Algebraic content of systems of SLEs: Conformal field theory, quantum groups, meanders, and Temperley-Lieb algebras

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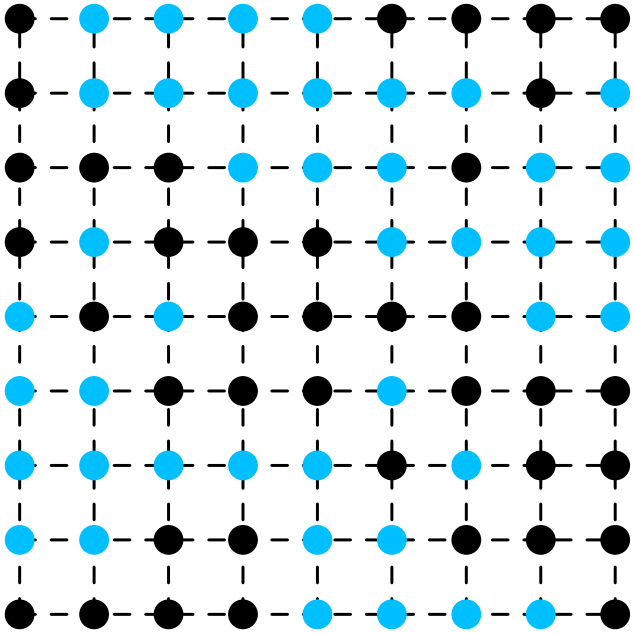
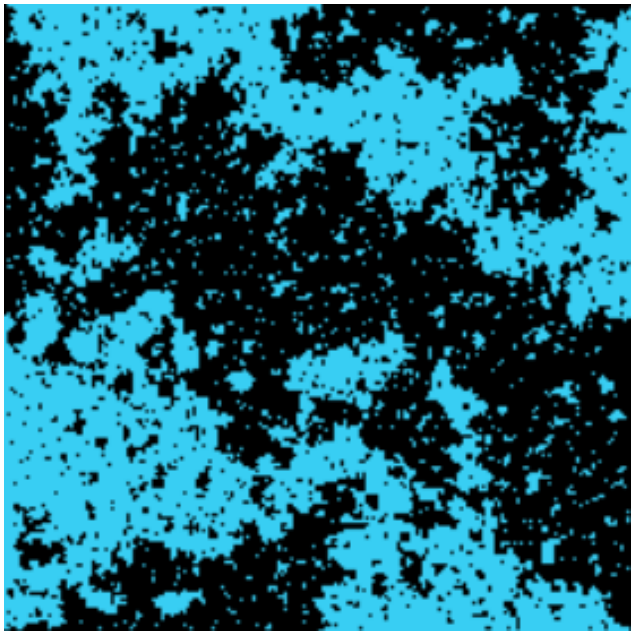
Conformal invariance "conjecture"

Any (reasonable) **critical lattice model** converges in the scaling limit to **a conformal field theory (CFT)**.

What is this supposed to mean?



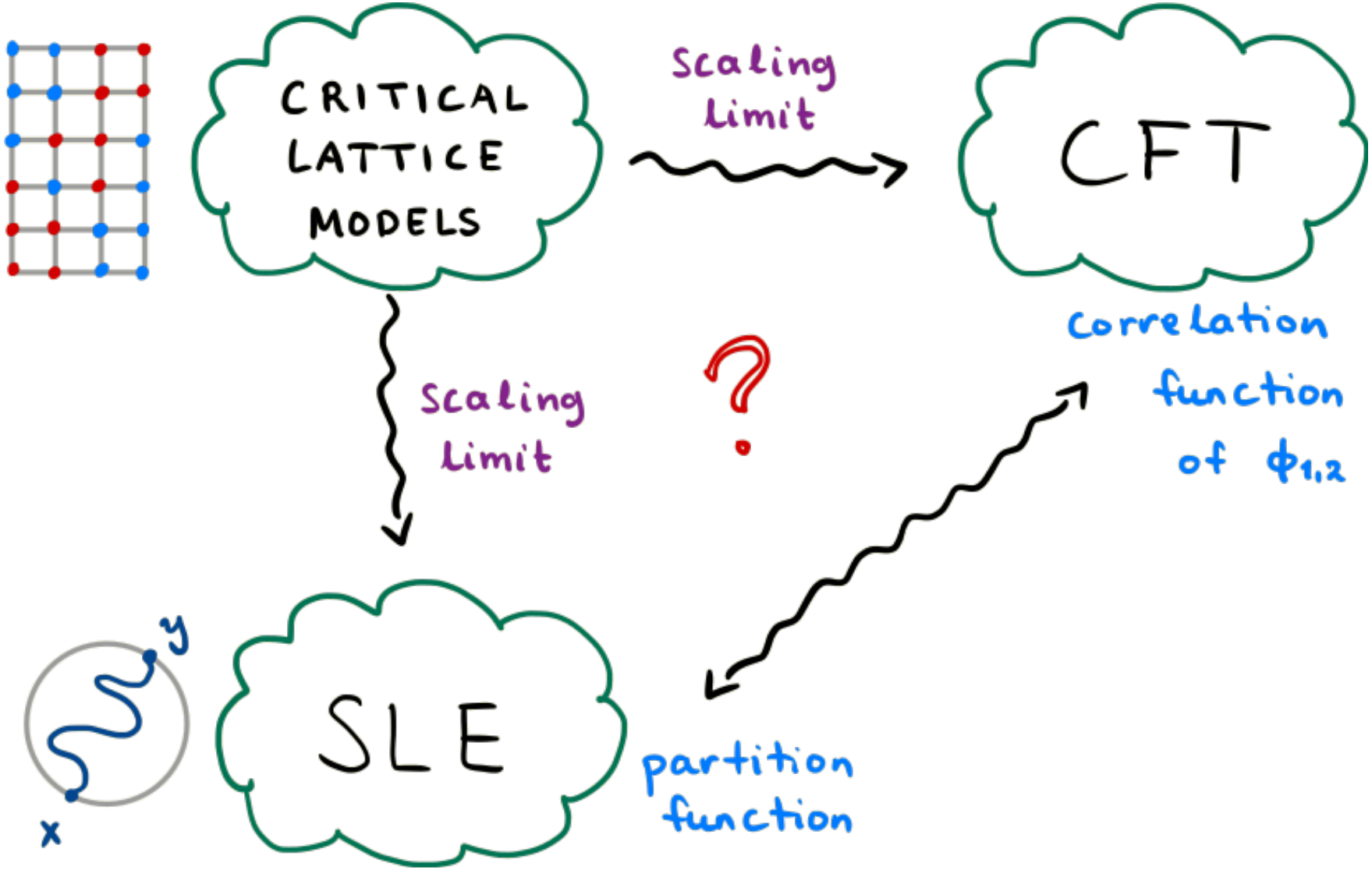
$\longrightarrow$



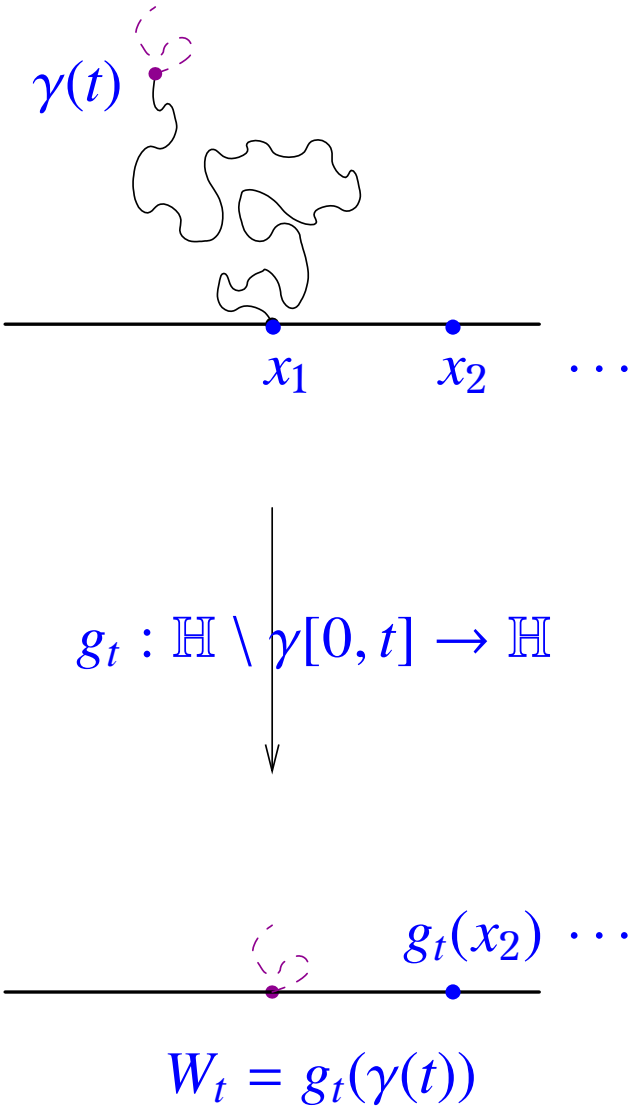
CFT: *conformally invariant* quantum field theory

[Polyakov '70s, BPZ '80s...]

2D critical models: emergent conformal symmetry



Growing SLE(κ) curves via Loewner evolution



- *driving process* of one curve γ: image of tip
- *randomness* from 1D Brownian motion B
- *interaction* encoded in **partition function** $\mathcal{Z}$
  - $\mathrm{d}W_t = \sqrt{\kappa}\,\mathrm{d}B_t + \kappa\,\partial_1 \log \mathcal{Z}(W_t, g_t(x_2), g_t(x_3), \dots)\,\mathrm{d}t$
  - $\mathrm{d}g_t(x_j) = \frac{2\,\mathrm{d}t}{g_t(x_j) - W_t}$

[Schramm '99; Bauer-Bernard-Kytölä '05; Dubédat '07; Kozdron-Lawler '07...]

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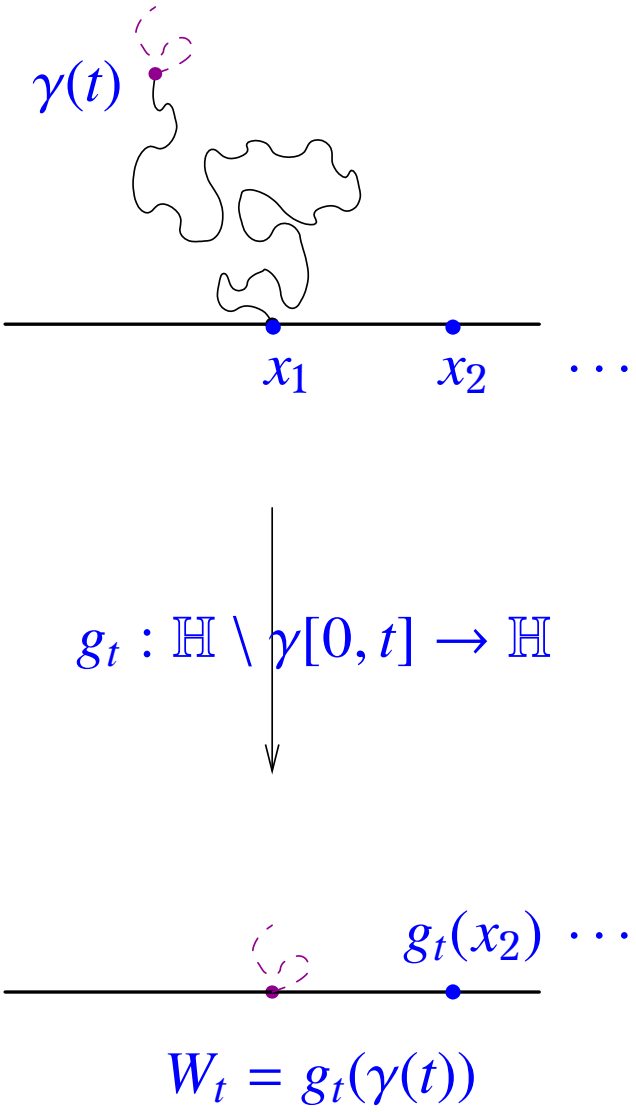

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- **defining properties:**

  $\mathcal{Z} = \mathcal{Z}^{(\kappa)}$ must satisfy system of **BPZ PDEs** $\forall j$

$$\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j} \left(\frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{6/\kappa - 1}{(x_i - x_j)^2} \right) \right\} \mathcal{Z}(x_1, \dots, x_{2N}) = 0$$

  and **Möbius covariance**

$$\mathcal{Z}(f(x_1), \dots, f(x_{2N})) = \Big(\prod_{1 \le j \le 2N} \left| f'(x_j) \right|^{\frac{\kappa - 6}{2\kappa}} \Big)\, \mathcal{Z}(x_1, \dots, x_{2N})$$

(for $f : \mathbb{H} \to \mathbb{H}$ s.t. $f(x_1) < \cdots < f(x_{2N})$)

[Schramm '99; Bauer-Bernard-Kytölä '05; Dubédat '07; Kozdron-Lawler '07...]

General heuristics for 2D critical models

- *universality class* labeled by $\kappa > 0$, central charge $c(\kappa) = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$
- discrete *crossing probabilities*
  $\xrightarrow{\delta\to 0}$ probabilities of connectivities of SLE(κ) curves:

$$\lim_{\delta\to 0} \mathbb{P}^{\delta}_{\beta}[\text{ connectivity} = \alpha\,] = \mathcal{M}_{\alpha,\beta}\,\frac{\mathcal{Z}_{\alpha}(x_1,\ldots,x_{2N})}{\mathcal{F}_{\beta}(x_1,\ldots,x_{2N})}$$

- amplitudes for α encoded in *pure* partition functions $\mathcal{Z}_{\alpha}$
- boundary conditions β encoded in *b.c.c.* partition functions $\mathcal{F}_{\beta}$

[Feĭgin-Fuchs '80s; Dotsenko-Fateev '80s; Cardy '80s; Bauer-Bernard-Kytölä '05; Dubédat '07; Flores-Kleban-Simmons-Ziff '17; P. & Wu (et al.) '18–22'; Izyurov '22...]

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  singled out via *fusion rules* $\rightsquigarrow$ works for $\kappa \in (0,8]$

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- *Coulomb gas* integral formulas

$$\mathcal{Z}_{\alpha}(x_1,\ldots,x_{2N}) = \prod_{i<j}(x_i-x_j)^{2/\kappa}\int_{\Gamma_{\alpha}}\prod_{r,j}(w_r-x_j)^{-4/\kappa}\prod_{r<s}(w_r-w_s)^{8/\kappa}\,\mathrm{d}w_1\cdots\mathrm{d}w_N$$

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Algebraic content of conformal symmetry

Virasoro algebra $\mathfrak{Vir}$: Lie algebra generated by $(L_n)_{n\in\mathbb{Z}}$ and central C

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(slightly more precisely, $v_\Phi := \lim_{z\to 0} \Phi(z)\,|\,0\rangle$ via "state-field correspondence")

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for some $r, s \in \mathbb{Z}_{>0}$ and $\kappa \in \mathbb{C} \setminus \{0\}$.

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$$w = P(L_{-m} : m \in \mathbb{N}).v_\Phi \in \mathrm{N} \qquad \Longrightarrow \qquad [w] = [0] \in \mathrm{V}_h^c/\mathrm{N}$$

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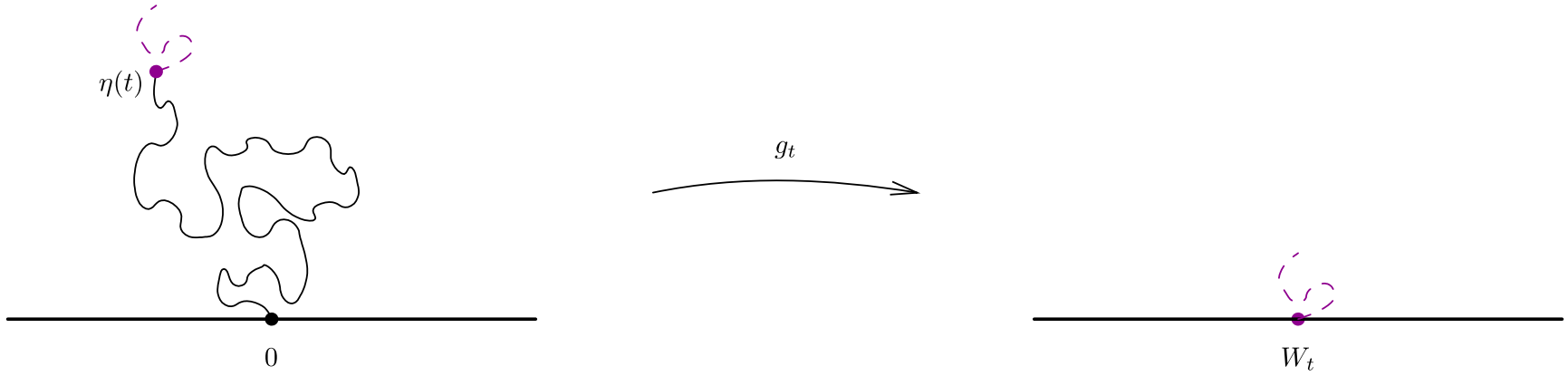
Upshot. **PDEs for correlation functions of Φ:**

$$\mathcal{D}^{(z)}\langle \Phi(z)\, \Phi_1(z_1) \cdots \Phi_d(z_d)\rangle = 0$$

with $\mathcal{D}^{(z)} = P(\mathcal{L}_{-m}^{(z)} : m \in \mathbb{N})$ and $\mathcal{L}_{-m}^{(z)} = -\sum_{j=1}^n \left(\frac{1}{(z_j-z)^{m-1}}\frac{\partial}{\partial z_j} + \frac{(1-m)\,h_j}{(z_j-z)^m}\right)$ and $h_j = h(\Phi_j)$

Example: "SLE(κ) field $\Phi_{1,2}$" of weight $h_{1,2} = \frac{6-\kappa}{2\kappa}$

"insert" $\Phi_{1,2}$ at points $x_1 < x_2 < \cdots < x_{2N}$ [Cardy '84; Bauer-Bernard '02]



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- parameter $\kappa > 0$, central charge $c = \frac{1}{2\kappa}(3\kappa - 8)(6-\kappa) = 13 - 6\left(\frac{\kappa}{4} + \frac{4}{\kappa}\right)$
- singular vector $\left(L_{-2} - \frac{3}{2(2h_{1,2}+1)} L_{-1}^2\right) v_{1,2}$
- (together with translation invariance) gives rise to **PDE system** $\forall\, i$

$$\left\{\frac{\kappa}{2}\frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^{n}\left(\frac{2}{x_j - x_i}\frac{\partial}{\partial x_j} - \frac{2h_{1,2}(\kappa)}{(x_j - x_i)^2}\right)\right\} \underbrace{\langle \Phi_{1,2}(x_1)\cdots\Phi_{1,2}(x_{2N})\rangle}_{\mathcal{Z}(x_1,x_2,\ldots,x_{2N})} = 0$$

Try to single out useful solutions ... fusion

- "multiplication of fields" given by *operator product expansion*: e.g.

$$\text{“ } \Phi_{1,2}(z)\,\Phi_{1,2}(w) \;\sim\; \frac{c_1}{(w-z)^{\Delta_{1,1}(\kappa)}}\,\Phi_{1,1}(w) \;+\; \frac{c_2}{(w-z)^{\Delta_{1,3}(\kappa)}}\,\Phi_{1,3}(w) \text{ ”}$$

  - $c_1, c_2 \in \mathbb{C}$ *structure constants*
  - $\Delta_{1,1}(\kappa) = 2h_{1,2}(\kappa) - h_{1,1}(\kappa) = \frac{6-\kappa}{\kappa}$ and $\Delta_{1,3}(\kappa) = 2h_{1,2}(\kappa) - h_{1,3}(\kappa) = -\frac{2}{\kappa}$

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- *anomaly*: e.g. $\kappa = 8 \implies \Delta_{1,1}(8) = \Delta_{1,3}(8)$

$\rightsquigarrow$ should use less restrictive form

$$\text{“ } \Phi_{1,2}(z)\,\Phi_{1,2}(w) \;\sim\; c_1(z,w)\,\Phi_{1,1}(w) \;+\; c_2(z,w)\,\Phi_{1,3}(w) \text{ ”}, \qquad |z-w| \to 0$$

for some functions $c_1(z_1, z_2)$ and $c_2(z_1, z_2)$ *allowing logarithms*

Fusion of "SLE(κ) fields $\Phi_{1,2}$" of weight $h_{1,2} = \frac{6-\kappa}{2\kappa}$

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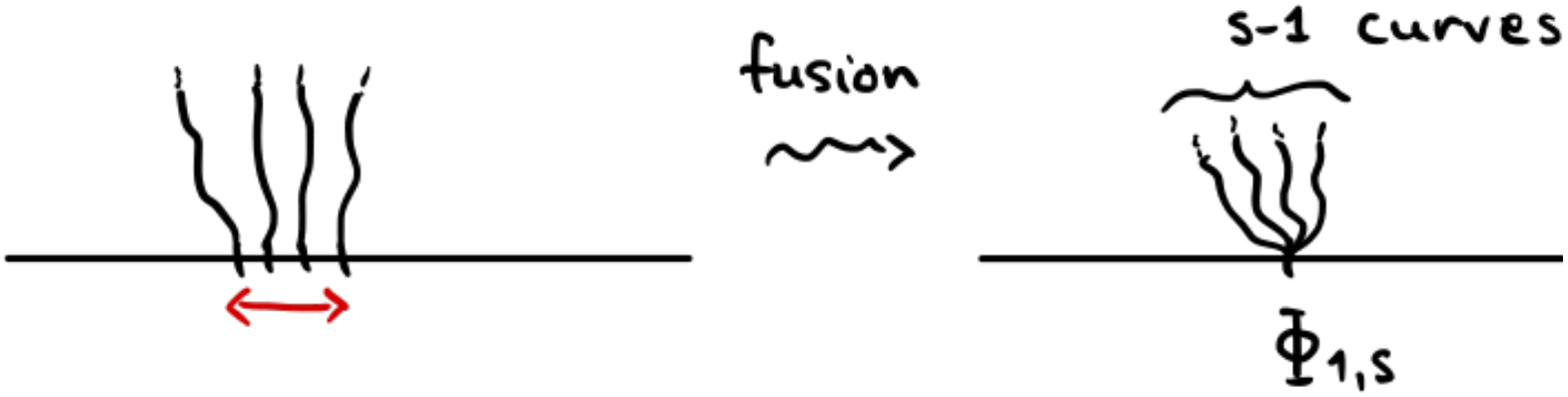

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- $s-1$ curves at x related to $\Phi_{1,s}(x)$ (fusion)

[e.g. Duplantier & Saleur '80s; Bauer & Saleur '80s]

Quantum group symmetry

on

functions of interest

$$\mathcal{Z}_\alpha(\boldsymbol{x}) \propto \prod_{i<j}(x_i - x_j)^{2/\kappa} \int_{\Gamma_\alpha} \prod_{r,\, j}(w_r - x_j)^{-4/\kappa} \prod_{r<s}(w_r - w_s)^{8/\kappa}\, \mathrm{d}w_1 \cdots \mathrm{d}w_N$$

Feĭgin, Fuchs (unpubl.); Dotsenko, Fateev '84; Felder, Fröhlich, Keller, Wieczerkowski '89–'91;

...

Kytölä & P. '14–'16; P. '16–'20, Flores & P. '18–

...

Quantum group ?!?!

Idea: quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ (*Hopf algebra*):

q-deformation of $\mathfrak{sl}_2(\mathbb{C})$:

generators:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

relations:

$$KK^{-1} = 1 = K^{-1}K, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

$$KE = q^2 EK, \qquad KF = q^{-2} FK$$

Quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ (Hopf algebra)

- **universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$:**
  - generators e, f, h
  - relations $ef - fe = [e, f] = h, \quad he - eh = [h, e] = 2e,$
    $hf - fh = [h, f] = -2f$
  - coproduct $\Delta : \mathcal{U}(\mathfrak{sl}_2) \to \mathcal{U}(\mathfrak{sl}_2) \otimes \mathcal{U}(\mathfrak{sl}_2),$
    $\Delta(x) = x \otimes 1 + 1 \otimes x$

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    morally, $K = q^h$ and $\lim\limits_{q \to 1} \frac{q^h - q^{-h}}{q - q^{-1}} = h$
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    $\Delta(K) = K \otimes K, \quad \Delta(E) = 1 \otimes E + E \otimes K,$
    $\Delta(F) = F \otimes 1 + K^{-1} \otimes F$

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  - generators E, F, K, K^{-1}, deformation parameter $q \in \mathbb{C} \setminus e^{\pi i \mathbb{Q}}$
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  - basis $v_0, \ldots, v_s$
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QUANTUM GROUP SYMMETRY ($r = 1, s \geq 1$) ($q = e^{4\pi \mathrm{i}/\kappa} \notin e^{\pi \mathrm{i}\mathbb{Q}}$)

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$$\overset{\mathcal{F}}{\longleftrightarrow}$$

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$$\mathrm{M}_\varsigma := \mathrm{M}_{(s_1)} \otimes \cdots \otimes \mathrm{M}_{(s_d)} \cong \bigoplus_s \mathrm{L}^{(s)} \otimes \mathrm{M}_{(s)}$$

carries two commuting actions, **quantum Schur-Weyl duality:**

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Specifically: $\mathrm{L}^{(s)} \cong \{ v \in \mathrm{M}_\varsigma \mid E.v = 0,\ K.v = q^s v \}$, so $\mathcal{S}_\varsigma \cong \mathrm{H}_\varsigma \cong \mathrm{L}^{(0)}$

Bulk correlation functions of (certain) primary fields

- **BPZ PDE system**: with $\varsigma = (s_1, \ldots, s_d)$,

$$\mathcal{D}^{(z_j)}_{s_j+1} F_\varsigma(z;\bar{z}) = 0, \qquad \mathcal{D}^{(\bar{z}_j)}_{s_j+1} F_\varsigma(z;\bar{z}) = 0, \qquad \text{for all } j = 1, \ldots, d$$

  where $s_j + 1 =$ degree of differential operator $\mathcal{D}^{(z_j)}_{s_j+1}$

- **covariance**: for all conformal maps $f\colon \mathbb{C} \to \mathbb{C}$,

$$F_\varsigma(f(z), f(\bar{z})) = \left(\prod_{i=1}^{n} \left(\partial f(z_i)\, \bar{\partial} f(\bar{z}_i)\right)^{-h_{1,s+1}} \right) F_\varsigma(z, \bar{z})$$

- F_ς is defined for $\left\{ (z_1, \ldots, z_d) \in \mathbb{C}^d \;\middle|\; z_i \neq z_j \text{ for } i \neq j \right\}$

Thm. [Flores, P. (in prep.)]

Suppose $\kappa \in (0,8) \setminus \mathbb{Q}$ (so $c = \frac{1}{2\kappa}(3\kappa - 8)(6 - \kappa) \leq 1$ irrational).

There exists a unique single-valued, conformally covariant, solution $F_\varsigma(z, \bar{z})$ to the above BPZ PDEs.

(Uniqueness holds up to normalization in a natural solution space.)

OPE and structure constants

Corollary. We get the expected asymptotics / OPE:

$$F_{\varsigma}(z;\bar{z}) \underset{z_j,\, z_{j+1}\to\zeta}{\sim} \sum_{s} C^{s}_{s_j,s_{j+1}} \, (z_{j+1}-z_j)^{-h_{1,s_j+1}-h_{1,s_{j+1}+1}+h_{1,s+1}} \times \text{c.c.}$$
$$\times\, F_{(s_1,\ldots,s,\ldots,s_d)}(\ldots, z_{j-1}, \zeta, z_{j+2}, \ldots; \text{c.c.})$$

where s belongs to a *finite* index set, $C^{s}_{r,t}$ are structure constants

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$$C^s_{r,t} := \frac{(B^s_{r,t})^2 B^0_{s,s}}{B^0_{r,r}B^0_{t,t}} \frac{([s]!)^2 \sqrt{[r+1][s+1][t+1]}[\frac{r+t-s}{2}]!}{[\frac{r+s+t}{2}+1]![\frac{s+r-t}{2}]![\frac{t+s-r}{2}]!}$$

and where $[n]! = [1][2]\cdots[n-1][n]$ and $[n] = \frac{\sin(4\pi n/\kappa)}{\sin(4\pi/\kappa)}$ and

$$B^s_{r,t} := \frac{1}{\left(\frac{r+t-s}{2}\right)!} \prod_{j=1}^{\frac{r+t-s}{2}} \frac{\Gamma(1-\frac{4}{\kappa}(r-j+1))\Gamma(1-\frac{4}{\kappa}(t-j+1))\Gamma(1+\frac{4}{\kappa}j)}{\Gamma\left(2-\frac{4}{\kappa}\left(\frac{r+s+t}{2}-j+2\right)\right)\Gamma(1+\frac{4}{\kappa})}$$

Compare with the work of Dotsenko & Fateev in the '80s:
they calculated the 4-point function and structure constants!

If we still have time...

let's add some concreteness!

Idea: discrete *crossing probabilities*

$\xrightarrow{\delta\to 0}$ probabilities of connectivities of SLE(κ) curves:

$$\lim_{\delta\to 0} \mathbb{P}^{\delta}_{\beta}[\text{ connectivity} = \alpha\,] = \mathcal{M}_{\alpha,\beta}\,\frac{\mathcal{Z}_{\alpha}(x_1,\ldots,x_{2N})}{\mathcal{F}_{\beta}(x_1,\ldots,x_{2N})}$$

$$\mathcal{Z}_{\alpha}(\boldsymbol{x}) \propto \prod_{i<j}(x_i - x_j)^{2/\kappa} \int_{\Gamma_\alpha} \prod_{r,\,j}(w_r - x_j)^{-4/\kappa} \prod_{r<s}(w_r - w_s)^{8/\kappa}\, \mathrm{d}w_1 \cdots \mathrm{d}w_N$$

Algebraic content ???

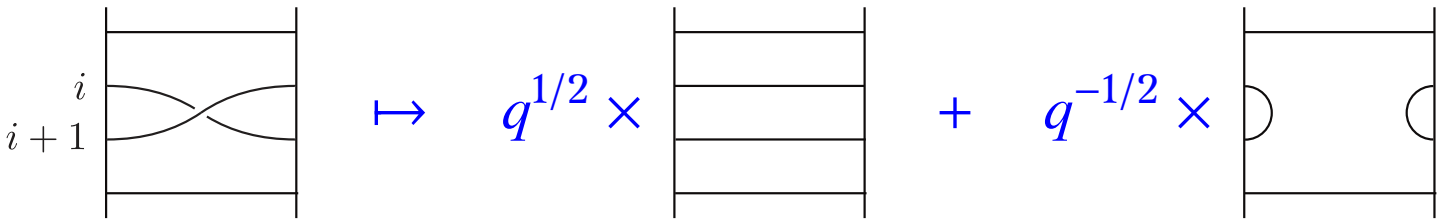
$(q = e^{4\pi i/\kappa} \notin e^{\pi i \mathbb{Q}})$

Setup. Let's look at the *special case* (relevant for multiple $\mathrm{SLE}(\kappa)$) where $\varsigma = (s_1, s_2, \ldots, s_n) = (1, 1, \ldots, 1)$ and $x_1, x_2, \ldots x_n \in \mathbb{R}$.

- *Quantum group* $\mathcal{U}_q(\mathfrak{sl}_2)$ (Hopf algebra): q-deformation of $\mathfrak{sl}_2(\mathbb{C})$:

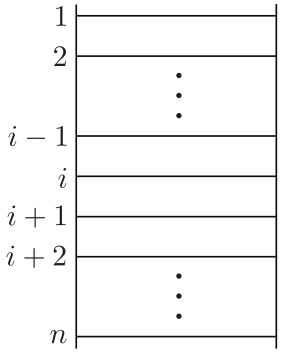
$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

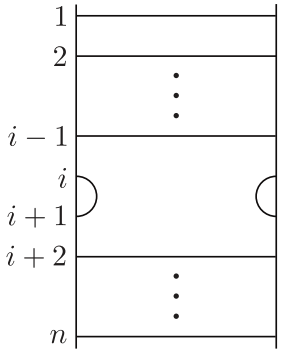
- *Braid group* $\mathbb{C}[\mathrm{Br}_n]$: gen. by coordinate switches of x_i, x_{i+1}
- Jones map $\mathbb{C}[\mathrm{Br}_n] \longrightarrow \mathrm{TL}_n(\nu)$ is surjective (with explicit kernel)



- *Temperley-Lieb algebra* $\mathrm{TL}_n(\nu)$ of fugacity $\nu = -q - q^{-1}$: algebra of planar diagrams...

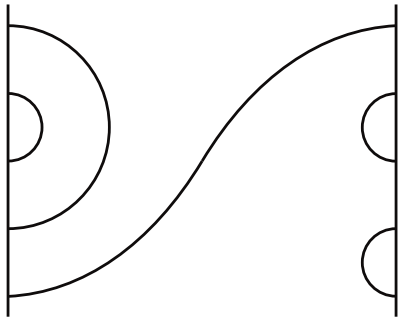
Temperley-Lieb algebra $\mathrm{TL}_n(\nu)$

generators: 

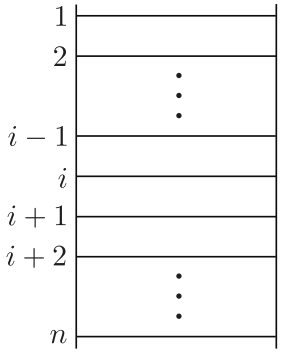
and 

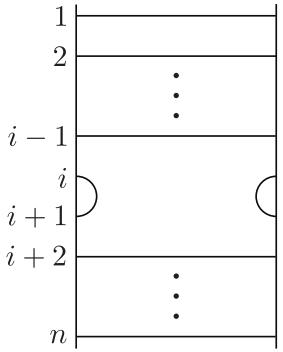
for $1 \leq i \leq n-1$

Thm. [Kauffman '80s]
$\mathrm{TL}_n(\nu)$ = algebra of **all** diagrams with $2n$ nodes connected by non-intersecting lines, product given by concatenation.



Temperley-Lieb algebra $\mathrm{TL}_n(\nu)$

generators: 

and 

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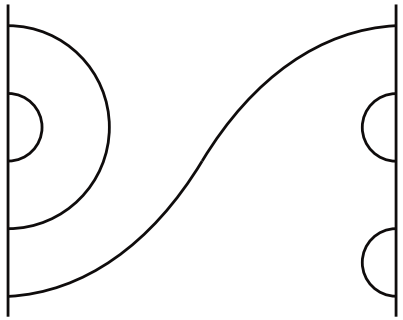
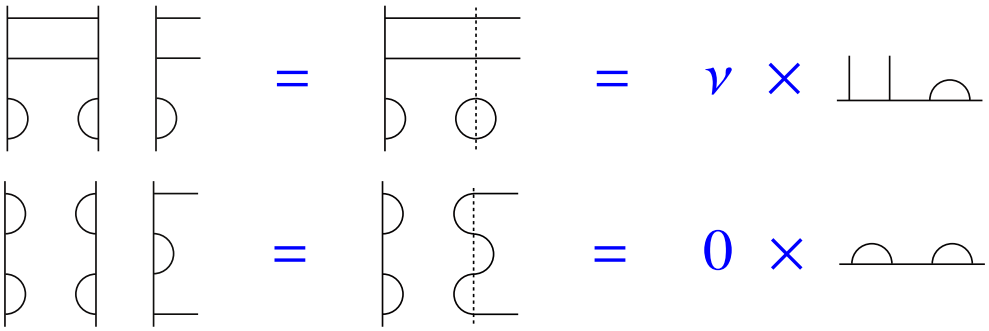
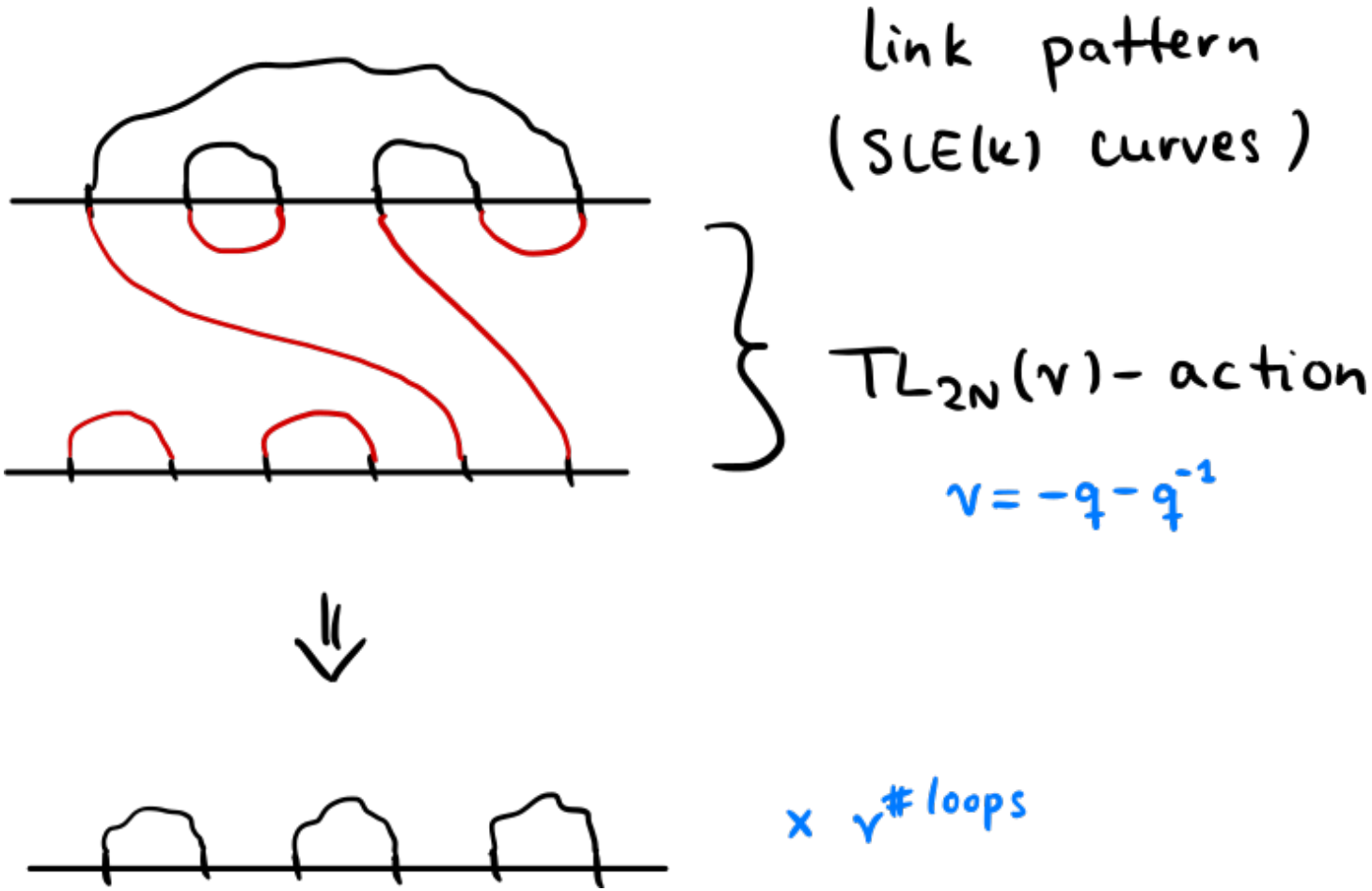


irreducible representations: *standard modules* $\mathrm{L}^{(s)}$



Temperley-Lieb action



Caveat: Simple $\mathrm{TL}_n(\nu)$-action for *b.c.c.* partition functions $\mathcal{F}_\beta$, *not* for *pure* partition functions $\mathcal{Z}_\alpha$!

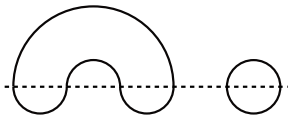
Change of basis via meander matrix

$$\lim_{\delta\to 0} \mathbb{P}^{\delta}_{\beta}[\text{ connectivity} = \alpha\,] = \mathcal{M}^{(\kappa)}_{\alpha,\beta}\,\frac{\mathcal{Z}^{(\kappa)}_{\alpha}(x_1,\ldots,x_{2N})}{\mathcal{F}^{(\kappa)}_{\beta}(x_1,\ldots,x_{2N})}$$

$$\mathcal{F}^{(\kappa)}_{\beta} = \sum_{\beta\in \mathrm{LP}_N} \mathcal{M}^{(\kappa)}_{\alpha,\beta}\,\mathcal{Z}^{(\kappa)}_{\alpha}$$

(meanders [DiFrancesco-Golinelli-Guitter '90s; Flores-Kleban-Simmons-Ziff '17])

$$\mathcal{M}^{(\kappa)}_{\alpha,\beta} := (2\cos(4\pi/\kappa))^{\#\text{loops in }(\alpha,\beta)} \quad \text{for} \quad (\alpha,\beta) =$$



- **Issue**: $\{\mathcal{F}^{(\kappa)}_{\beta} : \beta\in \mathrm{LP}_N\}$ not in general linearly independent!
- Pure partition functions $\{\mathcal{Z}^{(\kappa)}_{\alpha} : \alpha\in \mathrm{LP}_N\}$ *are linearly independent*. They are determined as solutions to PDE bdry value problem (BPZ equations, fusion asymptotics) — CFT heuristics in action!

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Thanks!