Conformally invariant fields out of Brownian loop soups

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Joint work with

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Paper I: arXiv:2303.03782   Paper II: will be posted soon

Random Conformal Geometry and Related Fields in Jeju

6th June, 2023

Background: Brownian loop soups, isomorphism, level lines

Brownian loop soups in the plane (Lawler-Werner)

The Brownian loop measure μ is a measure on parametrized loops

$$\mu = \int_{\mathbb{R}^2} \int_0^\infty \frac{1}{t} p(t, z, z) \mathbb{P}_{z,t} dt d\lambda(z)$$

- $p(t, \cdot, \cdot)$ is the transition density of the Brownian motion. In $\mathbb{R}^2$, we have
$$p(t, x, y) = \frac{1}{2\pi t} e^{-\|y-x\|^2/(2t)}.$$
- $\mathbb{P}_{z,t}$ is the probability measure on Brownian bridges in $\mathbb{R}^2$ rooted at z with time length t;
- $d\lambda$ is the Lebesgue measure on $\mathbb{R}^2$.

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A **Brownian loop soup** with intensity $\theta > 0$ is a Poisson point process of intensity $\theta\mu$ for $\theta > 0$. A different intensity parameter $c = 2\theta$ is also widely used.

For any domain $D \subset \mathbb{R}^2$, let μ_D be μ restricted to the loops that are contained in D. A Brownian loop soup in D with intensity θ is a Poisson point process of intensity $\theta\mu_D$.

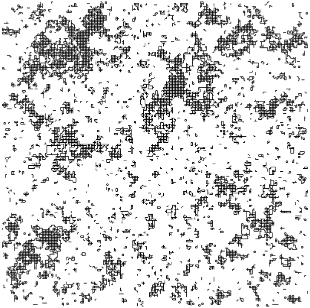

Figure: Brownian loop soup in the unit square

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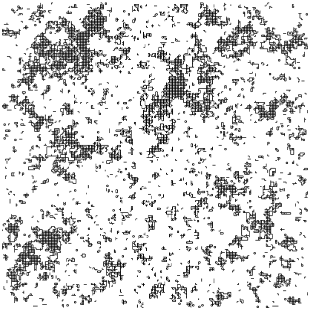


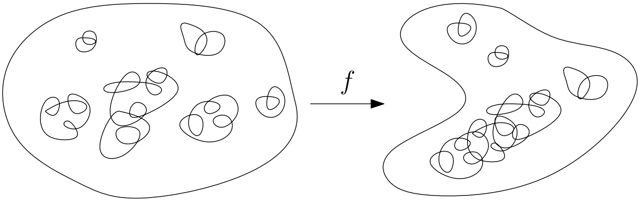
Figure: Brownian loop soup in the unit square

If D is a bounded domain, then inside a Brownian loop soup in D,

- there are a.s. finitely many big loops (time-length or dimeter $\geq \varepsilon$)
- and a.s. infinitely many small loops (time-length or dimeter $\leq \varepsilon$). In particular, the union of all loops in a loop soup is dense.

Conformal invariance

Let f be a conformal map from D_1 onto D_2. Let Γ_{D_1} be a Brownian loop soup in D_1. Then $f(\Gamma_{D_1})$ is a Brownian loop soup in D_2.



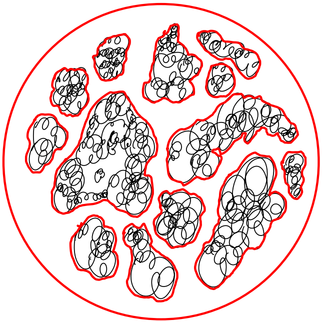
True for any intensity $\theta > 0$.

Brownian loop soup clusters

Two loops are in the same cluster if there is a finite chain of loops that connect them to each other.

By Sheffield-Werner [2012]

Phase transition at the intensity $c = 1$: A loop soup in D with $c > 1$ has a.s. one single cluster. A loop soup in D with intensity c has a.s. infinitely many clusters if $c \leq 1$.



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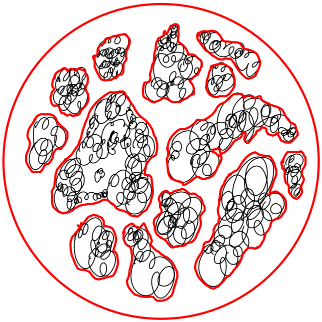
For $c \in (0, 1]$, the outer boundaries of the closures of the outermost clusters form a collection of simple loops which is distributed as a conformal loop ensemble (CLE).



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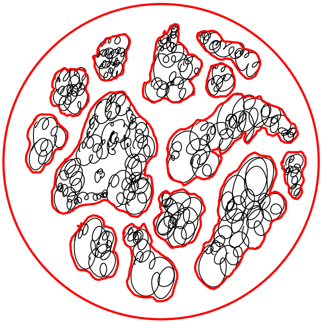
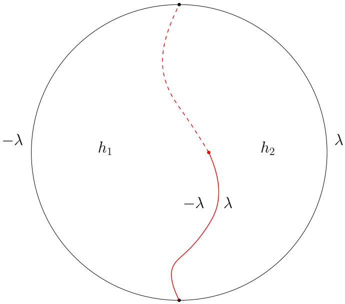


The parameter $c \in (0, 1]$ is also the central charge and $c(\kappa) = (6 - \kappa)(3\kappa - 8)/(2\kappa)$.. This corresponds to $\kappa \in (8/3, 4]$.

Level lines of the GFF: $\kappa = 4$

- Chordal version. Schramm-Sheffield [2010] introduced **local sets**. SLE_4 is coupled to the GFF as its level line.



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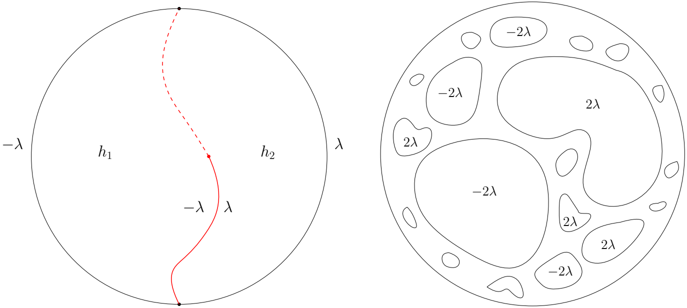

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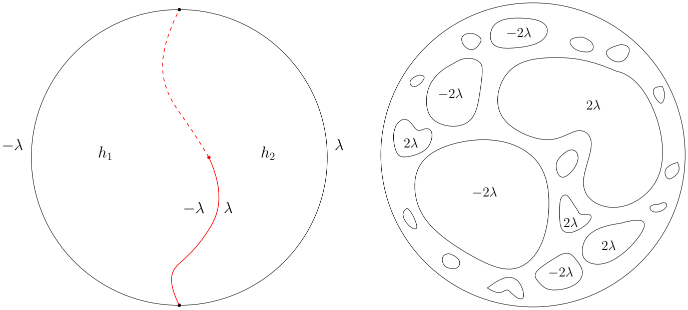

- Other variants by Aru-Sepúlveda-Werner (two-valued sets), Miller-Sheffield-Werner (boundary CLE), etc.

Isomorphism theory

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In $d = 1$ (continuum) and on a discrete graph, occupation time field is directly well defined as a function.

We will focus on $d = 2$, where we need to renormalize. The occupation time field T is a distribution. For an open set $O \subset D$,

$$T(O) := \lim_{\varepsilon \to 0} T_\varepsilon(O) - \mathbb{E}[T_\varepsilon(O)].$$

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The GFF φ is a distribution. The square of φ is defined via a renormalization procedure, and is called the Wick square

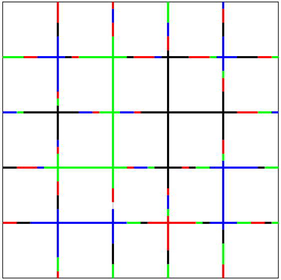
$$:\varphi^2:(f) = \lim_{\varepsilon \to 0} \int_D \left(\varphi_\varepsilon(x)^2 - \mathbb{E}[\varphi_\varepsilon(x)^2]\right) f(x) dx,$$

where $\varphi_\varepsilon(x) = \varphi(B(x,\varepsilon))/(\pi\varepsilon^2)$.

Lupu's cable graph

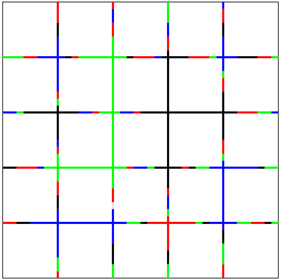
Take a discrete graph, and consider each edge as a continuous interval. Consider a loop soup made of 1D Brownian motions on the edges, which go to each of the adjacent edges with equal probabilities, when arriving at a vertex.

- The occupation time field of this loop soup is a continuous function.
- The isomorphism still holds on this graph, i.e., the occupation time field is equal to the square of the GFF on the cable graph.
- The GFF on the cable graph restricted to the vertices is a GFF on the discrete graph.



Lupu's cable graph

By Lupu [2016]: Given a loop soup on the cable graph, one can take the square root of its occupation time, and give i.i.d. signs to each cluster. This gives rise to a GFF on the cable graph.



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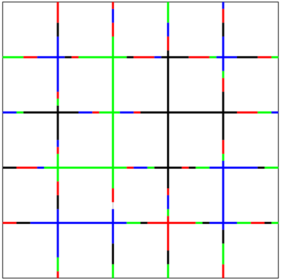

This recovers the GFF as a measurable function of the loop soup on the cable graph and the signs of the clusters.

In the 2D continuum, can we recover the GFF as a measurable function of the Brownian loop soup and the i.i.d. signs of its clusters?

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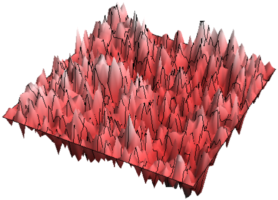
In 2D, the occupation time field is a distribution (not a function), hence we cannot take its square root.



Q.-Werner: Three couplings of different natures commute (proof using Lupu's cable graph)

$$
\begin{tikzcd}
\text{(critical) loop soup} \arrow[rr, leftrightarrow, "\text{Sheffield-Werner}"] \arrow[dr, leftrightarrow, "\text{Le Jan Isomorphism}"'] & & \mathrm{CLE}_4 \arrow[dl, leftrightarrow, "\text{Schramm-Sheffield, Miller-Sheffield, Local set, Level line}"] \\
& \mathrm{GFF} \arrow[u, phantom, "\text{Commutation}"] &
\end{tikzcd}
$$

Aru-Lupu-Sepúlveda: Describe the clusters using nested CLE_4. The GFF h on D can be decomposed via the clusters:

$$\varphi = \sum_{\mathcal{C} \text{ clusters}} \sigma_{\mathcal{C}} \nu_{\mathcal{C}}.$$

$\sigma_{\mathcal{C}} \in \{-1, 1\}$ i.i.d. fair coins.
$\nu_{\mathcal{C}}$ is Minkowski content of $\mathcal{C}$.

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Proof crucially relies on the couplings between the **critical** loop soup, CLE_4 and the GFF.

A conformally invariant field for each intensity $c \in (0, 1]$

What is a natural analogue for subcritical intensities?

$$
\begin{tikzcd}
\text{Loop soup} \arrow[rr, leftrightarrow, "\text{Sheffield-Werner}"] \arrow[dr, leftrightarrow, dashed, "\text{Isomorphism???}"'] & & \mathrm{CLE}_4 \arrow[dl, leftrightarrow, dashed, "\text{Level line???}"] \\
& \text{Field???} &
\end{tikzcd}
$$

A field h_θ for $\theta \in (0, 1/2]$

Theorem 2 (Jego, Lupu, Q., 2023+)

1. *Covariance: There exists $C_\theta : D \times D \to [0, \infty]$ such that for all test function f,*
$$\mathbb{E}(h_\theta, f)^2 = \int_{D\times D} f(x) C_\theta(x, y) f(y) dxdy.$$
The blow-up of the covariance C_θ on the diagonal is given by
$$C_\theta(x, y) = \left(\log \frac{1}{|x-y|}\right)^{2(1-\theta)+o(1)} \qquad \text{as} \quad x - y \to 0.$$
2. *Conformal invariance: Let $\psi : D \to \tilde{D}$ be a conformal map between two bounded simply connected domains We have*
$$h_{\theta,D} \circ \psi^{-1} \quad \overset{(\mathrm{d})}{=} \quad h_{\theta,\tilde{D}}.$$
3. *Symmetry: $h_\theta \overset{(\mathrm{d})}{=} -h_\theta$.*

A field h_θ for $\theta \in (0, 1/2]$

Theorem 3 (Jego, Lupu, Q., 2023+)

For $\theta \in (0, 1/2]$, h_θ is a measurable function of the loop soup together with i.i.d. signs $(\sigma_{\mathcal{C}})$ of the clusters. The field h_θ can be decomposed via the clusters in a loop soup of intensity θ:

$$\varphi = \sum_{\mathcal{C} \text{ clusters}} \sigma_{\mathcal{C}} \nu_{\mathcal{C}}.$$

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When $\theta = 1/2$, $h = h_\theta$ has the law of a GFF in D, so that

$$: L := \frac{1}{2} : h^2 : .$$

We recover a result by Aru, Lupu and Sepúlveda, but we use a different approach.

Isomorphism for $\theta \in (0, 1/2)$

A conjecture that we plan to prove in a subsequent paper:

Conjecture 1 (Construction of h_θ from a discrete loop soup)

Sample a random walk loop soup on a discrete lattice, with occupation time field $L_{\theta,N}(x)$. Define

$$h_{\theta,N}(x) := \frac{\Gamma(\theta)}{2^{1-\theta}\Gamma(2-\theta)}\sigma_{\mathcal{C}_x}(2\pi L_{\theta,N}(x))^{1-\theta}.$$

Then $h_{\theta,N}$ converges to a constant times h_θ.

Permanental field: by Marcus-Rosen [2010], before loop soup! Then identified with the loop-soup occupation time field by Le Jan-Marcus-Rosen [2012].

Our field: fractional power of the permanental field. Morally, we expect $L = |h_\theta|^{1/(1-\theta)}$. Name: Fractional permanental field?

One-dimensional analogue

Le Jan, Lupu: The occupation time field of a 1D loop soup of intensity θ on $\mathbb{R}^+$ is distributed as a $\mathsf{BESQ}_{2\theta}$ on $\mathbb{R}^+$.

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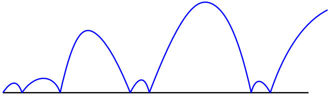
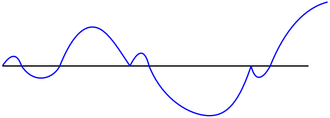
For $\theta \in (0,1]$, $BESQ_{2\theta}$ a.s. hits 0 infinitely many times. For $\theta > 1$, $BESQ_{2\theta}$ a.s. never hits 0.

Naively, if we take $T_\theta = BESQ_{2\theta}$, take the square root, and give i.i.d. signs to each cluster (or excursion), then we get a Bessel process with dimension 2θ.



Square Bessel



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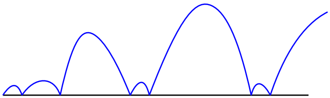

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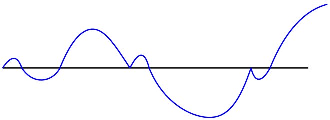

Bessel

However, surprisingly, the correct thing is to take the fractional power $1-\theta$! This means taking the square root for $\theta = 1/2$.

Let T_θ be the occupation time field of a loop soup on $[0, \infty)$ with intensity $\theta \in (0, 1)$, so that T_θ is distributed as a square Bessel process with dimension 2θ. Let h_θ be obtained from T by taking the power $1 - \theta$, and then choosing i.i.d. signs for each cluster (excursion).

Lemma 4

For all $0 \leq x \leq y$, and all bounded measurable function F,

$$\begin{aligned} & \mathbb{E}\left[h_\theta(x) h_\theta(y) F(T_\theta(z), z \in [x, y])\right] \\ = & \frac{\Gamma(\theta^*)}{\Gamma(\theta)} G(x, y)^{2(1-\theta)} \mathbb{E}\left[F(T_{\theta^*}(z), z \in [x, y])\right], \end{aligned}$$

where $\theta^* = 2 - \theta$ is the dual intensity, and $G(x, y) = 2x \wedge y$ is the Green's function on $[0, \infty)$.

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where $\theta^* = 2-\theta$ is the dual intensity, and $G(x,y) = 2x \wedge y$ is the Green's function on $[0,\infty)$.

Tilting the probability by $h_\theta(x)h_\theta(y)$ amounts to changing the dimension of the Bessel process (intensity of the loop soup) to its dual.

Lemma 5

Let $\theta \in (0,1)$. The process h_θ is a martingale.

An analogous result in dimension two

Lemma 6 (Jego, Lupu, Q., 2023+)

For $x, y \in D$, we can construct a probability measure $\mathbb{P}_{x \leftrightarrow y}$ on a loop soup in D conditioned to have a cluster that goes through both x and y.

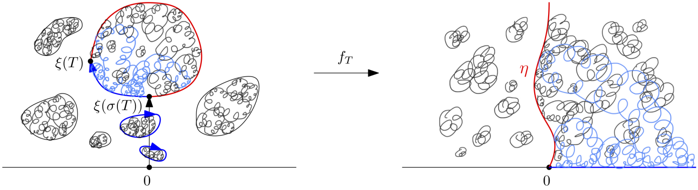
Theorem 7 (Jego, Lupu, Q., 2023+)

Let $F : D \times D \times \mathfrak{L} \to \mathbb{R}$ be a bounded measurable function such that for all $\mathcal{L} \in \mathfrak{L}$, $F(\cdot, \cdot, \mathcal{L})$ is smooth. Then

$$\mathbb{E}\left[\int_D F(x, y, \mathcal{L}_D^\theta) h_\theta(x) h_\theta(y) dxdy\right] = \int_{D \times D} C_\theta(x, y) \mathbb{E}_{x \leftrightarrow y}\left[F(x, y, \mathcal{L})\right] dxdy.$$

Level line of our field

Partial exploration of the loop soup (using the partial exploration of the CLE, by Sheffield-Werner).



We show that SLE_κ is a level line of h_θ with constant boundary conditions on both sides, where

$$2\theta(\kappa) = c(\kappa) = (6-\kappa)(3\kappa-8)/(2\kappa).$$

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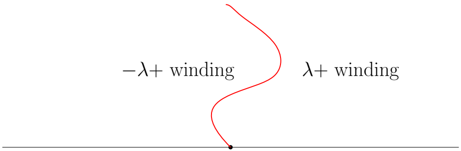

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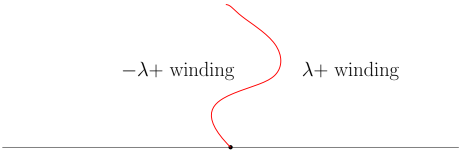

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- Our coupling does not have any winding term.

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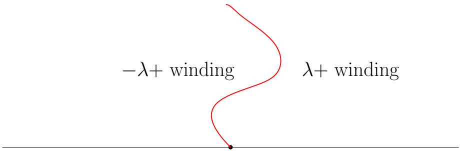

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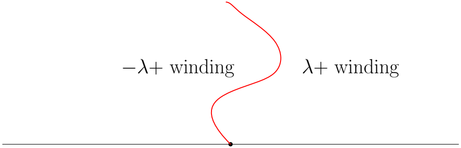

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Attention: Our field is not equal to the CLE nesting field!

Crossing exponent in the Brownian loop soup

Key ingredients in the proof of Theorem 2

The correlation function is given by the following limit:

$$C_\theta(x,y) := \lim_{\gamma\to 0} \frac{1}{Z_\gamma^2} \mathbb{P}[x \xleftrightarrow{\mathcal{L}_D^\theta \wedge \Xi_a^x \wedge \Xi_a^y} y],$$

where

$$Z_\gamma = \mathbb{P}\left(e^{-1}\partial\mathbb{D} \xleftrightarrow{\mathcal{L}_{\mathbb{D}}^\theta} \Xi_a^0\right), \quad \Xi_a^x \approx B(x, e^{-c/\gamma}), \quad \gamma = \sqrt{2a}.$$

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- Main input: We compute the crossing exponent

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As a consequence, the correlation function blows up as

$$C_\theta(x,y) = \left(\log \frac{C}{|x-y|} \right)^{2(1-\theta)+o(1)} \quad \text{as } x-y \to 0.$$

Crossing exponent of a cluster

Theorem 8 (Jego, Lupu, Q.)

Let $\theta \in (0, 1/2]$ and $r \in (0,1)$. The probability that a cluster in a Brownian loop soup $\mathcal{L}_{\mathbb{D}}^{\theta}$ intersects both the microscopic circle $\varepsilon\partial\mathbb{D}$ and the macroscopic circle $r\partial\mathbb{D}$ decays like

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This cannot be obtained using CLE. The probability that $n \geq 2$ clusters in $\mathcal{L}_{\mathbb{D}}^{\theta}$ intersect both the microscopic circle $\varepsilon\partial\mathbb{D}$ and the macroscopic circle $r\partial\mathbb{D}$ should decay like

$$\varepsilon^{\alpha_{2n}+o(1)} \quad \text{as} \quad \varepsilon \to 0,$$

where α_{2n} is the $2n$-arm exponent of SLE. Polynomial in ε!

Exact scaling limit of the crossing probability

Theorem 9 (Jego, Lupu, Q.)

Let $\theta \in (0, 1/2]$. *For all* $s > 1$, *the following crossing probability converges*

$$\mathbb{P}\left(\delta \partial \mathbb{D} \overset{\mathcal{L}_{\mathbb{D}}^{\theta}}{\longleftrightarrow} \delta^s \partial \mathbb{D} \right) \xrightarrow[\delta \to 0]{} f_\infty(s) \tag{1}$$

where

$$f_\infty(s) = \frac{\sin(\pi\theta)}{\pi} \int_{s-1}^{\infty} t^{\theta - 1} (t+1)^{-1} dt. \tag{2}$$

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We can observe that

$$f_\infty(s) \sim \frac{\sin(\pi\theta)}{\pi(1-\theta)} s^{\theta-1}, \qquad \text{as} \quad s \to \infty. \tag{3}$$

A fixed-point equation

If f_∞ exists, then it should satisfy

$$f_\infty(s) = 1 - \left(1 - \frac{1}{s}\right)^\theta + \theta(s-1)^\theta \int_1^\infty (s+t-1)^{-\theta-1} f_\infty(t)\,dt, \quad s \geq 1. \quad (4)$$

- One loop crossing $\delta\partial\mathbb{D} \leftrightarrow \delta^s\partial\mathbb{D}$: $1 - \left(1 - \frac{1}{s}\right)^\theta$
- The first loop crosses exactly $\delta\partial\mathbb{D} \leftrightarrow \delta^{1+(s-1)/t}\partial\mathbb{D}$, and the rest is crossed by a cluster in $\delta\mathbb{D}$.

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For all $\alpha \in (0, 1-\theta)$, the equation (4) admits a unique solution belonging to

$$\mathcal{F}_\alpha := \left\{ f : [1,\infty) \to [0,1] \text{ measurable}, \|f\|_\alpha := \sup_{s\geq 1} s^\alpha |f(s)| < \infty \right\}.$$

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Tightness + Uniqueness $\implies$ Existence of f_∞.

Polar sets of a cluster

Let $D \subset \mathbb{C}$ be a bounded open simply connected domain and let $A \subset D$ be a Borel set. We will say that A is polar for the clusters of $\mathcal{L}_D^\theta$ if

$$\mathbb{P}\left(\exists \mathcal{C} \text{ cluster of } \mathcal{L}_D^\theta : \overline{\mathcal{C}} \cap A \neq \varnothing\right) = 0.$$

Let $K : \mathbb{C} \times \mathbb{C} \to [0, \infty]$ be a measurable function and let

$$\mathrm{Cap}_K(A) := \left(\inf \left\{ \int K(x,y)\mu(dx)\mu(dy) : \mu \text{ probability measure on } A \right\}\right)^{-1}$$

be the capacity of A with respect to the kernel K.

Theorem 10 (Jego-Lupu-Q.)

Let $\theta \in (0, 1/2]$, $D \subset \mathbb{C}$ be a bounded open simply connected domain and let $A \subset D$ be a closed set. If A is polar for the clusters of $\mathcal{L}_D^\theta$, then $\mathrm{Cap}_{\log^\alpha}(A) = 0$ for all $\alpha > 1 - \theta$. Conversely, if A is not polar for the clusters of $\mathcal{L}_D^\theta$, then $\mathrm{Cap}_{\log^\alpha}(A) > 0$ for all $\alpha < 1 - \theta$.

Kakutani's theorem: A closed set A is polar for Brownian motion iff $\mathrm{Cap}_{\log^\alpha}(A) = 0$ for $\alpha = 1$.

Minkowski content

Theorem 11 (Jego, Lupu, Q.)

Let $\theta \in (0, 1/2]$. A cluster $\mathcal{C}_k$ in a Brownian loop soup $\mathcal{L}_{\mathbb{D}}^{\theta}$ a.s. has Minkowski gauge function $t^2 G(t)$ for some function $G(t) = |\log t|^{1-\theta+o(1)}$.

The Minkowski content μ_k of $\mathcal{C}_k$ under the gauge function $t^2 G(t)$ is a radon measure.

For all bounded measurable function $f : \mathbb{C} \to \mathbb{R}$,

$$\sum_{j=1}^{k} \sigma_{\mathcal{C}_j}(\mu_j, f) \underset{k\to\infty}{\longrightarrow} (h_\theta, f).$$

Thank you for your attention !