theory.

1. Introduction.

Let g be an element in a finite group G, and let id be the identity element in G. Let ρ be an ordinary representation of G. The character $\chi_\rho(g)$ is the trace of $\rho(g)$. The *normalized character* of ρ at g, denoted by $r_\rho(g)$, is the ratio $\frac{\chi_\rho(g)}{\chi_\rho(id)}$.

The normalized characters of the irreducible representations of finite groups play an important role in the study of random walks on these groups [D ch. 3], expander graphs [Lu1 ch. 8] and many other areas. Unfortunately, there are no general explicit formulas for the irreducible characters of the symmetric groups. The Murnaghan-Nakayama rule (Theorem 2.6) is a recursive method to compute the characters. Frobenius formula (Theorem 2.4) presents the characters as coefficients of a complicated polynomial. Explicit formulas for characters of very simple conjugacy classes (e.g. cycles of length ≤ 6) follow from this formula. See [Fs], [In] and [Su].

From Magee and Puder

$$\mathrm{Wg}_L(\sigma) = \frac{1}{(L!)^2} \sum_{\lambda \vdash L} \frac{\chi_\lambda(e)^2}{d_\lambda(n)} \chi_\lambda(\sigma),$$

where λ runs over all partitions of L, χ_λ is the character of S_L corresponding to λ, and $d_\lambda(n)$ is the number of semistandard Young tableaux with shape λ, filled with numbers from $[n]$. A well known formula for $d_\lambda(n)$ is $d_\lambda(n) = \frac{\chi_\lambda(e)}{L!} \prod_{(i,j)\in\lambda} (n + j - i)$, where (i, j) are the coordinates of cells in the Young diagram with shape λ (e.g. [Ful97, Section 4.3, Equation (9)]). Thus,

Corollary 2.3. *For $\sigma \in S_L$, $\mathrm{Wg}_L(\sigma)$ may have poles only at integers n with $-L < n < L$.*

Below we use the following properties of the Weingarten function. The standard norm of $\rho \in S_L$, denoted $\|\rho\|$, is the shortest length of a product of transpositions giving ρ, and is equal to $L - \#\mathrm{cycles}(\rho)$.

Theorem 2.4. *Let $\pi \in S_L$ be a permutation.*

1. *[CŚ06, Corollary 2.7] Leading term:*

$$\mathrm{Wg}_L(\pi) = \frac{\mathrm{Möb}(\pi)}{n^{L+\|\pi\|}} + O\left(\frac{1}{n^{L+\|\pi\|+2}}\right), \tag{2.1}$$

where

$$\mathrm{Möb}(\pi) = \mathrm{sgn}(\pi) \prod_{i=1}^{k} c_{|C_i|-1}, \tag{2.2}$$

with[6] $C_1, \ldots, C_k$ the cycles composing π, and $c_m = \frac{(2m)!}{m!(m+1)!}$ being the m-th Catalan number.

2. *[Col03, Theorem 2.2] Asymptotic expansion:*

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- In general there are lots of symmetries that allow cancellation (e.g. moves that swap sign but don't change genus). In 2D, weighted sum over all surfaces is equivalent to weighted sum over flat surfaces. Do the battling insects cancel themselves out in higher dimensions? At least when $N = \infty$?

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- **AD:** Magee and Puder offer more context about Weingarten function.

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- In two dimensions, gauge fixing simplifies problem tremendously. Two dimensions can be place to test theories believed to hold in general dimension.

Classical matrix-map stories

- Many names: Balaban, Brézin, Brydges, Chatterjee, Di Franceso, Eynard, Feynman, Fr´olich, Guionnet, Harer, Itzykson, Kazakov, Kostov, Mehta, Parisi, Seiler, ‘t Hooft, Wilson, Witten, Zagier, Zeitouni, Zinn-Justin, Zuber... (This list is far from exhaustive.)

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- Sample A from N-dim. GUE. How do you compute $E[\mathbf{Tr}A^4\mathbf{Tr}A^6\mathbf{Tr}A^8]$?

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- Sample A from N-dim. GUE. How do you compute $E[\mathbf{Tr}A^4\mathbf{Tr}A^6\mathbf{Tr}A^8]$?
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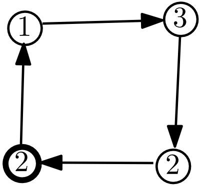
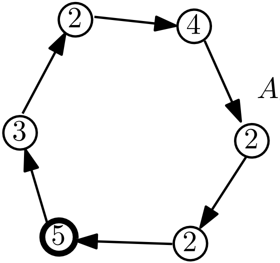
- Do same thing for $\mathbf{Tr}A^6$ and $\mathbf{Tr}A^8$. Get hexagon and octagon.

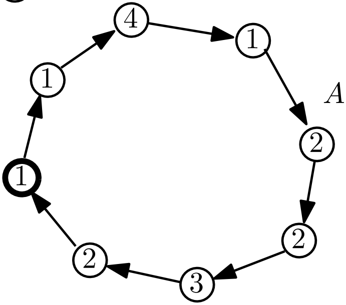
Graphical representation of a term of $\mathbf{Tr}(A^4)\mathbf{Tr}(A^6)\mathbf{Tr}(A^8)$

$A_{2,1}A_{1,3}A_{3,2}A_{2,2}$





$A_{5,3}A_{3,2}A_{2,4}A_{4,2}A_{2,2}A_{2,5}$



$A_{1,1}A_{1,4}A_{4,1}A_{1,2}A_{2,2}A_{2,3}A_{3,2}A_{2,1}$

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- Similar story for GOE but maps not orientable, weights are signed.

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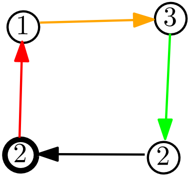
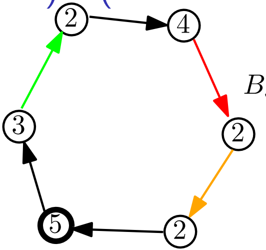
- Expansion in powers of N enumerates within genus classes.
- What if you have more than one matrix?

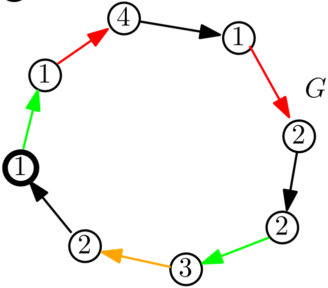
Graphical representation of a term of $\mathbf{Tr}(ROGB)\mathbf{Tr}(BGBROB)\mathbf{Tr}(GRBRBGOB)$



$R_{2,1}O_{1,3}G_{3,2}B_{2,2}$



$B_{5,3}G_{3,2}B_{2,4}R_{4,2}O_{2,2}B_{2,5}$



$G_{1,1}R_{1,4}B_{4,1}R_{1,2}B_{2,2}G_{2,3}O_{3,2}B_{2,1}$

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- Imagine assigning a matrix $A^{v,w}$ with i.i.d. complex Gaussian entries to each directed edge (v, w) of a lattice. Actually, let's impose constraint that $A^{v,w}$ is conjugate tranpose of $A^{w,v}$. So we have one matrix of information for each edge.

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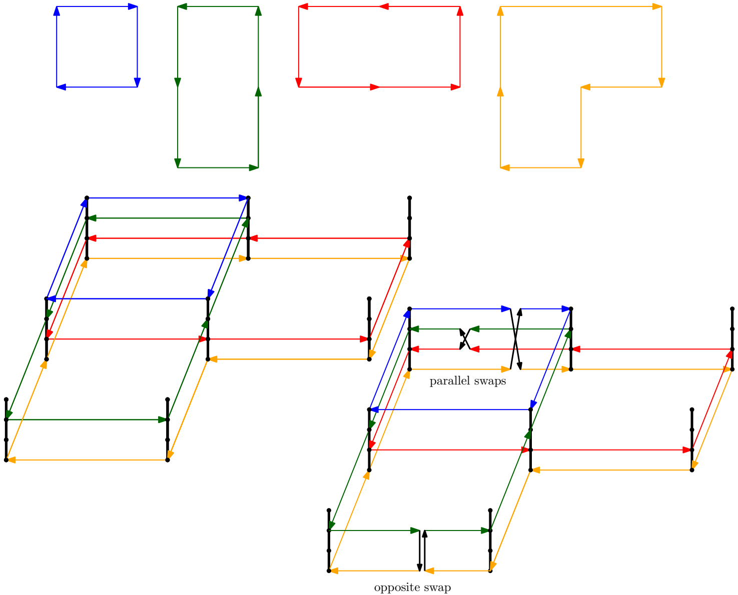
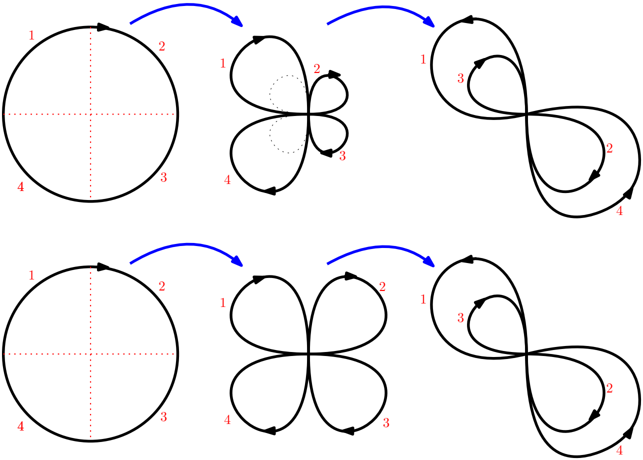
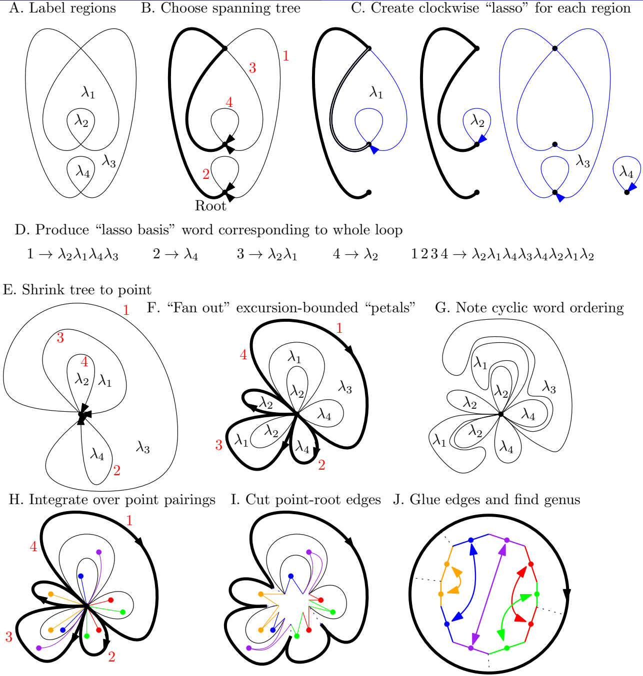
- For any oriented plaquette P can write $\mathbf{Tr}P$ for trace of corresponding product of matrices. Now we can formally compute $E[e^{\sum \mathbf{Tr}(p)}]$ where sum ranges over all oriented plaquettes. Using Wick's theorem, we get a sum of surfaces built out of oriented plaquettes.

We can divide each edge into approximate Gaussians

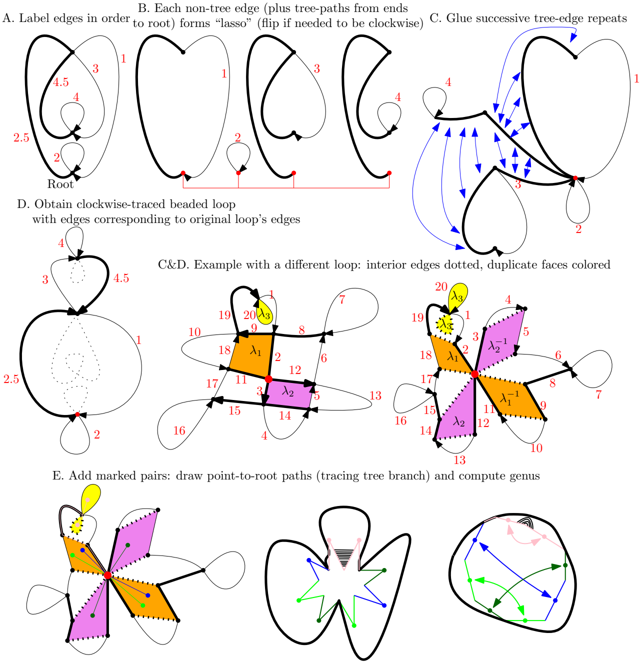


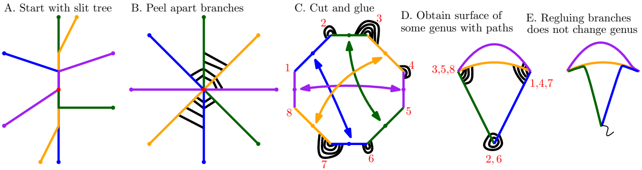
Surface interpretations for 2D Wilson loop expectations





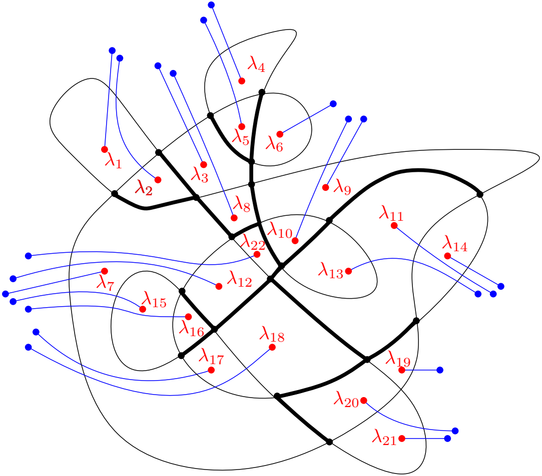
A. Label regions
B. Choose spanning tree
C. Create clockwise "lasso" for each region
λ_1
λ_2
λ_3
λ_4
1
2
3
4
Root
D. Produce "lasso basis" word corresponding to whole loop
$1 \to \lambda_2\lambda_1\lambda_4\lambda_3$ $2 \to \lambda_4$ $3 \to \lambda_2\lambda_1$ $4 \to \lambda_2$ $1\,2\,3\,4 \to \lambda_2\lambda_1\lambda_4\lambda_3\lambda_4\lambda_2\lambda_1\lambda_2$
E. Shrink tree to point
F. "Fan out" excursion-bounded "petals"
G. Note cyclic word ordering
H. Integrate over point pairings
I. Cut point-root edges
J. Glue edges and find genus


A. Label edges in order
B. Each non-tree edge (plus tree-paths from ends to root) forms "lasso" (flip if needed to be clockwise)
C. Glue successive tree-edge repeats
Root
D. Obtain clockwise-traced beaded loop with edges corresponding to original loop's edges
C&D. Example with a different loop: interior edges dotted, duplicate faces colored
E. Add marked pairs: draw point-to-root paths (tracing tree branch) and compute genus


A. Start with slit tree
B. Peel apart branches
C. Cut and glue
D. Obtain surface of some genus with paths
E. Regluing branches does not change genus
1
2
3
4
5
6
7
8
3,5,8
1,4,7
2, 6

A. Draw spanning tree in black, one red vertex for each face

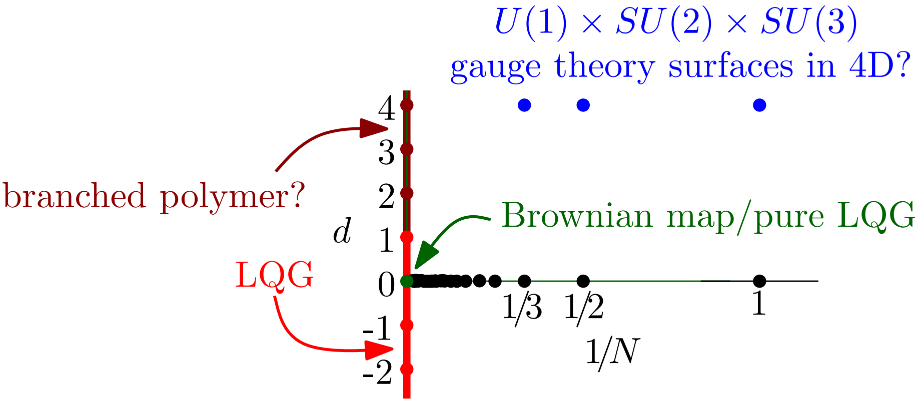
B. Draw blue dual tree path from each red vertex to outer face



C. Form word by recording λ_i whenever corresponding blue edge is crossed clockwise, λ_i^{-1} if counterclockwise

D. Form surface containing k copies of λ_i face if corresponding blue path crosses k black edges.

Continuum scaling limits of random surfaces?



Can interpret d as a lattice dimension **or** (as we will later see) weight factor for planar maps (based on determinant of Laplacian). Can interpret N as a matrix dimension **or** as a weight factor (based on surface genus). Non-integer values of d and N make sense. Need to handle oscillatory weighting and cancellation.

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- The DGFF partition function can be be written $\int_R (2\pi)^{-|V-1|/2} e^{-(f,-\Delta f)/2} df = \alpha^{-1/2}$.