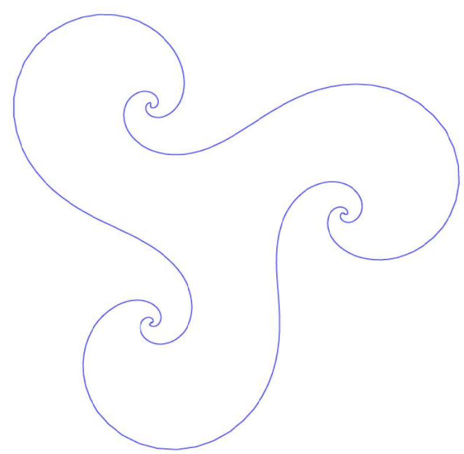
Piecewise geodesic Jordan curves

Steffen Rohde, UW

Jeju Island, June 6, 2023



joint with Yilin Wang, Don Marshall, Mario Bonk, Janne Junnila
2019 2022 2023

Motivation:

- How to "conformally naturally" embed planar graphs into $\hat{\mathbb{C}}$

- $\lim_{\kappa \to 0}$ SLE_κ conditioned to pass through given $z_1, z_2, \dots, z_n$

- interesting (projective) structure on $\hat{\mathbb{C}} \setminus \{z_1, \dots, z_n\}$ related to Teichmüller theory

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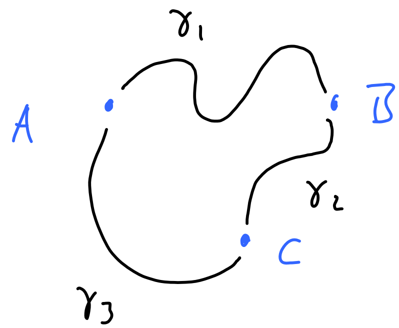
Def.: A Jordan curve $\Gamma = \bigcup_{i=1}^{n} \Gamma_i$ is piecewise geodesic if each arc Γ_i is a hyperbolic geodesic in $\hat{\mathbb{C}} \setminus \bigcup_{j \neq i} \Gamma_j$.

Illustrating application: Given A, B, C, curve $\gamma_1\gamma_2\gamma_3$:



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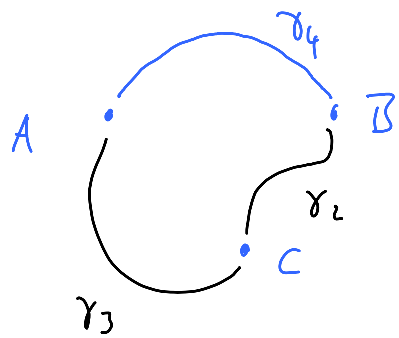

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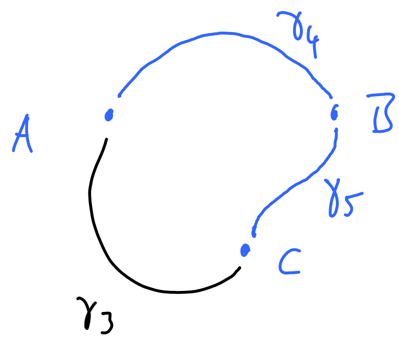

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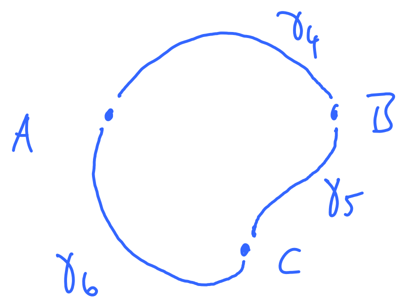

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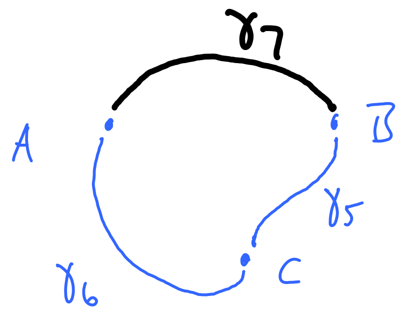

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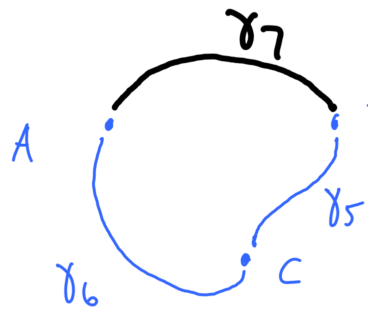

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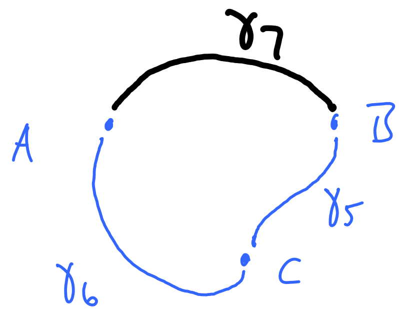

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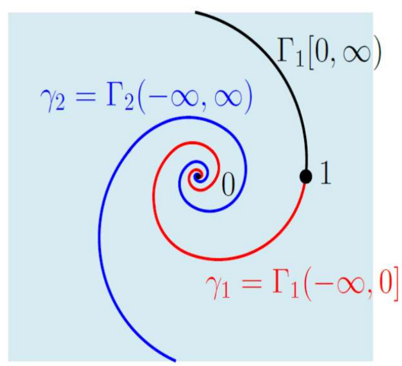

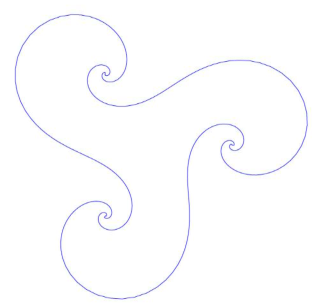

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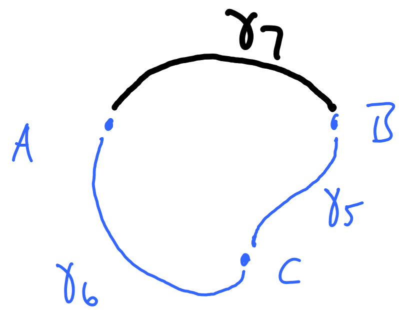

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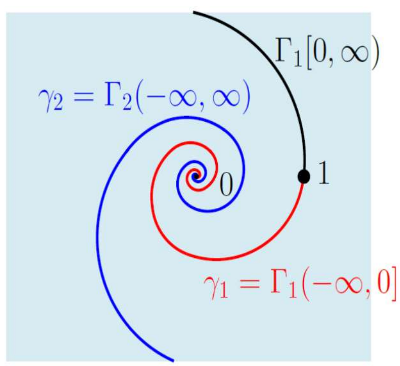

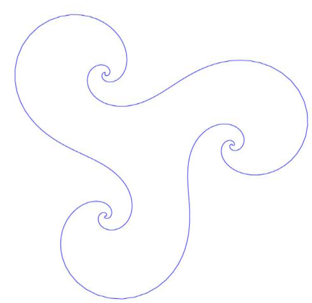

Thm (Bonk, Junnila, MRW): If $\gamma_1\gamma_2\cdots\gamma_k$ smooth $\Rightarrow$ $\gamma_n\cdots\gamma_{n+k-1} \to$ p.g. J c.

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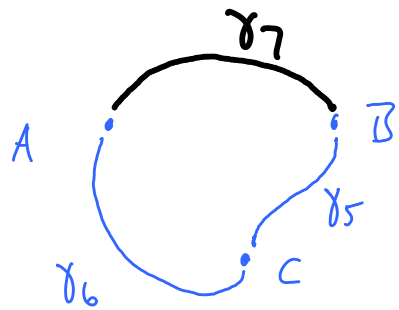

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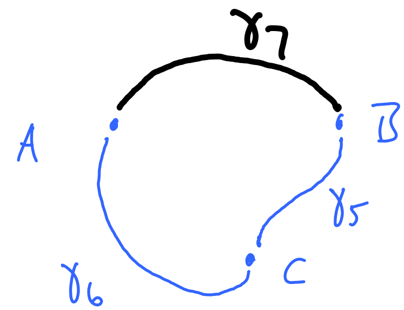

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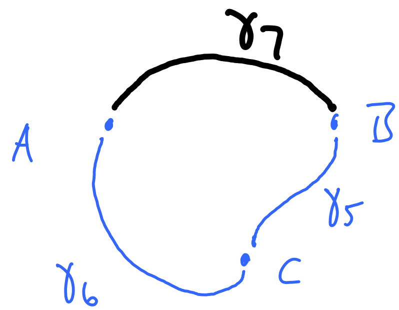

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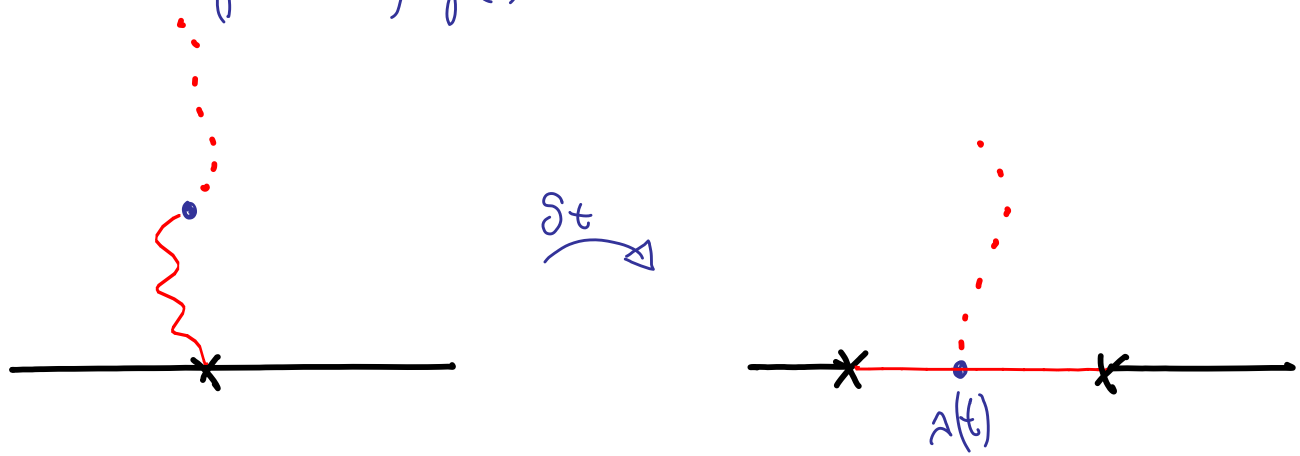

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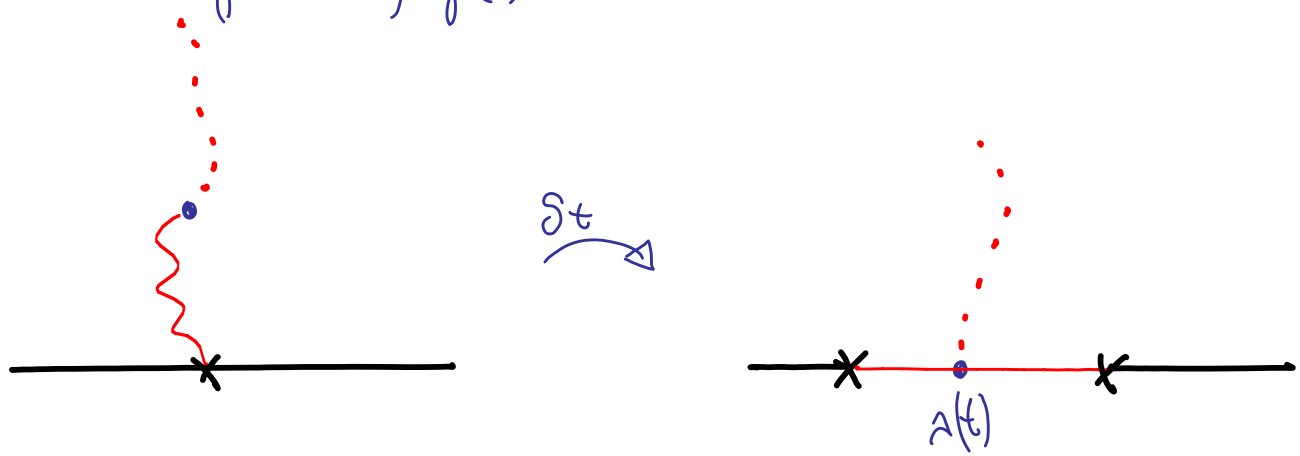

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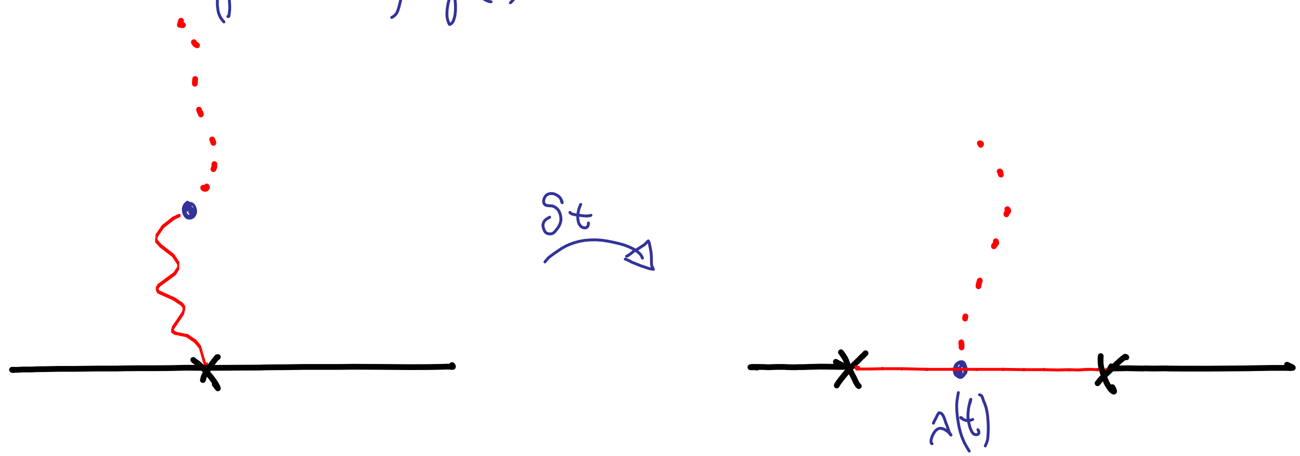

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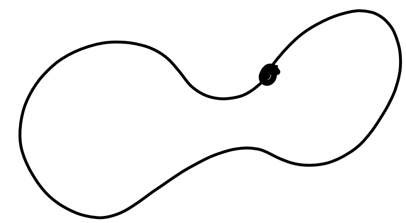
Thm (Wang): $E(\gamma)$ is invariant under "reversal": $E(\gamma) = E(T\gamma)$ where $T(z) = -\frac{1}{z} \in \mathrm{Aut}\,\mathbb{H}$

Loop Loewner energy : $\gamma : [0,1] \to \hat{\mathbb{C}}$ simple closed curve



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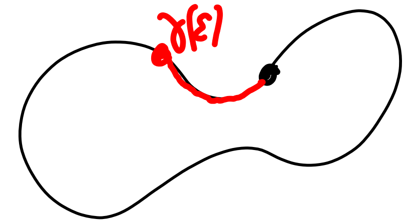

$\gamma[\varepsilon, 1]$ chord in $\hat{\mathbb{C}} \setminus \gamma[0,\varepsilon]$

$E(\gamma[\varepsilon,1]) \searrow$ in ε

Def: $E(\gamma; \gamma(0)) = \lim_{\varepsilon \to 0} E(\gamma[\varepsilon, 1])$.

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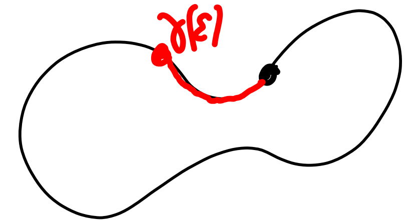

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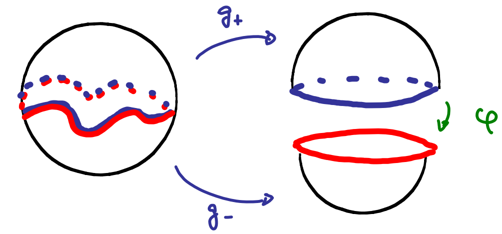
Thm. (R, Wang): The loop energy is root-independent:

$$E(\gamma; \gamma(0)) = E(\gamma; \gamma(t)) \quad \forall t.$$

More on loop Loewner energy :



- Wang : $E(\gamma) = \frac{1}{\pi} \iint_{S^2} \left| \frac{g''}{g'} \right|^2 dx\,dy$

- Takhtajan - Teo : Developed Teichmüller theory of $T_0(1) = \{ \gamma : E(\gamma) < \infty \}$.

- Viklund - Wang : Deterministic version of Miller-Sheffield imaginary geometry (flow lines of e^{ih}, $h \in GFF$ (MS) or $\in H^1$ (VW)).

- Bishop : Geometric / analytic conditions for $E(\gamma) < \infty$

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Thm (Marshall, R., Wang): If $\gamma_1\gamma_2\gamma_3$ sufficiently smooth $\Rightarrow \gamma_n\gamma_{n+1}\gamma_{n+2} \to$ circle

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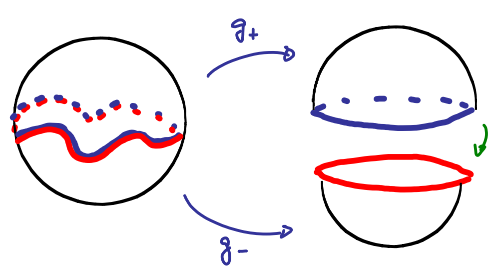
3) (Characterize conformal welding of (*))

$\gamma \subset S^2$ closed



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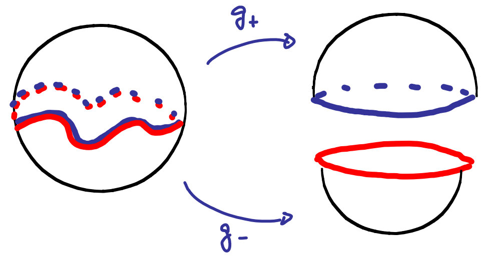
$\varphi = g_- \circ g_+^{-1}$



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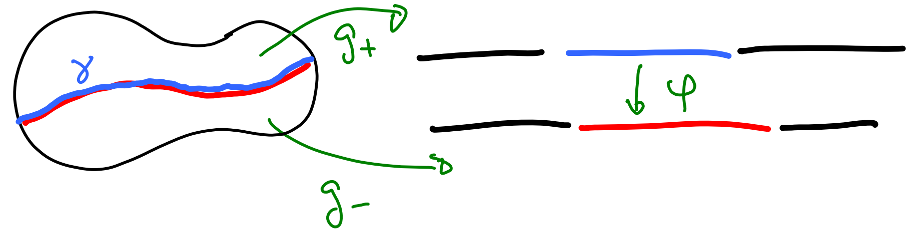
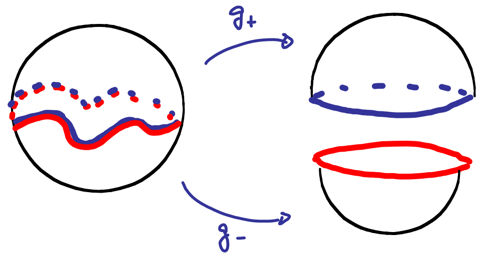


γ geodesic $\iff$ φ Möbius



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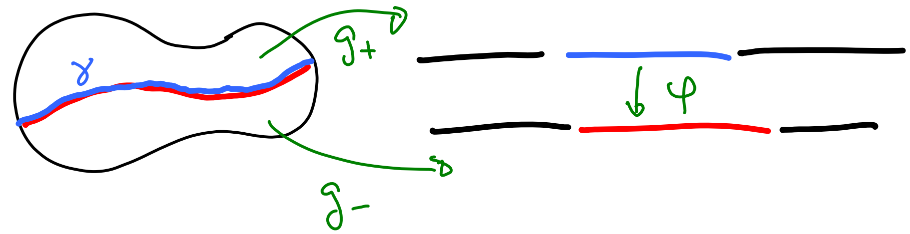
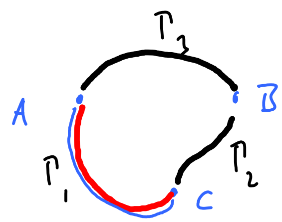


γ geodesic $\Longleftrightarrow$ φ Mobius

Cor. (*) for



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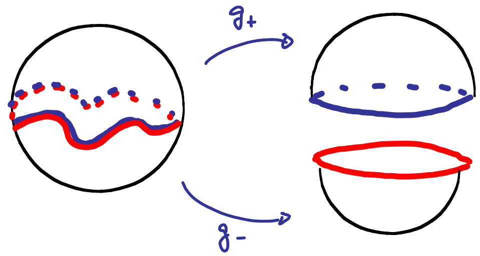

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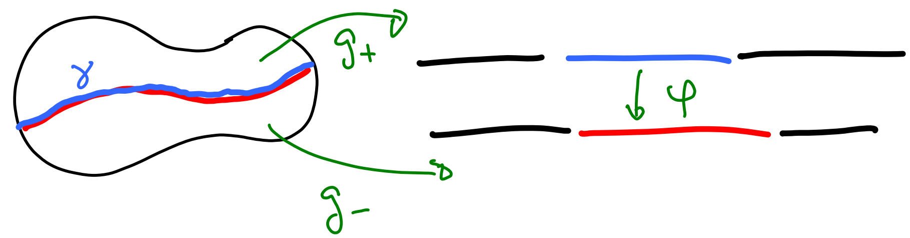

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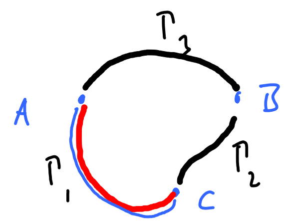

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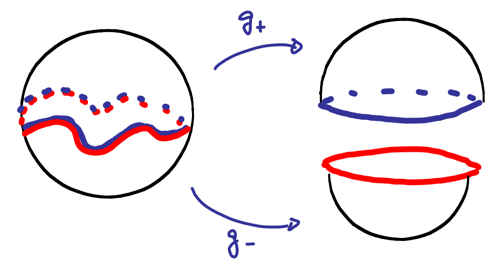
Lemma φ piecewise Möbius, $E(\gamma) < \infty$ $\implies$ $\varphi \in C^1$



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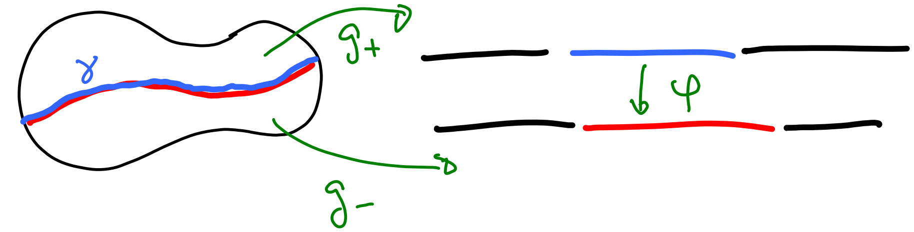

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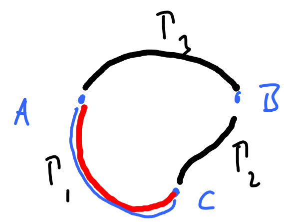

$\Rightarrow$ φ piecewise Mobius

Lemma φ piecewise Mobius, $E(\gamma) < \infty$ $\Rightarrow$ $\varphi \in C^1$

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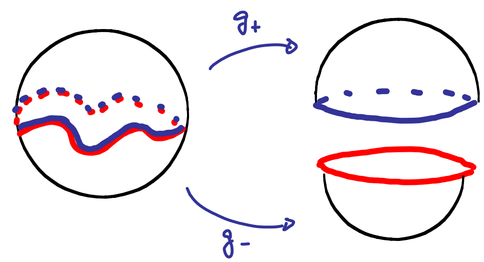

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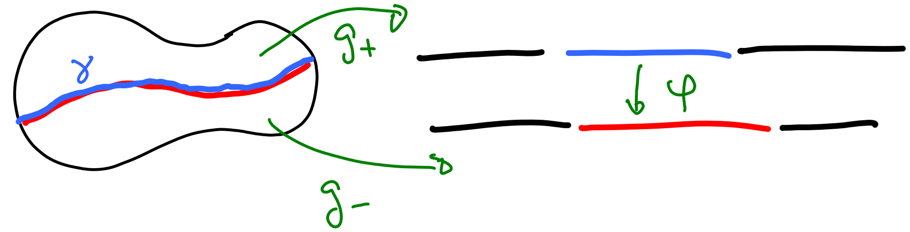

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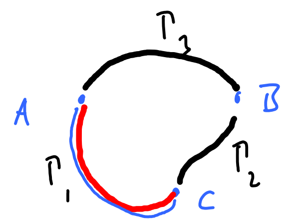

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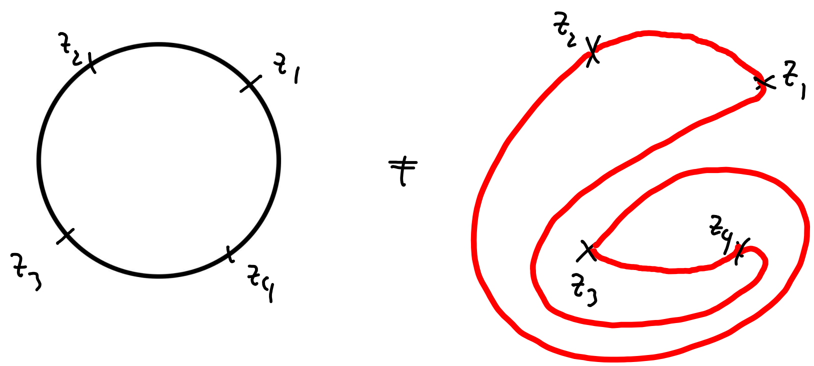
If $n = 3$: φ Mobius

More than 3 points:



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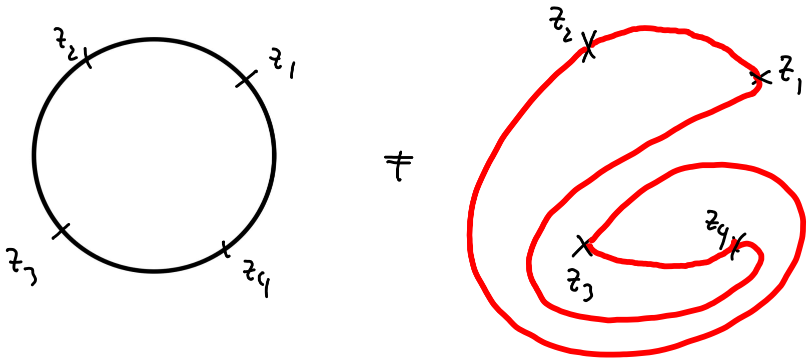

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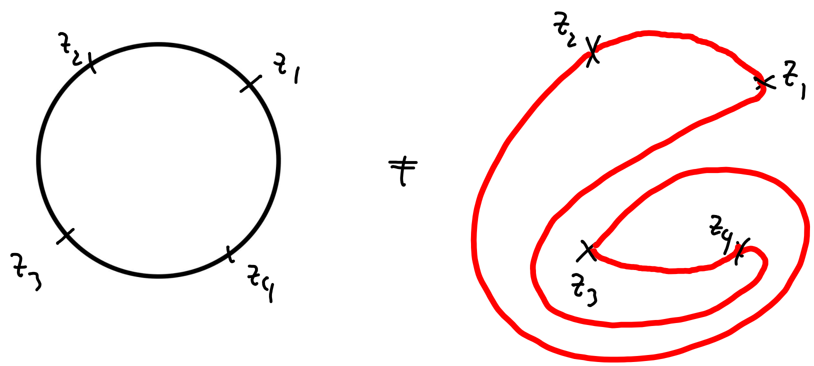
Prop. (BJMRW): $\{$isotopy classes of Jordan curves through $z_1, \dots, z_{n-3}, 0, 1, \infty\}$

$$= T_{0,n}$$

More than 3 points:



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Prop. (BJMRW): $\{$isotopy classes of Jordan curves through $z_1, \dots, z_{n-3}, 0, 1, \infty\}$
$$= T_{0,n}$$

Lemma: For every Jordan curve γ through given $z_1, \dots, z_n$ there is $\Gamma = \bigcup_i^n \Gamma_i$ isotopic to γ (fixing $z_1, \dots, z_n$) and minimizing E, and Γ is a smooth piecewise geodesic Jordan curve:

Each arc Γ_i is a hyperbolic geodesic in $\hat{\mathbb{C}} \setminus \bigcup_{j \neq i} \Gamma_j$.

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Cor.: $\{(x_1, \dots, x_{n-3}, y_1, \dots, y_{n-3}) : x_1 < \dots < x_{n-3} < 0, y_1 < \dots < y_{n-3} < 0\} \subset \mathbb{R}^{2n-6}$

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Cor.: Existence and uniqueness of p.g.J.c. with prescribed spiraling rates.

Projective structures and the Schwarzian Diff. eq.

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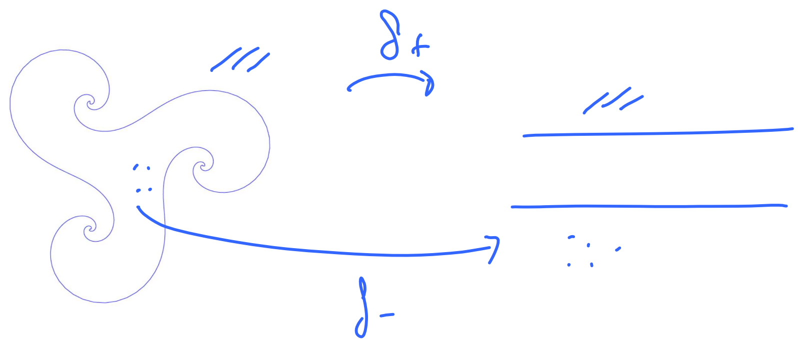

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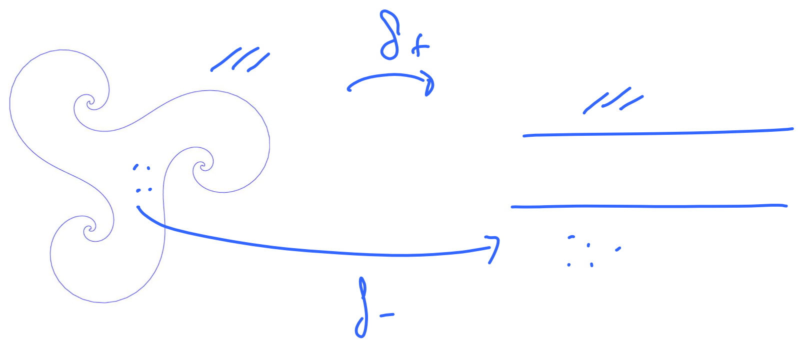

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Each projective structure uniquely defines a quadratic differential $q = S[h \circ \pi^{-1}]$

($Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$, π universal covering map, h developing map)

here: $$q = \begin{cases} S(g_+) & \text{on } \Omega^+ \\ S(g_-) & \text{on } \Omega^- \end{cases}.$$

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$$q = \sum_{j=1}^{n} \left(\frac{1}{2} \frac{1}{(z - z_j)^2} + \frac{d_j}{z - z_j} \right), \quad \text{"accessory parameters" } d_j = \partial_{z_j} (\text{classical Liouville action})$$

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Rmk.: If γ p.g.J.c., $q = \sum_{1}^{n} \left(\frac{r_j}{(z-z_j)^2} + \frac{c_j'}{z - z_j} \right)$.

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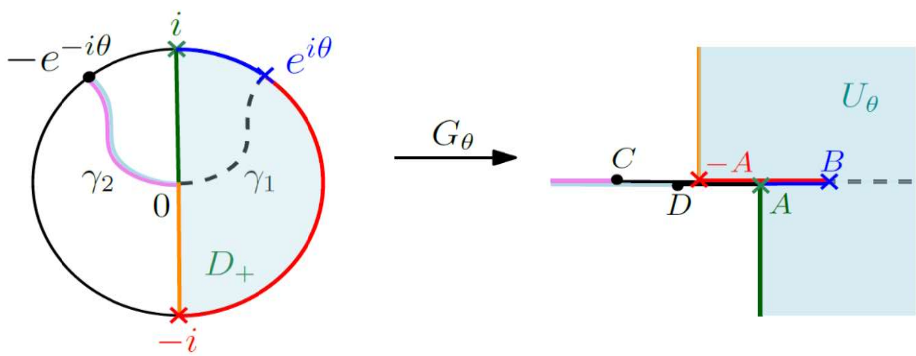
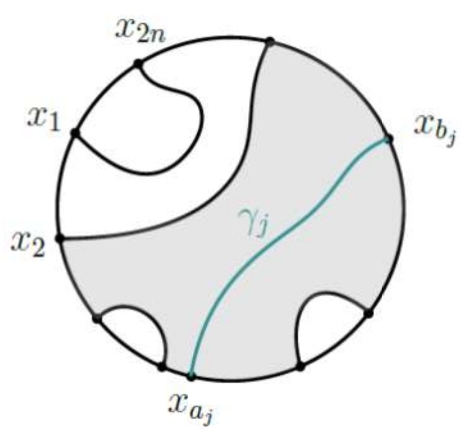


$$G_\theta(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) - ic \log z,$$

Other graphs:

Peltola - Wang, Large deviations of multichordal SLE_{0+} ...



Thank you !