Measures on random fractals

A conformally covariant measure of CLE

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Random Conformal Geometry and Related Fields in Jeju
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Outline

1. Introduction
   - Measures on fractals
   - Constructing measures using the GFF

2. The conformally covariant measure on CLE
   - Construction and a measure on CLE_4
   - Uniqueness

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Measures on fractals

A central aspect of the analysis of fractals is the construction of measures on top of them. They can provide us with

- Hausdorff dimension
- Ways of sampling points on the fractal

Measures on fractals

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  Say that K is the set we are interested in. Typically such measures are defined as a subsequential limit of measures:

$$\mu_n(A) = \int_A \sum_{z \in D_n} \frac{E_n(z)}{\mathbb{E}[E_n(z)]} 1_{J_n(z)}(w) dw$$

  Here E_n is a random variable s.t. if $E_n(z) > 0$ for all $n \in \mathbb{N}$, then $z \in K$, D_n grid of points, $J_n(z)$ neighbourhood of z.

Measures on fractals

- **Minkowski content:** The d-dimensional Minkowski content of a set A is defined as

$$\mathcal{M}^d(A) = \lim_{r\to 0} r^{d-2} \int_{\mathbb{C}} 1_{\{\text{dist}(z,A)<r\}} dz.$$

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Good luck.

Properties and relations

- Hausdorff measure and Minkowski content agree on nice sets (for example $\mathcal{H}^1(\gamma) = \mathcal{M}^1(\gamma)$ if γ is a rectifiable curve). In general, there is a constant $C = C(s)$

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- Minkowski contents satisfy the following conformal covariance relation. Let $A \subset \overline{D}$ and $f : D \to \widetilde{D}$ be conformal. Then,

$$\mathcal{M}^d(f(A)) = \int_A |f'(z)|^d d\mathcal{M}^d(z).$$

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- However, the key step in getting the up to constants asymptotics often consists of showing that the limit exists with $\text{dist}(z, K)$ replaced by $\mathrm{crad}_{D \setminus K}(z)$, that is,

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$\Rightarrow$ In a good position to construct the conformal Minkowski content!

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- Minkowski content of Brownian cut points in dimensions 2 and 3 [Holden-Lawler-Li-Sun, 2022]

Natural parametrisation of SLE

- The natural parametrisation of SLE is conjecturally the parametrisation which arises when SLE is constructed as the scaling limit of discrete interfaces normalised by step size. (And proved by Lawler and Viklund for the convergence of LERW to SLE_2!)
- Constructed by Lawler and Sheffield for chordal SLE and shown by Lawler and Rezaei to be (a constant times) the Minkowski content of SLE.

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- Let $\mathbb{A}_{-a}$ be the first passage set of h of level $-a$, let $h_{\mathbb{A}_{-a}}$ be the distribution that is harmonic off $\mathbb{A}_{-a}$ that arises with the coupling, and let $\mathfrak{h}_{\mathbb{A}_{-a}}$ be the harmonic function (i.e. $h_{\mathbb{A}_{-a}} = \mathfrak{h}_{\mathbb{A}_{-a}}$ in $D \setminus \mathbb{A}_{-a}$).

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- Define the measure $\nu_{\mathbb{A}_{-a}} = h_{\mathbb{A}_{-a}} - \mathfrak{h}_{\mathbb{A}_{-a}}$. Then, $\nu_{\mathbb{A}_{-a}}$ defines a positive measure on $\mathbb{A}_{-a}$ and

$$\nu_{\mathbb{A}_{-a}}(f) = \lim_{\varepsilon \to 0} \frac{1}{2} |\log \varepsilon|^{1/2} \int_D f(z) 1_{\{\mathrm{dist}(z, \mathbb{A}_{-a}) \le \varepsilon\}}(z) dz.$$

That is, $\nu_{\mathbb{A}_{-a}}$ is (proportional to) the Minkowski content "in the gauge" $\varepsilon \mapsto \varepsilon^2 |\log \varepsilon|^{1/2}$. [Aru-Lupu-Sepúlveda, 2019]

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This means that $\mathcal{M}^2(\mathbb{A}_{-a}) = 0$, that is, we should maybe not expect the Minkowski content of each interesting fractal to exist.

Open problems

We are yet to construct/show the existence of the Minkowski contents or conformal Minkowski contents of:

- **SLE**: double points, cut points, exceptional sets (e.g. the multifractal spectra)
- **CLE**: CLE and its pivotal points
- **GFF**: thick points, two-valued sets

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For many choices of X_h, this yields a measure on the random fractal K which exhibits conformal covariance with the "right" exponent.
The compensator F is typically a power of the conformal radius, to handle the conformal radius terms that naturally occur when considering the image/pushforward of X_h.

Natural parametrisation of SLE

Let $\eta \sim \mathrm{SLE}_\kappa$ and (f_t) be the centred Loewner chain of η. Let

$$\eta^t(u) = f_t(\eta(t+u)), \quad t \geq 0.$$

Let $d_\kappa = 1 + \kappa/8$ and for a measure μ on η, define the measure μ^t on η^t by

$$\mu^t(dz) = |(f_t^{-1})'|^{-d_\kappa} \mu^0 \circ f_t^{-1}(dz).$$

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If (η, μ) and (η^t, μ^t) have the same law for each $t > 0$, then μ is the natural parametrisation of η.

Natural parametrisation of SLE

Let $\eta \sim \mathrm{SLE}_\kappa$ in $\mathbb{H}$, $\kappa \in (0,8)$. Let h be a zero-boundary GFF in $\mathbb{H}$ and let ξ_h be the LQG length measure associated with η^0 (subcritical if $\kappa < 4$, critical if $\kappa = 4$, generalised quantum length if $\kappa \in (4,8)$).

$$\mu(dz) = \mathrm{crad}_{\mathbb{H}}(z)^{q(\kappa)} \mathbb{E}[\xi_h(dz)|\eta]$$

Theorem

*The measure μ is (a deterministic constant times) the natural parametrisation of SLE_κ. ($\kappa \in (0,4)$: **[Benoist, 2018]**, $\kappa = 4$: **[Margarint-S., in preparation]**, $\kappa \in (4,8)$: **[Miller-S.,2023]***

Natural measure on CLE

Fix $\kappa \in (8/3, 8)$ and let $\mathbb{P}_D$ be a law on pairs (Γ, Ξ) where $\Gamma \sim \mathrm{CLE}_\kappa$ on D and Ξ is a measure supported on the carpet/gasket of Γ.

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- $(\mathbb{P}_D)$ is said to be conformally covariant with exponent d if the following is true. Assume that $\varphi : D \to \widetilde{D}$ is conformal and $(\Gamma, \Xi) \sim \mathbb{P}_D$ and $(\widetilde{\Gamma}, \widetilde{\Xi}) \sim \mathbb{P}_{\widetilde{D}}$ are coupled together so that $\widetilde{\Gamma} = \varphi(\Gamma)$. Then, for all Borel $A \subseteq D$,

$$\widetilde{\Xi}(\varphi(A)) = \int_A |\varphi'(z)|^d d\Xi(z).$$

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For $U \subseteq D$, let U^Γ be the points in U which are not on or surrounded by a loop in Γ which intersects U^c and Γ_U be the loops of Γ, contained in $\overline{U}$.

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- We say that $(\mathbb{P}_D)$ satisfies the CLE_κ Markov property if the following is true. Suppose that $(\Gamma, \Xi) \sim \mathbb{P}_D$, $U \subseteq D$ open and V is a component of U^Γ. Then, the conditional law of $(\Gamma_V, \Xi|_V)$ given $\Gamma \setminus \Gamma_V$ and $\Xi|_{D \setminus V}$ is that of $\mathbb{P}_V$.

Natural measure on CLE

Let d_κ be the a.s. Hausdorff dimension of CLE_κ

Theorem (Miller-S. 2023)

Fix $\kappa \in (8/3, 8)$. Then there is a family of probability measures $(\mathbb{P}_D)$ as on the previous slide, which are conformally covariant with exponent d_κ, satisfies the CLE_κ Markov property, and such that if $(\Gamma, \Xi) \sim \mathbb{P}_D$, then $\mathbb{E}_D[\Xi(K)] < \infty$ for each compact $K \subset D$ and $\mathbb{P}(\Xi(D) > 0) = 1$. Let $(\widetilde{\mathbb{P}}_D)$ be another such family of measures. Then there is a constant $C > 0$ such that if $(\Gamma, \Xi) \sim \mathbb{P}_D$, then $(\Gamma, C\Xi) \sim \widetilde{\mathbb{P}}_D$.

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- If the d_κ-dimensional Minkowski content of CLE_κ exists (and its intensity is locally finite), then it is equal to (a constant times) the measure in the theorem.
- We expect that the assumption $\mathbb{E}_D[\Xi(K)] < \infty$ is not necessary.

Other measures via GFF

With a similar strategy, the natural measures have been constructed and proved to be unique (under the assumption of finiteness of intensities on compacts) on the following sets [Cai-Li, 2023]:

- $\mathrm{SLE}_\kappa(\rho)$ for $\kappa \in (0,4)$,
- CLE_κ, $\kappa \in (4,8)$ pivotal points,
- CLE_κ carpet/gasket, $\kappa \in (8/3,4) \cup (4,8)$,
- Cut points of SLE

Moreover, the measure on the cut points of SLE was also constructed and moment bounds were derived in [Kavvadias-Miller-S., 2023].

Conformal Minkowski content on TVS..?

We also mention the following measure constructed using a GFF [S.-Sepúlveda-Viklund, 2020]. Let h be a zero-boundary GFF on $\mathbb{D}$ and $\mathbb{A}_{-a,b}$ be the TVS of levels $-a$ and b and let $\sigma_c = 2\lambda/(a+b)$. Define

$$\mathcal{V}^{i\sigma} = \lim_{\varepsilon \to 0} \varepsilon^{-\sigma^2/2} e^{i\sigma h_\varepsilon(z)}.$$

Then $\mathbb{E}[\mathrm{Re}(\mathcal{V}^{i\sigma_c})|\mathbb{A}_{-a,b}](dz)$ defines a measure on $\mathbb{A}_{-a,b}$.

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More precisely, define for each $\delta > 0$

$$\mu_\delta(dz) = \delta \mathrm{crad}_{\mathbb{D}\setminus\mathbb{A}_{-a,b}}(z)^{-(\sigma_c-\delta)^2/2} dz.$$

Then, the limit μ of μ_δ exists in the weak topology and for each continuous and bounded f, $\mathbb{E}[\mathrm{Re}(\mathcal{V}^{i\sigma_c}, f)|\mathbb{A}_{-a,b}] = \frac{a+b}{2}\mu(f)$.

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Furthermore, if the conformal Minkowski content of $\mathbb{A}_{-a,b}$ exists, then it is (a constant times) μ.

Differences between measures

We remark that the measures which are constructed on other sets that the FPS and TVS use GFFs which are independent of the fractal we construct a measure on, whereas the measures on the local sets of the GFF use the very GFF that they are coupled with.

Differences between measures

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Furthermore, note that while μ is also a measure on CLE_4 (provided that $a = b = 2\lambda$), it is different from the CLE measure constructed in the other work. The CLE measure should be the Minkowski content, whereas the TVS measure should be the conformal Minkowski content.

Outline

Constructing measures using the GFF

Often, above mentioned measures are too hard to construct. A natural way to construct measures on GFF-related fractals is via the GFF.

1. For a GFF h on a domain D, pick/construct object $X_h(z)$ such that X_h is supported and non-negative on the desired random element $K \subseteq \overline{D}$.
2. Take the conditional expectation $\mathbb{E}[X_h(dz)|K]$, possibly multiplied by some compensator $F(z) : \overline{D} \to \mathbb{R}$.

For many choices of X_h, this yields a measure on the random fractal K which exhibits conformal covariance with the "right" exponent.
The compensator F is typically a power of the conformal radius, to handle the conformal radius terms that naturally occur when considering the image/pushforward of X_h.

Construction

- Measure Ξ on CLE_κ carpet/gasket $\kappa \in (8/3, 4) \cup (4, 8)$ constructed as a conditional expectation of the natural LQG measure ([Miller-Sheffield-Werner, 2020]) on CLE_κ, given the CLE_κ, with compensator being an appropriate power of the conformal radius.

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- By some LQG-type calculations, it follows that the associated family of laws $(\mathbb{P}_D)$ is conformally covariant with exponent d_κ and respects the CLE_κ Markov property.

Existence of measure on CLE_4

- Pick sequence $\kappa_n \nearrow 4$ and couple each CLE_{κ_n} with the same family of Brownian loop soups with intensity constants $c_n \nearrow 1$ and let Ξ^{κ_n} be the constructed measures normalised to have expected total mass 1.

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- By monotonicity, the CLE_{κ_n} carpets Γ^{κ_n} converges to the CLE_4 carpet Γ^4 in the topology of the Hausdorff distance. By the Skorokhod representation theorem, $(\Gamma^{\kappa_n}, \Xi^{\kappa_n})$ converge a.s. to (Γ^4, Ξ^4) where Ξ^4 is a measure on Γ^4.

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- Conformal covariance with exponent d_4 and the property that the associated laws satisfy the CLE_4 Markov property are inherited from the corresponding properties of the measures Ξ^{κ_n}.

Outline

1. Introduction
   - Measures on fractals
   - Constructing measures using the GFF
2. The conformally covariant measure on CLE
   - Construction and a measure on CLE_4
   - Uniqueness

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2. Explore the CLE_κ piece by piece and use the property of respecting the CLE_κ Markov property.

Form of the intensity

Let $(\Gamma, \Xi) \sim \mathbb{P}_{\mathbb{D}}$.

- By the conformal covariance, we have that if $\phi : \mathbb{D} \to \mathbb{D}$ is conformal and we set $\widetilde{\Gamma} = \phi(\Gamma)$ and

$$\widetilde{\Xi}(dz) = |\phi'(z)|^{-d} \Xi \circ \phi(dz),$$

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  then $(\widetilde{\Gamma}, \widetilde{\Xi}) \stackrel{d}{=} (\Gamma, \Xi)$.
- Taking expectations, we have

$$\mathbb{E}[\Xi](dz) = |\phi'(z)|^{-d} \mathbb{E}[\Xi] \circ \phi(dz)$$

  for all $\phi : \mathbb{D} \to \mathbb{D}$ conformal. This completely determines the form of $\mathbb{E}[\Xi]$.

Structure of intensity

Let $\mathscr{B}(D)$ denote the σ-algebra of Borel subsets of D.

Lemma

Let m be a locally finite measure on $(\mathbb{D}, \mathscr{B}(\mathbb{D}))$ for which there exists a $d \in \mathbb{R}$ such that for each Möbius transformation $\phi : \mathbb{D} \to \mathbb{D}$, we have

$$m(dz) = |\phi'(z)|^{-d} m \circ \phi(dz).$$

Then there exists a constant C_m such that

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Proof consists of first showing that m is absolutely continuous with respect to Lebesgue measure and performing a calculation and letting $\phi(w) = \frac{w-z}{1-\bar{z}w}$.

Uniqueness

Let $(\mathbb{P}_D)$ and $(\widetilde{\mathbb{P}}_D)$ be as in the statement of the existence/uniqueness theorem and assume that $(\Gamma, \Xi) \sim \mathbb{P}_{\mathbb{D}}$ and $(\Gamma, \widetilde{\Xi}) \sim \widetilde{\mathbb{P}}_{\mathbb{D}}$ are coupled so that the CLE_κ is the same and $\mathbb{E}[\Xi(\mathbb{D})] = \mathbb{E}[\widetilde{\Xi}(\mathbb{D})] = 1$

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6. Deduce that $\Xi = \widetilde{\Xi}$. Uniqueness is proved!

Thanks for listening!