Field Theory of the Massive Ising Model

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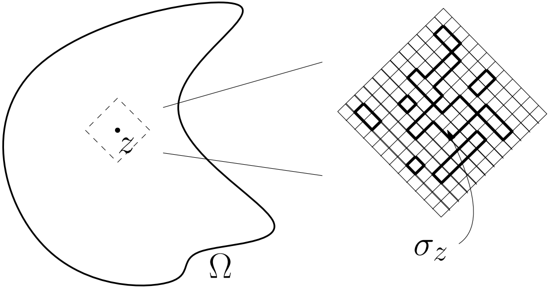
Planar Ising Model



- Spin-**Ising** model: randomly assign ± 1 *spins* σ_v on vertices (or faces) v of discretisation $\Omega^\delta = \Omega \cap \delta e^{i\pi/4}\mathbb{Z}^2$ of (smooth) simply connected $\Omega \subset \mathbb{C}$
- Define probability measure through **low**-temperature expansion at any *inverse temperature* $\beta > 0$: weight σ by $\exp[-2\beta L(\sigma)]$, where $L(\sigma)$ is the *domain wall* length $\leftrightarrow$ **high**-temperature expansion
  - spins σ on faces, **disorders** μ on vertices $\leftrightarrow$ spins on vertices
  - boundary condition: **plus** $\leftrightarrow$ **free**

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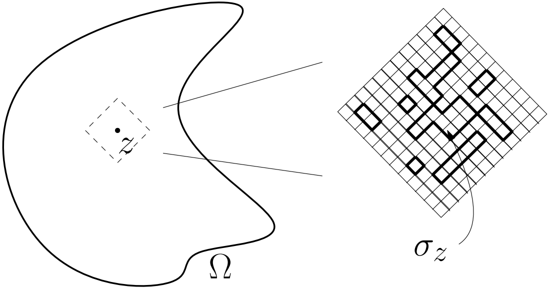

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Classical Results

- Onsager ('44): exact solution; phase transition at $\beta_c = \frac{1}{2}\ln(1+\sqrt{2})$
- Yang ('52): exact formula for $\mathbb{E}^{+,\beta}_{\mathbb{C}^\delta}[\sigma_0] \asymp (\beta-\beta_c)_+^{1/8}$
- Wu: at β_c, for $r>0$,

$$\delta^{-1/4}\mathbb{E}^{+}_{\mathbb{C}^\delta}[\sigma_0\sigma_r] \xrightarrow{\delta\downarrow 0} \frac{\mathscr{C}_\sigma^2}{r^{1/4}},$$

where $\mathscr{C}_\sigma = 2^{1/6}e^{3\zeta'(-1)/2}$.

- Wu, McCoy, Tracy, Barouch ('76): at $\beta = \beta_c - \frac{m}{\sqrt{2}}\delta$ for $m\in\mathbb{R}$

$$\delta^{-1/4}\mathbb{E}^{+}_{\mathbb{C}^\delta}[\sigma_0\sigma_z] \xrightarrow{\delta\downarrow 0} \mathscr{C}_\sigma^2\mathbb{III}^m(|z|),$$

where $\mathbb{III}^m$ is given explicitly in terms of a Painlevé III transcendent and it has an exponential length scale $2|m|$.

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Critical Convergence

Theorem

For the model at β_c on Ω^δ, write $\varepsilon_{(v_1 v_2)} := \sqrt{2}\sigma_{v_1}\sigma_{v_2} - 1$ for the centred energy density at the edge $(v_1 v_2)$. Then as $\delta \downarrow 0$,

- $\delta^{-n}\mathbb{E}^+_{\Omega^\delta}[\varepsilon_{z_1}\cdots\varepsilon_{z_n}] \to \mathscr{C}_\varepsilon{}^n \langle \varepsilon_{z_1}\cdots\varepsilon_{z_n}\rangle^+_\Omega$ *(Hongler '10 + Hongler, Smirnov '13);*
- $\delta^{-n/8}\mathbb{E}^+_{\Omega^\delta}[\sigma_{z_1}\cdots\sigma_{z_n}] \to \mathscr{C}_\sigma^n \langle \sigma_{z_1}\cdots\sigma_{z_n}\rangle^+_\Omega$ *(Chelkak, Hongler, Izyurov '15);*
- $\delta^{-n_1/8-n_2}\mathbb{E}^+_{\Omega^\delta}\left[\sigma_{z_1}\cdots\sigma_{z_{n_1}}\varepsilon_{z_1}\cdots\varepsilon_{z_{n_2}}\right] \to$ $\mathscr{C}_\sigma^{n_1}\mathscr{C}_\varepsilon^{n_2}\left\langle \sigma_{z_1}\cdots\sigma_{z_{n_1}}\varepsilon_{z_1}\cdots\varepsilon_{z_{n_2}}\right\rangle^+_\Omega$ *(Chelkak, Hongler, Izyurov '21),*

where the limits are smooth function of distinct points $z_k \in \Omega$ which are ***conformally covariant****: with explicit function on $\mathbb{H}$,*

$$\langle\cdots\rangle^+_\Omega = (|\varphi'(z_1)\cdots\varphi'(z_{n_1})|)^{1/8}(|\varphi'(z_1)\cdots\varphi'(z_{n_2})|)\langle\varphi(\cdots)\rangle^+_{\mathbb{H}}.$$

Massive Convergence

Theorem

For the model at $\beta = \beta_c - \frac{m}{\sqrt{2}}\delta$ *on* Ω^δ, *write*

$\varepsilon_{(v_1 v_2)} := \sqrt{2}\left(\sigma_{v_1}\sigma_{v_2} - \mathbb{E}^+_{\mathbb{C}^\delta}[\sigma_{v_1}\sigma_{v_2}]\right)$. *Then as* $\delta \downarrow 0$,

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- $\delta^{-n}\mathbb{E}^+_{\Omega^\delta}[\varepsilon_{z_1}\cdots\varepsilon_{z_n}] \to {\mathscr{C}_\varepsilon}^n \langle \varepsilon_{z_1}\cdots\varepsilon_{z_n}\rangle^{+,m}_\Omega$ *(Chelkak, P., Wan '22);*
- $\delta^{-n_1/8-n_2}\mathbb{E}^+_{\Omega^\delta}\left[\sigma_{z_1}\cdots\sigma_{z_{n_1}}\varepsilon_{z_1}\cdots\varepsilon_{z_{n_2}}\right] \to$
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where the limits are uniquely determined smooth functions of distinct points $z_k \in \Omega$, *and satisfy massive* ***operator product expansions*** *(OPEs).*

Cf. regarding interface: for any m and general Ω,

- FK-Ising interface (fermionic) martingale converges (P. '22);
- Spin-Ising interface (fermionic) martingale converges (Chelkak, P., et al. '23+).

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Fermions and Bers-Vekua Theory

- Guided by Kadanoff, Ceva ('71) and Smirnov ('10): write $\psi^{[\eta]} = \sigma_u \mu_v$ where $i\eta^{-2}$ is the unit complex number pointing from face u to adjacent vertex v
- Define complex fermion $\psi = \psi^{[1]} + i\psi^{[i]}$
- Near-critical complex fermions satisfy the equation (within even correlations)

$$\overline{\partial}\psi = -im\overline{\psi}.$$

- Under conformal change of coordinates φ, we have covariance rule

$$\psi \to (\psi \circ \varphi) \cdot (\varphi')^{1/2}; \quad m \to m|\varphi'|.$$

- Bers-Vekua theory ('60s): *generalised analyticity* $\overline{\partial} f = \alpha \overline{f}$, $\alpha \in C^\infty(\Omega)$,
  - series expansion in explicit *formal powers* $Z_n^\eta \sim \eta z^n$. written in terms of modified Bessel functions $I_\nu(x) \sim x^\nu, K_\nu \sim x^{-\nu}$ (say for $x, \nu > 0$).
  - 1-1 correspondence with suitably defined *holomorphic part* $\underline{f}$ modulo a $C^{\alpha<1}$ factor e^s, non-explicit but uniquely characterisable.

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Characterising the Limit (I)

- 2-point fermion correlation $f(z) = \dfrac{\left\langle \psi_z \psi_w^{[\eta]} \sigma_{z_1} \cdots \sigma_{z_n} \right\rangle_\Omega^{+,m}}{\left\langle \sigma_{z_1} \cdots \sigma_{z_n} \right\rangle_\Omega^{+,m}}$ (with spin insertions):
  - $\overline{\partial} f = -im\bar{f}$ away from insertions;
  - has -1 multiplicative monodromy around $z_1, \ldots, z_n$, with behaviour

$$f(z)\sqrt{z - z_j} \xrightarrow{z \to z_j} e^{i\pi/4} r \in e^{i\pi/4}\mathbb{R};$$

  - has a simple pole at w:

$$f(z)(z-w) \xrightarrow{z \to w} \bar{\eta};$$

  - has the *Riemann-Hilbert* boundary condition

$$f(z) \in \tau_z^{-1/2}\mathbb{R}, \quad z \in \partial\Omega,$$

    where τ_z is the tangent vector as a complex number;

- $2n$-point fermion correlations are given as Pfaffian of 2-point correlations:

$$\mathbb{E}\left[\frac{\langle \psi_{w_1} \cdots \psi_{w_{2n}} \sigma_A \rangle_\Omega^{+,m}}{\langle \sigma_A \rangle_\Omega^{+,m}}\right] = \mathrm{Pf}\left(\mathbb{E}\left[\frac{\langle \psi_{w_j} \psi_{w_k} \sigma_A \rangle_\Omega^{+,m}}{\langle \sigma_A \rangle_\Omega^{+,m}}\right]\right)_{j,k=1}^{2n}$$

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- Merging fermion and spin, we get disorder insertions

$$\psi_z \sigma_{z_k} = \mu_z e^{i\pi/4}(z - z_k)^{-1/2} + o\left((z - z_k)^{-1/2}\right);$$

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  - only thing to fix is normalisation: uniquely fixed by
    - as $z_k \to \partial\Omega$, n-point correlations break up into multiple of $(n-1)$- and 1-point correlations;
    - Wu's asymptotic, i.e. $\sim 1/|z_1 - z_2|^{1/4}$ as $z_1 \to z_2$ in 2-point correlation.
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  - from $\frac{\langle \psi_z \psi_w \sigma_{z_1} \cdots \sigma_{z_n} \rangle_{\Omega}^{+,m}}{\langle \sigma_{z_1} \cdots \sigma_{z_n} \rangle_{\Omega}^{+,m}}$, we have $\partial_{z_k} \log \langle \sigma_{z_1} \cdots \sigma_{z_n} \rangle_{\Omega}^{+,m}$;
  - only thing to fix is normalisation: uniquely fixed by
    - ★ as $z_k \to \partial\Omega$, n-point correlations break up into multiple of $(n-1)$- and 1-point correlations;
    - ★ Wu's asymptotic, i.e. $\sim 1/|z_1 - z_2|^{1/4}$ as $z_1 \to z_2$ in 2-point correlation.
- This characterisation can be used to *construct* $\langle \cdot \rangle_{\Omega}^{+,\tilde{m}}$ for generic (say, smooth) functions $\tilde{m} : \Omega \cup \partial\Omega \to \mathbb{R}$.

Massive OPEs

Uniformly away from boundary and other insertions, we get, as $z \to w$,

$$\begin{aligned}
\psi_z \psi_w &= 2Z^{1}_{-1}(z-w) + O(z-w) \\
\psi_z \psi^{\star}_w &= -2i\varepsilon_w + O(z-w) \\
\psi_z \psi^{[\eta]}_w &= Z^{\bar{\eta}}_{-1}(z-w) - i\eta\varepsilon_w + O(z-w) \\
\psi_z \sigma_w &= \mu_z Z^{e^{i\pi/4}}_{-1/2}(z-w) + 4\partial_w \mu_w (z-w)^{1/2} + O((z-w)^{3/2}) \\
\varepsilon_z \sigma_w &= \frac{i}{2}\frac{\sigma_w}{|z-w|} + O(1) \\
\sigma_z \sigma_w &= \mathbb{III}^m(|z-w|) + \frac{1}{2}\varepsilon_w |z-w|^{3/4} + o(|z-w|^{3/4}),
\end{aligned}$$

where Z^{η}_n are explicit (in terms of modified Bessel functions) and $\mathbb{III}^m(|\cdot|)$ is the full plane correlation given in terms of the Painlevé III transcendent.

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Massless Sine-Gordon Theory

- *Massless* **Sine-Gordon**: construct a random field ϕ with measure

$$\mathbb{P}_{\mathsf{SG}}[\phi] \propto \exp\left[\mu \int_{\Omega} :\cos\left(\sqrt{\beta}\phi\right):\right] \mathbb{P}_{\mathsf{GFF}}[\phi]$$

- Coleman ('75): should be bosonisation of massive **Thirring model** ($\ni$ massive free fermions)
- Extensive literature on construction starting with Fröhlich ('75)
- Emergent length scale (exponential decay): cf. Bernard, LeClair ('94), Zamolodchikov ('95), ...
- We will focus on $\beta = 4\pi$ case with $\mu = m$ ($\leftrightarrow$ m-massive Ising) on nice simply connected domains Ω

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Correlation Construction

- Work with *regularised GFF* ϕ_ε with correlation $C_\varepsilon(x,y) = \int_{\varepsilon^2}^{\infty} p_\Omega(s,x,y)ds$;
- Work with truncated mass, i.e. $\mu(z) = m\rho(z) \in C_c^\infty(\Omega)$;
- Work with renormalisations $:\cos(\sqrt{4\pi}\phi_\varepsilon):_\varepsilon = \varepsilon^{-1}\cos(\sqrt{4\pi}\phi_\varepsilon)$, etc.;
- Send to limit *smeared correlations*, i.e. area integrals of fields against $f \in C_c^\infty(\Omega)$, extract integral kernels after the limit, then send $\rho \to 1$.

Theorem (P., Virtanen, Webb '23)

The limits of Sine-Gordon smeared correlations exist, and the integral kernels, written as pointwise correlations of fields, may be identified with massive $\text{Ising}_m \times \text{Ising}_m$*' correlations, under bosonisation rules, e.g.:*

$$\sigma\sigma' \leftrightarrow \sqrt{2} :\cos(\sqrt{\pi}\phi):, \quad \psi\psi' \leftrightarrow 4\sqrt{\pi} i\partial\phi, \quad \varepsilon + \varepsilon' \leftrightarrow :\cos(\sqrt{4\pi}\phi):$$

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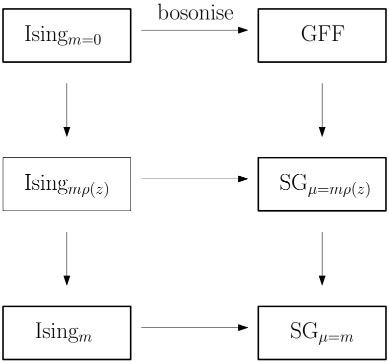
Massive Bosonisation

- Construct continuum $\text{Ising}_{m\rho(z)}$ correlations, then expand both correlations in the powers of m:
- Explicit identification of (critical!) correlations near $m = 0$ and analyticity around every $m \in \mathbb{R}$.



What's More?

- Massive CFT?
  - Cardy, Mussardo ('90) conjectures that there is a pair of Viraroso algebras acting on massive Ising fields.
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Thank you!