Conformal removability of SLE_κ for $\kappa \in [4, 8)$

Jason Miller

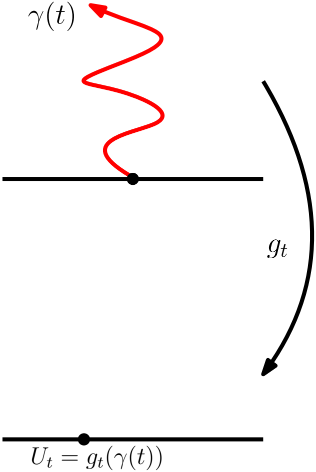
Cambridge

joint with **Konstantinos Kavvadias** *and* **Lukas Schoug**

June 7, 2023

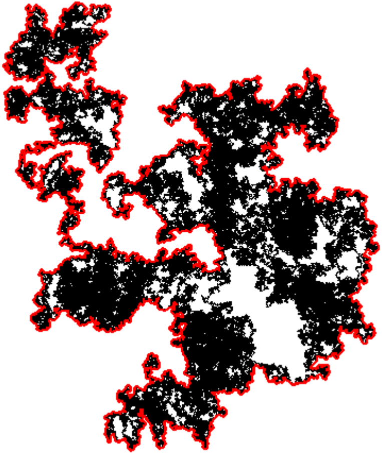
Constructions of SLE

- Loewner equation (**Schramm**)



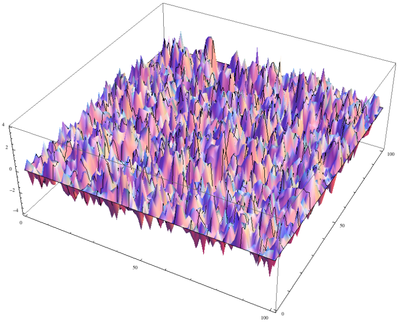
Constructions of SLE

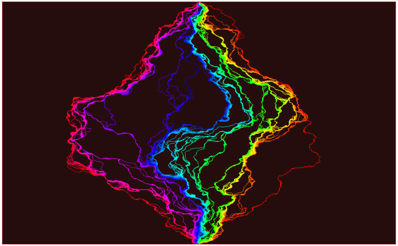
- Loewner equation (**Schramm**)
- Outer boundary of Brownian motion and Brownian loop soup clusters (**Lawler-Schramm-Werner**, **Sheffield-Werner**)



Constructions of SLE

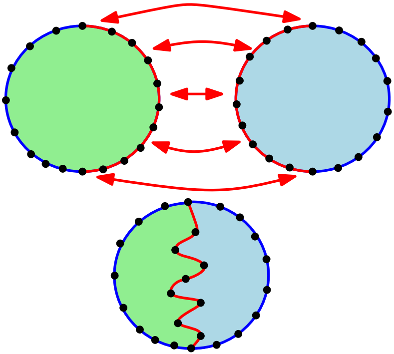
- Loewner equation (**Schramm**)
- Outer boundary of Brownian motion and Brownian loop soup clusters (**Lawler-Schramm-Werner**, **Sheffield-Werner**)
- Level and flow lines of the GFF (**Schramm-Sheffield**, **Dubédat**, **M.-Sheffield**, **Wang-Wu**)





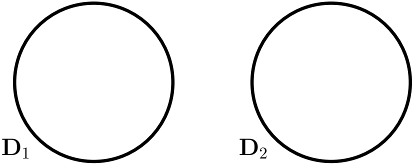
Constructions of SLE

- Loewner equation (**Schramm**)
- Outer boundary of Brownian motion and Brownian loop soup clusters (**Lawler-Schramm-Werner**, **Sheffield-Werner**)
- Level and flow lines of the GFF (**Schramm-Sheffield**, **Dubédat**, **M.-Sheffield**, **Wang-Wu**)
- Conformal welding of Liouville quantum gravity surfaces (**Sheffield**, **Duplantier-M.-Sheffield**)
  - Continuous analog of gluing planar maps with boundary



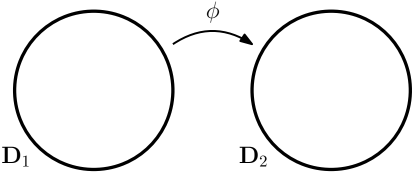
Conformal welding

- Suppose that $\mathbf{D}_1$, $\mathbf{D}_2$ are two copies of the unit disk.



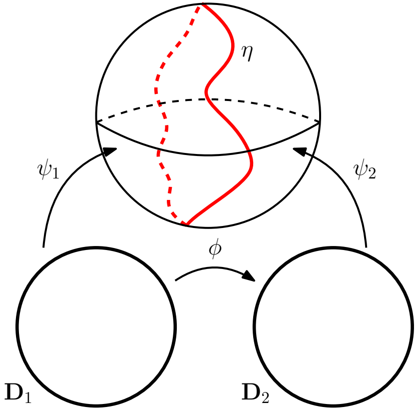
Conformal welding

- Suppose that $\mathbf{D}_1$, $\mathbf{D}_2$ are two copies of the unit disk.
- Let $\phi\colon \partial\mathbf{D}_1 \to \partial\mathbf{D}_2$ be a homeomorphism.



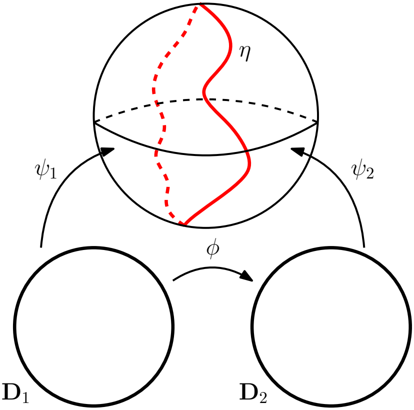
Conformal welding

- Suppose that $\mathbf{D}_1$, $\mathbf{D}_2$ are two copies of the unit disk.
- Let $\phi\colon \partial\mathbf{D}_1 \to \partial\mathbf{D}_2$ be a homeomorphism.
- A **conformal welding** with welding homeomorphism ϕ consists of a simple loop η on $\mathbf{S}^2$ and a pair of conformal transformations ψ_i from $\mathbf{D}_i$ to the two components of $\mathbf{S}^2 \setminus \eta$ so that $\phi = \psi_2^{-1} \circ \psi_1|_{\partial\mathbf{D}_1}$.



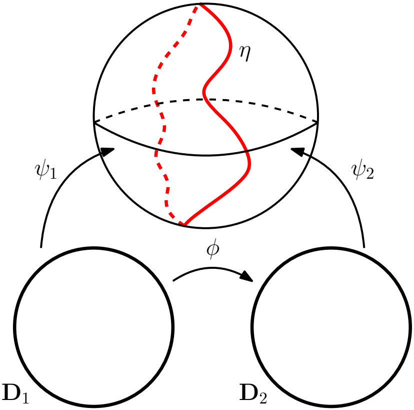
Conformal welding

- Suppose that $\mathbf{D}_1$, $\mathbf{D}_2$ are two copies of the unit disk.
- Let $\phi \colon \partial \mathbf{D}_1 \to \partial \mathbf{D}_2$ be a homeomorphism.
- A **conformal welding** with welding homeomorphism ϕ consists of a simple loop η on $\mathbf{S}^2$ and a pair of conformal transformations ψ_i from $\mathbf{D}_i$ to the two components of $\mathbf{S}^2 \setminus \eta$ so that $\phi = \psi_2^{-1} \circ \psi_1|_{\partial \mathbf{D}_1}$.
- Existence is not obvious.



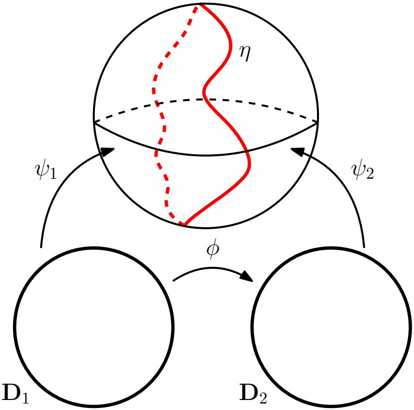
Conformal welding

- Suppose that $\mathbf{D}_1$, $\mathbf{D}_2$ are two copies of the unit disk.
- Let $\phi\colon \partial \mathbf{D}_1 \to \partial \mathbf{D}_2$ be a homeomorphism.
- A **conformal welding** with welding homeomorphism ϕ consists of a simple loop η on $\mathbf{S}^2$ and a pair of conformal transformations ψ_i from $\mathbf{D}_i$ to the two components of $\mathbf{S}^2 \setminus \eta$ so that $\phi = \psi_2^{-1} \circ \psi_1|_{\partial \mathbf{D}_1}$.
- Existence is not obvious.
- **Sheffield** proved that SLE curves arise as (random) conformal weldings where the welding homeomorphism comes from $e^{\gamma \mathrm{GFF}}$



Conformal welding

- Suppose that $\mathbf{D}_1$, $\mathbf{D}_2$ are two copies of the unit disk.
- Let $\phi\colon \partial\mathbf{D}_1 \to \partial\mathbf{D}_2$ be a homeomorphism.
- A **conformal welding** with welding homeomorphism ϕ consists of a simple loop η on $\mathbf{S}^2$ and a pair of conformal transformations ψ_i from $\mathbf{D}_i$ to the two components of $\mathbf{S}^2 \setminus \eta$ so that $\phi = \psi_2^{-1} \circ \psi_1|_{\partial\mathbf{D}_1}$.
- Existence is not obvious.
- **Sheffield** proved that SLE curves arise as (random) conformal weldings where the welding homeomorphism comes from $e^{\gamma \mathrm{GFF}}$
- Uniqueness is connected to **conformal removability**



Conformal removability

- A compact set $K \subseteq \mathbf{C}$ is **conformally removable** if whenever $\varphi \colon \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus K$ then φ is conformal on $\mathbf{C}$.

Conformal removability

- A compact set $K \subseteq \mathbf{C}$ is **conformally removable** if whenever $\varphi\colon \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus K$ then φ is conformal on $\mathbf{C}$.
- **Example**: $K = \{z\}$

Conformal removability

- A compact set $K \subseteq \mathbf{C}$ is **conformally removable** if whenever $\varphi\colon \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus K$ then φ is conformal on $\mathbf{C}$.
- **Example**: $K = \{z\}$
- **Jones-Smirnov** proved that if D is a Hölder domain then ∂D is conformally removable

Conformal removability

- A compact set $K \subseteq \mathbf{C}$ is **conformally removable** if whenever $\varphi \colon \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus K$ then φ is conformal on $\mathbf{C}$.
- **Example**: $K = \{z\}$
- **Jones-Smirnov** proved that if D is a Hölder domain then ∂D is conformally removable
- Combined with the work of **Rhode-Schramm**, gives that the boundary complementary component of an SLE_κ for $\kappa \neq 4$ is conformally removable

Conformal removability

- A compact set $K \subseteq \mathbf{C}$ is **conformally removable** if whenever $\varphi \colon \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus K$ then φ is conformal on $\mathbf{C}$.
- **Example**: $K = \{z\}$
- **Jones-Smirnov** proved that if D is a Hölder domain then ∂D is conformally removable
- Combined with the work of **Rhode-Schramm**, gives that the boundary complementary component of an SLE_κ for $\kappa \neq 4$ is conformally removable
- **Sheffield**: is SLE_4 conformally removable? What about SLE_κ for $\kappa \in (4, 8)$?

Conformal removability

- A compact set $K \subseteq \mathbf{C}$ is **conformally removable** if whenever $\varphi \colon \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus K$ then φ is conformal on $\mathbf{C}$.
- **Example**: $K = \{z\}$
- **Jones-Smirnov** proved that if D is a Hölder domain then ∂D is conformally removable
- Combined with the work of **Rhode-Schramm**, gives that the boundary complementary component of an SLE_κ for $\kappa \neq 4$ is conformally removable
- **Sheffield**: is SLE_4 conformally removable? What about SLE_κ for $\kappa \in (4,8)$?
- **Jones-Smirnov** actually proved that it suffices for the uniformizing map to have modulus of continuity

$$\exp(-\sqrt{(\log \delta^{-1})(\log \log \delta^{-1})}/o(1)) \quad \text{as} \quad \delta \to 0.$$

Conformal removability

- A compact set $K \subseteq \mathbf{C}$ is **conformally removable** if whenever $\varphi \colon \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus K$ then φ is conformal on $\mathbf{C}$.
- **Example**: $K = \{z\}$
- **Jones-Smirnov** proved that if D is a Hölder domain then ∂D is conformally removable
- Combined with the work of **Rhode-Schramm**, gives that the boundary complementary component of an SLE_κ for $\kappa \neq 4$ is conformally removable
- **Sheffield**: is SLE_4 conformally removable? What about SLE_κ for $\kappa \in (4,8)$?
- **Jones-Smirnov** actually proved that it suffices for the uniformizing map to have modulus of continuity

$$\exp(-\sqrt{(\log \delta^{-1})(\log\log \delta^{-1})}/o(1)) \quad \text{as} \quad \delta \to 0.$$

Kavvadias-M.-Schoug: Not satisfied by SLE_4; modulus of continuity is $(\log \delta^{-1})^{-1/3+o(1)}$

Conformal removability

- A compact set $K \subseteq \mathbf{C}$ is **conformally removable** if whenever $\varphi \colon \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus K$ then φ is conformal on $\mathbf{C}$.
- **Example**: $K = \{z\}$
- **Jones-Smirnov** proved that if D is a Hölder domain then ∂D is conformally removable
- Combined with the work of **Rhode-Schramm**, gives that the boundary complementary component of an SLE_κ for $\kappa \neq 4$ is conformally removable
- **Sheffield**: is SLE_4 conformally removable? What about SLE_κ for $\kappa \in (4,8)$?
- **Jones-Smirnov** actually proved that it suffices for the uniformizing map to have modulus of continuity

$$\exp(-\sqrt{(\log \delta^{-1})(\log\log \delta^{-1})}/o(1)) \quad \text{as} \quad \delta \to 0.$$

  Kavvadias-M.-Schoug: Not satisfied by SLE_4; modulus of continuity is $(\log \delta^{-1})^{-1/3+o(1)}$
- Not possible to completely characterize conformally removable sets (see **Bishop**'s paper *Conformal removability is hard*)

How to prove conformal non-removability?

- **Fact 1**: conformal removability $\longleftrightarrow$ quasiconformal removability
- **Fact 2**: if X has positive Lebesgue measure then X is not conformally removable

To show that X is not conformally removable, look for a homeomorphism $f \colon \mathbf{C} \to \mathbf{C}$ which is (quasi)conformal on $\mathbf{C} \setminus X$ so that $f(X)$ has positive Lebesgue measure

How to prove conformal non-removability?

- **Fact 1**: conformal removability $\longleftrightarrow$ quasiconformal removability
- **Fact 2**: if X has positive Lebesgue measure then X is not conformally removable

To show that X is not conformally removable, look for a homeomorphism $f \colon \mathbf{C} \to \mathbf{C}$ which is (quasi)conformal on $\mathbf{C} \setminus X$ so that $f(X)$ has positive Lebesgue measure

Example: $X = \text{Cantor Set} \times [0, 1]$

How to prove conformal non-removability?

- **Fact 1**: conformal removability $\longleftrightarrow$ quasiconformal removability
- **Fact 2**: if X has positive Lebesgue measure then X is not conformally removable

To show that X is not conformally removable, look for a homeomorphism $f \colon \mathbf{C} \to \mathbf{C}$ which is (quasi)conformal on $\mathbf{C} \setminus X$ so that $f(X)$ has positive Lebesgue measure

Example: $X = \text{Cantor Set} \times [0,1]$

- $h \colon \mathbf{R} \to \mathbf{R}$ the "Devil's staircase map" which is non-decreasing, piecewise constant off the Cantor Set, and maps the Cantor Set to a set of positive Lebesgue measure.

How to prove conformal non-removability?

- **Fact 1**: conformal removability $\longleftrightarrow$ quasiconformal removability
- **Fact 2**: if X has positive Lebesgue measure then X is not conformally removable

To show that X is not conformally removable, look for a homeomorphism $f : \mathbf{C} \to \mathbf{C}$ which is (quasi)conformal on $\mathbf{C} \setminus X$ so that $f(X)$ has positive Lebesgue measure

Example: $X = \text{Cantor Set} \times [0, 1]$

- $h \colon \mathbf{R} \to \mathbf{R}$ the "Devil's staircase map" which is non-decreasing, piecewise constant off the Cantor Set, and maps the Cantor Set to a set of positive Lebesgue measure.
- $\phi \colon \mathbf{R} \to \mathbf{R}$ a non-zero bump function supported on $[1/4, 3/4]$.

How to prove conformal non-removability?

- **Fact 1**: conformal removability $\longleftrightarrow$ quasiconformal removability
- **Fact 2**: if X has positive Lebesgue measure then X is not conformally removable

To show that X is not conformally removable, look for a homeomorphism $f\colon \mathbf{C} \to \mathbf{C}$ which is (quasi)conformal on $\mathbf{C} \setminus X$ so that $f(X)$ has positive Lebesgue measure

Example: $X = \text{Cantor Set} \times [0,1]$

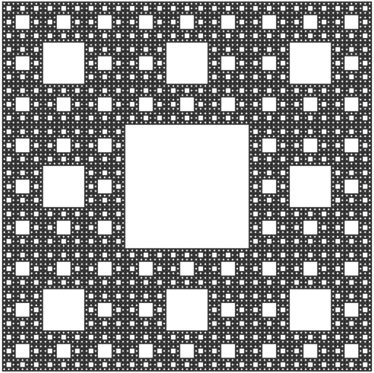
- $h\colon \mathbf{R} \to \mathbf{R}$ the "Devil's staircase map" which is non-decreasing, piecewise constant off the Cantor Set, and maps the Cantor Set to a set of positive Lebesgue measure.
- $\phi\colon \mathbf{R} \to \mathbf{R}$ a non-zero bump function supported on $[1/4, 3/4]$.
- $f(x,y) = (h(x)\phi(y) + x, y)$ is a homeomorphism, quasiconformal off X, $f(X)$ has positive Lebesgue measure $\to$ X is *not conformally removable*.

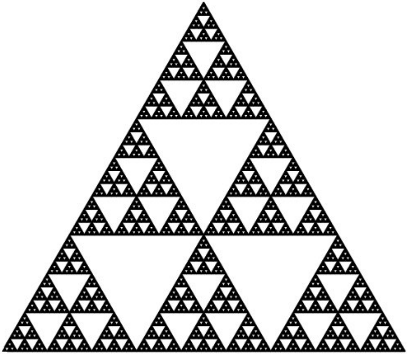
Conformal removability in the disconnected setting

SLE_4 is **almost** self-intersecting and conformal removability for boundaries of domains which are not connected is not well understood.

Conformal removability in the disconnected setting

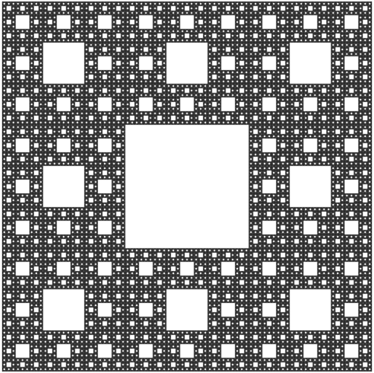
SLE_4 is **almost** self-intersecting and conformal removability for boundaries of domains which are not connected is not well understood.

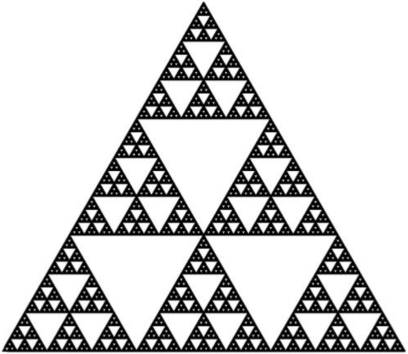




Conformal removability in the disconnected setting

SLE_4 is **almost** self-intersecting and conformal removability for boundaries of domains which are not connected is not well understood.

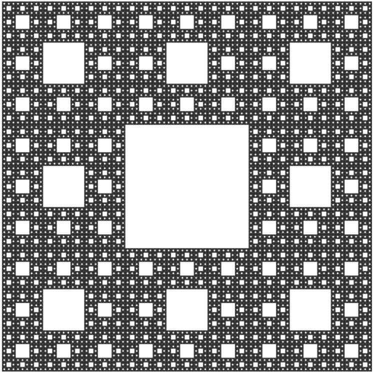


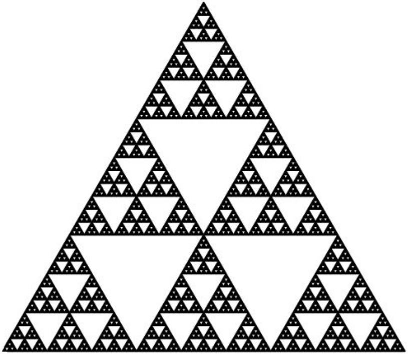


Not difficult to show that the Sierpinski carpet is not conformally removable (contains Cantor Set $\times$ $[0, 1]$).

Conformal removability in the disconnected setting

SLE_4 is **almost** self-intersecting and conformal removability for boundaries of domains which are not connected is not well understood.



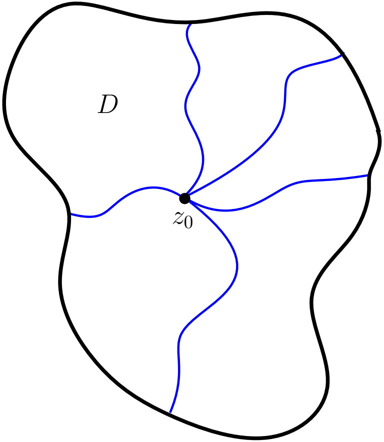


Not difficult to show that the Sierpinski carpet is not conformally removable (contains Cantor Set $\times$ $[0, 1]$).

Ntalampekos recently proved that the Sierpinski gasket and all topological Sierpinski carpets are not conformally removable.

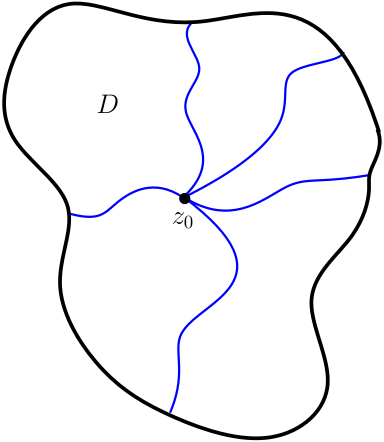
Jones-Smirnov condition

- Suppose $D \subseteq \mathbf{C}$ is simply connected, $z_0 \in D$, and Γ is a family of paths in D all of which start from z_0 and whose accumulation set is ∂D.



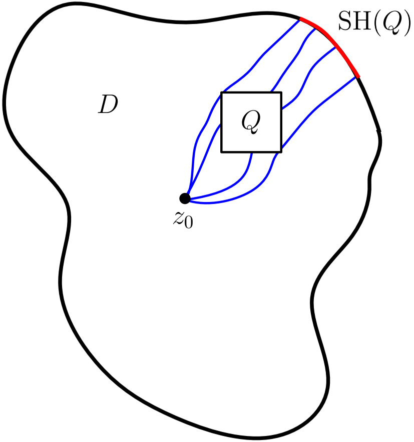
Jones-Smirnov condition

- Suppose $D \subseteq \mathbf{C}$ is simply connected, $z_0 \in D$, and Γ is a family of paths in D all of which start from z_0 and whose accumulation set is ∂D.
- Suppose that $\mathcal{W}$ is a Whitney cube decomposition of D.



Jones-Smirnov condition

- Suppose $D \subseteq \mathbf{C}$ is simply connected, $z_0 \in D$, and Γ is a family of paths in D all of which start from z_0 and whose accumulation set is ∂D.
- Suppose that $\mathcal{W}$ is a Whitney cube decomposition of D.
- The **shadow** $\mathrm{sh}(Q)$ is the accumulation set in ∂D of the curves in Γ which pass through Q.

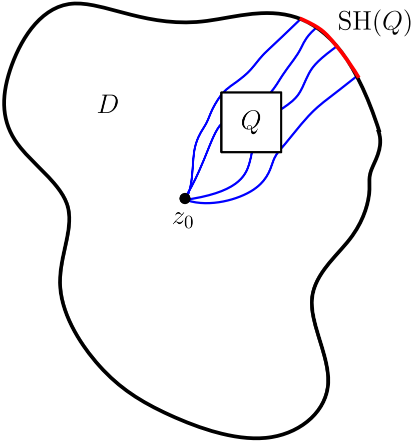


Jones-Smirnov condition

- Suppose $D \subseteq \mathbf{C}$ is simply connected, $z_0 \in D$, and Γ is a family of paths in D all of which start from z_0 and whose accumulation set is ∂D.
- Suppose that $\mathcal{W}$ is a Whitney cube decomposition of D.
- The **shadow** $\mathrm{sh}(Q)$ is the accumulation set in ∂D of the curves in Γ which pass through Q.
- **Jones-Smirnov** proved that if

$$\sum_{Q \in \mathcal{W}} \mathrm{diam}(\mathrm{sh}(Q))^2 < \infty$$

then ∂D is conformally removable.

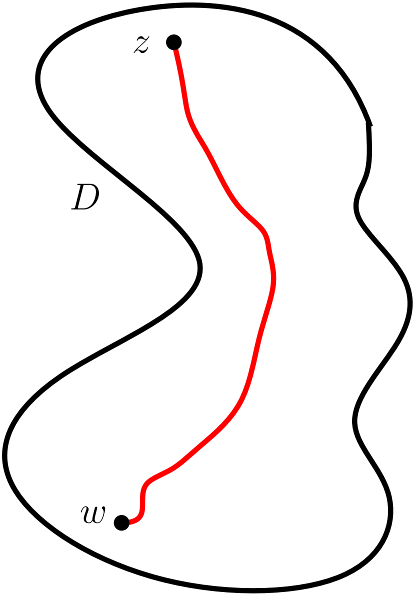


Jones-Smirnov condition

- **Jones-Smirnov** proved that if

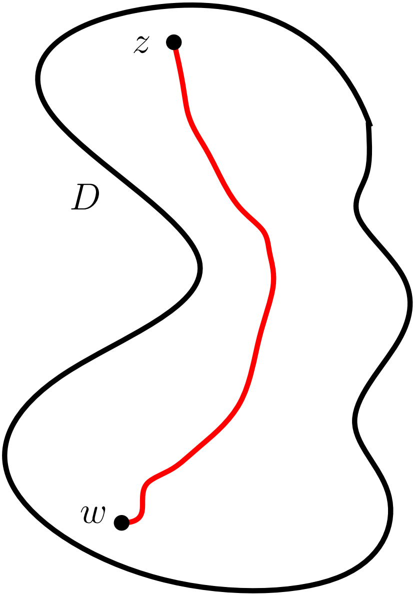
$$\sum_{Q \in \mathcal{W}} \mathsf{diam}(\mathrm{sh}(Q))^2 < \infty$$

then ∂D is conformally removable.



Jones-Smirnov condition

- **Jones-Smirnov** proved that if
$$\sum_{Q \in \mathcal{W}} \mathrm{diam}(\mathrm{sh}(Q))^2 < \infty$$
then ∂D is conformally removable.
- The **quasihyperbolic** metric on D is defined by
$$\mathrm{qh}(z,w) = \inf \left\{ \int \frac{|\gamma'(t)|}{\mathrm{dist}(\gamma(t), \partial D)} \right\}$$



Jones-Smirnov condition

- **Jones-Smirnov** proved that if

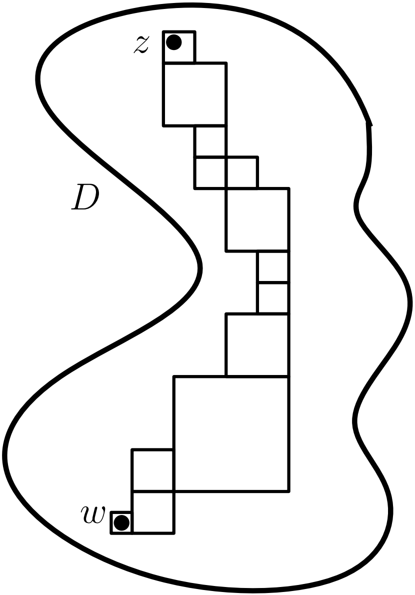
$$\sum_{Q \in \mathcal{W}} \text{diam}(\mathrm{sh}(Q))^2 < \infty$$

then ∂D is conformally removable.

- The **quasihyperbolic** metric on D is defined by

$$\mathrm{qh}(z, w) = \inf \left\{ \int \frac{|\gamma'(t)|}{\text{dist}(\gamma(t), \partial D)} \right\}$$

- $\mathrm{qh}(z, w) \asymp$ the number of Whitney cubes necessary to get from z to w



Jones-Smirnov condition

- **Jones-Smirnov** proved that if

$$\sum_{Q \in \mathcal{W}} \mathrm{diam}(\mathrm{sh}(Q))^2 < \infty \quad (\bigstar)$$

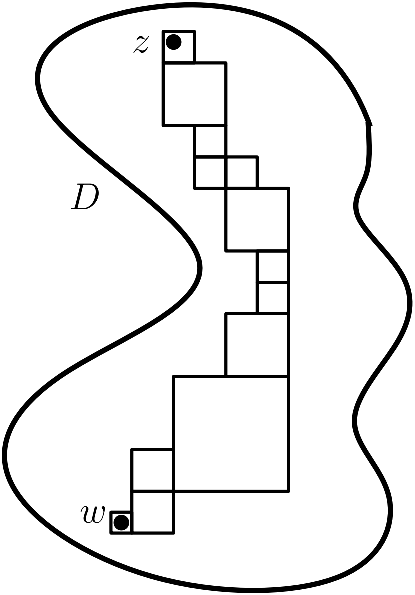
  then ∂D is conformally removable.

- The **quasihyperbolic** metric on D is defined by

$$\mathrm{qh}(z,w) = \inf \left\{ \int \frac{|\gamma'(t)|}{\mathrm{dist}(\gamma(t), \partial D)} \right\}$$

- $\mathrm{qh}(z,w) \asymp$ the number of Whitney cubes necessary to get from z to w

- **Jones-Smirnov** proved that if $z \mapsto \mathrm{qh}(z_0, z)$ is not in L^1 then $(\bigstar)$ does not hold and if $z \mapsto \mathrm{qh}(z_0, z)$ is in L^2 then $(\bigstar)$ holds



Jones-Smirnov condition

- **Jones-Smirnov** proved that if

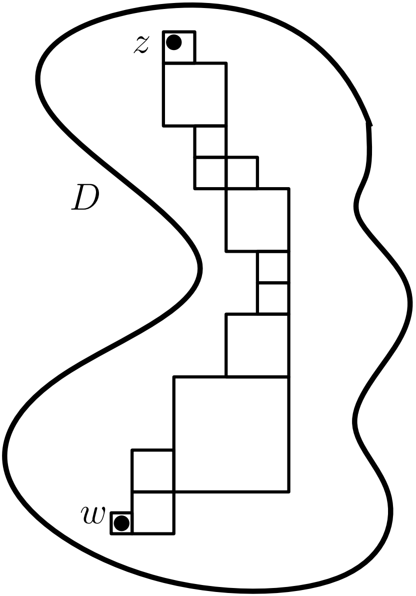
$$\sum_{Q \in \mathcal{W}} \mathrm{diam}(\mathrm{sh}(Q))^2 < \infty \quad (\star)$$

then ∂D is conformally removable.

- The **quasihyperbolic** metric on D is defined by

$$\mathrm{qh}(z,w) = \inf \left\{ \int \frac{|\gamma'(t)|}{\mathrm{dist}(\gamma(t), \partial D)} \right\}$$

- $\mathrm{qh}(z,w) \asymp$ the number of Whitney cubes necessary to get from z to w
- **Jones-Smirnov** proved that if $z \mapsto \mathrm{qh}(z_0, z)$ is not in L^1 then $(\star)$ does not hold and if $z \mapsto \mathrm{qh}(z_0, z)$ is in L^2 then $(\star)$ holds
- **Kavvadias-M.-Schoug**: $\mathrm{qh}(z_0, z) \notin L^1$ for D a complementary component of an SLE_4



New results

- Recall that $\kappa = 4$ is the critical value at or below which an SLE_κ curve is simple and $\kappa \in (4,8)$ is the range so that an SLE_κ curve is self-intersecting but not space-filling.

New results

- Recall that $\kappa = 4$ is the critical value at or below which an SLE_κ curve is simple and $\kappa \in (4,8)$ is the range so that an SLE_κ curve is self-intersecting but not space-filling.

Theorem (Kavvadias-M.-Schoug)

SLE_4 *curves are conformally removable.*

New results

- Recall that $\kappa = 4$ is the critical value at or below which an SLE_κ curve is simple and $\kappa \in (4, 8)$ is the range so that an SLE_κ curve is self-intersecting but not space-filling.

Theorem (Kavvadias-M.-Schoug)

SLE_4 *curves are conformally removable.*

- Suppose that $D \subseteq \mathbf{C}$ is open and $X \subseteq D$.

New results

- Recall that $\kappa = 4$ is the critical value at or below which an SLE_κ curve is simple and $\kappa \in (4, 8)$ is the range so that an SLE_κ curve is self-intersecting but not space-filling.

Theorem (Kavvadias-M.-Schoug)

SLE_4 *curves are conformally removable.*

- Suppose that $D \subseteq \mathbf{C}$ is open and $X \subseteq D$.
- Say that the graph of complementary components of X in D is **connected** if for every pair of components U, V of $D \setminus X$ there exist components $U_1, \ldots U_n$ of $D \setminus X$ so that $\partial U_i \cap \partial U_{i+1} \neq \emptyset$ for each $1 \leq i \leq n-1$.

New results

- Recall that $\kappa = 4$ is the critical value at or below which an SLE_κ curve is simple and $\kappa \in (4,8)$ is the range so that an SLE_κ curve is self-intersecting but not space-filling.

Theorem (Kavvadias-M.-Schoug)

SLE_4 *curves are conformally removable.*

- Suppose that $D \subseteq \mathbf{C}$ is open and $X \subseteq D$.
- Say that the graph of complementary components of X in D is **connected** if for every pair of components U, V of $D \setminus X$ there exist components $U_1, \ldots U_n$ of $D \setminus X$ so that $\partial U_i \cap \partial U_{i+1} \neq \emptyset$ for each $1 \leq i \leq n-1$.
- Let $\mathcal{K}$ be the set of $\kappa \in (4,8)$ so that the graph of complementary components of the range of an SLE_κ curve in $\mathbf{H}$ is connected.

New results

- Recall that $\kappa = 4$ is the critical value at or below which an SLE_κ curve is simple and $\kappa \in (4,8)$ is the range so that an SLE_κ curve is self-intersecting but not space-filling.

Theorem (Kavvadias-M.-Schoug)

SLE_4 *curves are conformally removable.*

- Suppose that $D \subseteq \mathbf{C}$ is open and $X \subseteq D$.
- Say that the graph of complementary components of X in D is **connected** if for every pair of components U, V of $D \setminus X$ there exist components $U_1, \ldots U_n$ of $D \setminus X$ so that $\partial U_i \cap \partial U_{i+1} \neq \emptyset$ for each $1 \leq i \leq n-1$.
- Let $\mathcal{K}$ be the set of $\kappa \in (4,8)$ so that the graph of complementary components of the range of an SLE_κ curve in $\mathbf{H}$ is connected.

Theorem (Kavvadias-M.-Schoug)

SLE_κ *for* $\kappa \in (4,8)$ *is conformally removable for all* $\kappa \in \mathcal{K}$*.*

New results

- Recall that $\kappa = 4$ is the critical value at or below which an SLE_κ curve is simple and $\kappa \in (4,8)$ is the range so that an SLE_κ curve is self-intersecting but not space-filling.

Theorem (Kavvadias-M.-Schoug)

SLE_4 *curves are conformally removable.*

- Suppose that $D \subseteq \mathbf{C}$ is open and $X \subseteq D$.
- Say that the graph of complementary components of X in D is **connected** if for every pair of components U, V of $D \setminus X$ there exist components $U_1, \ldots U_n$ of $D \setminus X$ so that $\partial U_i \cap \partial U_{i+1} \neq \emptyset$ for each $1 \leq i \leq n-1$.
- Let $\mathcal{K}$ be the set of $\kappa \in (4,8)$ so that the graph of complementary components of the range of an SLE_κ curve in $\mathbf{H}$ is connected.

Theorem (Kavvadias-M.-Schoug)

SLE_κ *for* $\kappa \in (4,8)$ *is conformally removable for all* $\kappa \in \mathcal{K}$.

- Proved by **Gwynne-Pfeffer** that $\mathcal{K} \neq \emptyset$.

New results

- Recall that $\kappa = 4$ is the critical value at or below which an SLE_κ curve is simple and $\kappa \in (4,8)$ is the range so that an SLE_κ curve is self-intersecting but not space-filling.

Theorem (Kavvadias-M.-Schoug)

SLE_4 *curves are conformally removable.*

- Suppose that $D \subseteq \mathbf{C}$ is open and $X \subseteq D$.
- Say that the graph of complementary components of X in D is **connected** if for every pair of components U, V of $D \setminus X$ there exist components $U_1, \ldots U_n$ of $D \setminus X$ so that $\partial U_i \cap \partial U_{i+1} \neq \emptyset$ for each $1 \leq i \leq n-1$.
- Let $\mathcal{K}$ be the set of $\kappa \in (4,8)$ so that the graph of complementary components of the range of an SLE_κ curve in $\mathbf{H}$ is connected.

Theorem (Kavvadias-M.-Schoug)

SLE_κ *for* $\kappa \in (4,8)$ *is conformally removable for all* $\kappa \in \mathcal{K}$.

- Proved by **Gwynne-Pfeffer** that $\mathcal{K} \neq \emptyset$. Not known if $\mathcal{K} = (4,8)$.

New results

- Recall that $\kappa = 4$ is the critical value at or below which an SLE_κ curve is simple and $\kappa \in (4,8)$ is the range so that an SLE_κ curve is self-intersecting but not space-filling.

Theorem (Kavvadias-M.-Schoug)

SLE_4 *curves are conformally removable.*

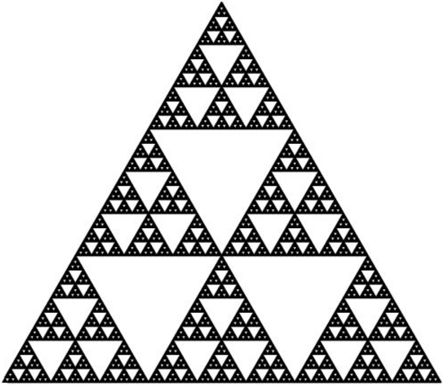
- Suppose that $D \subseteq \mathbf{C}$ is open and $X \subseteq D$.
- Say that the graph of complementary components of X in D is **connected** if for every pair of components U, V of $D \setminus X$ there exist components $U_1, \ldots U_n$ of $D \setminus X$ so that $\partial U_i \cap \partial U_{i+1} \neq \emptyset$ for each $1 \leq i \leq n-1$.
- Let $\mathcal{K}$ be the set of $\kappa \in (4,8)$ so that the graph of complementary components of the range of an SLE_κ curve in $\mathbf{H}$ is connected.

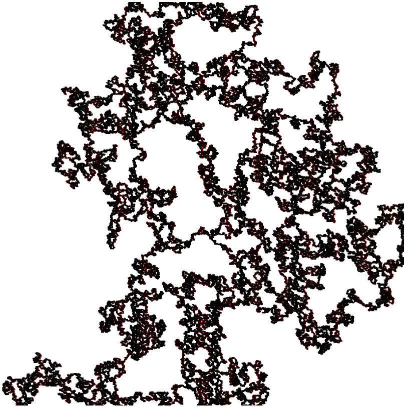
Theorem (Kavvadias-M.-Schoug)

SLE_κ *for* $\kappa \in (4,8)$ *is conformally removable for all* $\kappa \in \mathcal{K}$.

- Proved by **Gwynne-Pfeffer** that $\mathcal{K} \neq \emptyset$. Not known if $\mathcal{K} = (4,8)$.
- Based on a new geometric condition for conformal removability.

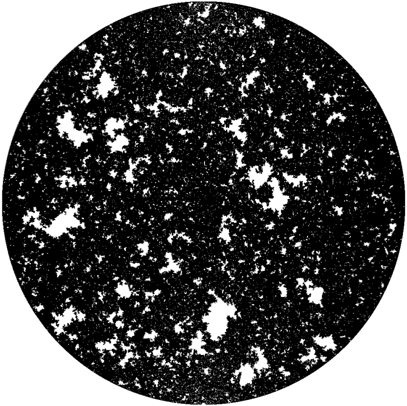
What's different?

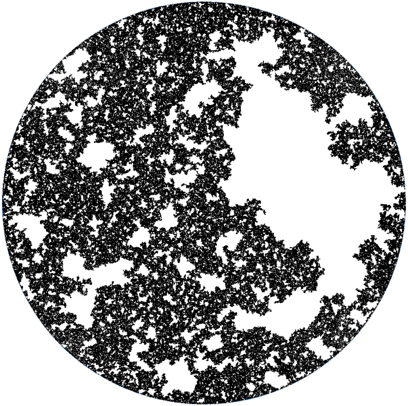




- The graph of components of a Sierpinski gasket is connected but it is not conformally removable (**Ntalampekos**).
- One of the main differences with SLE_κ with $\kappa \in \mathcal{K}$ is that two components in the complement of the latter intersect in uncountably many places.

What about related processes?





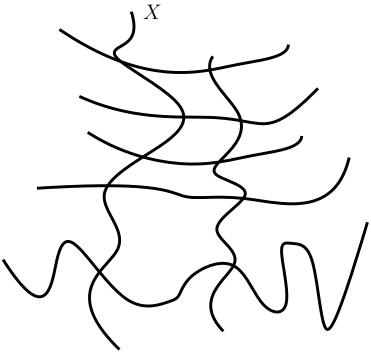
- The conformal loop ensemble (CLE_κ) is the loop version of SLE_κ. Describes the joint scaling limit of all of the interfaces. Defined for $\kappa \in (8/3, 8)$.
- Shown by **Ntalampekos** that topological Sierpinski carpets are not conformally removable. Therefore CLE_κ is not conformally removable as it is a topological Sierpinski carpet.

How does one prove conformal removability?

- Suppose that $X \subseteq \mathbf{C}$ and $f\colon \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus X$.
- To show that f is conformal everywhere, suffices to show that it satisfies the ACL property (absolutely continuous on lines):
  - for Lebesgue a.e. horizontal or vertical line L, $f|_L$ is absolutely continuous.

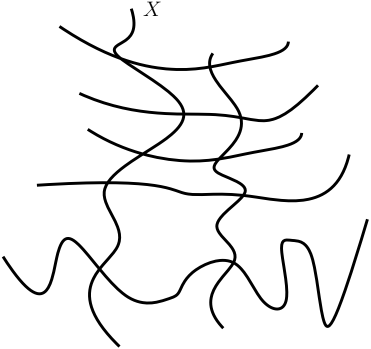
Annulus condition

- $X \subseteq \mathbf{C}$ with $d = \overline{\dim}_{\mathrm{M}}(X) < 2$, $0 < a \ll 2 - d$



Annulus condition

- $X \subseteq \mathbf{C}$ with $d = \overline{\dim}_{\mathrm{M}}(X) < 2$, $0 < a \ll 2 - d$
- Suppose that $f : \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus X$

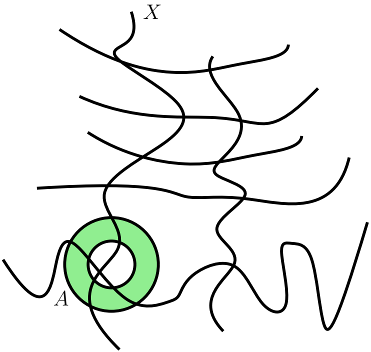


Annulus condition

- $X \subseteq \mathbf{C}$ with $d = \overline{\dim}_{\mathrm{M}}(X) < 2$, $0 < a \ll 2 - d$
- Suppose that $f\colon \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus X$
- **Claim**: If for every $k \in \mathbf{N}$ we can cover X by annuli A of size $\asymp 2^{-k}$ such that there is a path in A which disconnects the inside/outside with

  $$\mathrm{diam}(f(\gamma)) \lesssim 2^{(3a-1)k} + 2^{(1-a)k} \int_{A \setminus X} |f'(u)|^2 du$$

  then f is ACL.

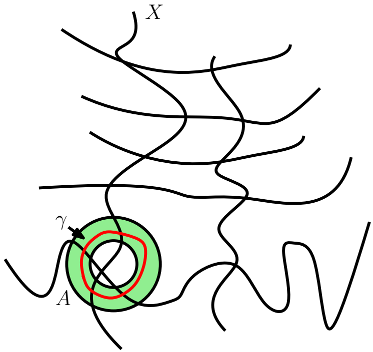


Annulus condition

- $X \subseteq \mathbf{C}$ with $d = \overline{\dim}_{\mathrm{M}}(X) < 2$, $0 < a \ll 2 - d$
- Suppose that $f : \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus X$
- **Claim**: If for every $k \in \mathbf{N}$ we can cover X by annuli A of size $\asymp 2^{-k}$ such that there is a path in A which disconnects the inside/outside with

$$\mathrm{diam}(f(\gamma)) \lesssim 2^{(3a-1)k} + 2^{(1-a)k} \int_{A \setminus X} |f'(u)|^2 du$$

  then f is ACL.

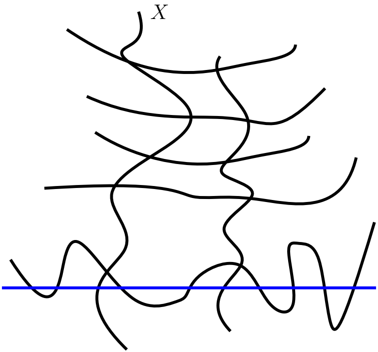


Annulus condition

- $X \subseteq \mathbf{C}$ with $d = \overline{\dim}_{\mathrm{M}}(X) < 2$, $0 < a \ll 2 - d$
- Suppose that $f : \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus X$
- **Claim**: If for every $k \in \mathbf{N}$ we can cover X by annuli A of size $\asymp 2^{-k}$ such that there is a path in A which disconnects the inside/outside with

  $$\mathrm{diam}(f(\gamma)) \lesssim 2^{(3a-1)k} + 2^{(1-a)k} \int_{A \setminus X} |f'(u)|^2 du$$

  then f is ACL.
- Why? For a Lebesgue typical line L:

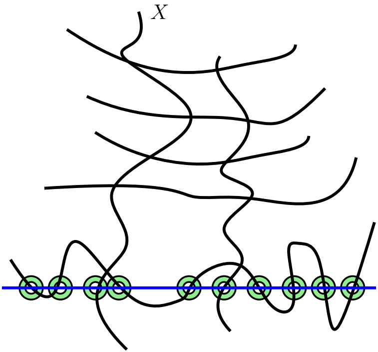


Annulus condition

- $X \subseteq \mathbf{C}$ with $d = \overline{\dim}_{\mathrm{M}}(X) < 2$, $0 < a \ll 2 - d$
- Suppose that $f : \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus X$
- **Claim**: If for every $k \in \mathbf{N}$ we can cover X by annuli A of size $\asymp 2^{-k}$ such that there is a path in A which disconnects the inside/outside with

  $$\mathrm{diam}(f(\gamma)) \lesssim 2^{(3a-1)k} + 2^{(1-a)k} \int_{A \setminus X} |f'(u)|^2 du$$

  then f is ACL.
- Why? For a Lebesgue typical line L:
  - the number of annuli of size $\asymp 2^{-k}$ needed to cover $L \cap X$ is $O(2^{(d-1)k})$.

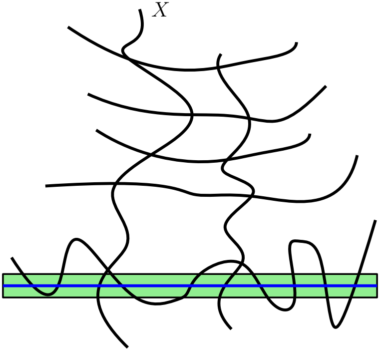


Annulus condition

- $X \subseteq \mathbf{C}$ with $d = \overline{\dim}_{\mathrm{M}}(X) < 2$, $0 < a \ll 2 - d$
- Suppose that $f : \mathbf{C} \to \mathbf{C}$ is a homeomorphism which is conformal on $\mathbf{C} \setminus X$
- **Claim**: If for every $k \in \mathbf{N}$ we can cover X by annuli A of size $\asymp 2^{-k}$ such that there is a path in A which disconnects the inside/outside with

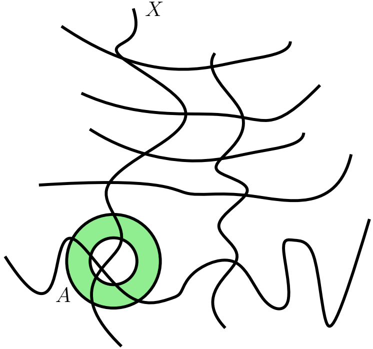
  $$\mathrm{diam}(f(\gamma)) \lesssim 2^{(3a-1)k} + 2^{(1-a)k} \int_{A \setminus X} |f'(u)|^2 du$$

  then f is ACL.
- Why? For a Lebesgue typical line L:
  - the number of annuli of size $\asymp 2^{-k}$ needed to cover $L \cap X$ is $O(2^{(d-1)k})$.
  - the Lebesgue measure of the image under f of the 2^{-k}-neighborhood of L is $O(2^{-k})$



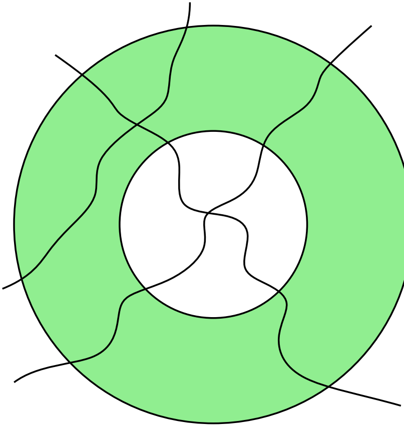
Conformal removability condition

Suppose for each $k \in \mathbf{N}$ we can cover X by annuli of size $\asymp 2^{-k}$ so that the following hold for some $M \in \mathbf{N}$.



Conformal removability condition

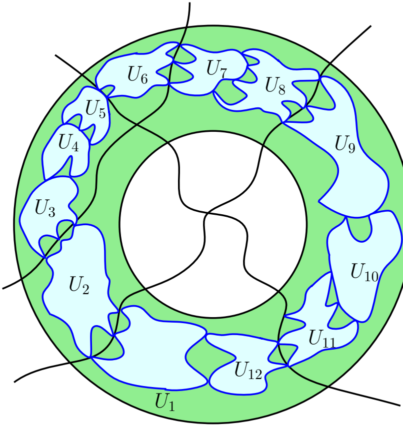
Suppose for each $k \in \mathbf{N}$ we can cover X by annuli of size $\asymp 2^{-k}$ so that the following hold for some $M \in \mathbf{N}$.



Conformal removability condition

Suppose for each $k \in \mathbf{N}$ we can cover X by annuli of size $\asymp 2^{-k}$ so that the following hold for some $M \in \mathbf{N}$.

(I) There are at most M pairwise disjoint simply connected subsets $U_1, \ldots, U_m$ of $A \setminus X$ which disconnect the inside/outside of A.



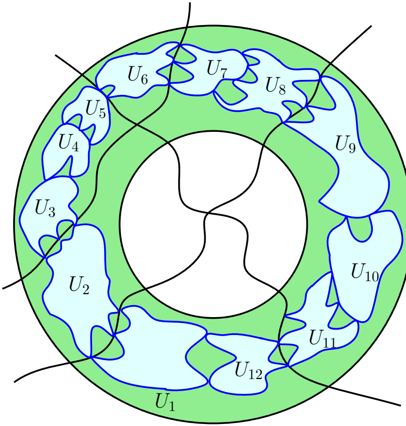
Conformal removability condition

Suppose for each $k \in \mathbf{N}$ we can cover X by annuli of size $\asymp 2^{-k}$ so that the following hold for some $M \in \mathbf{N}$.

(I) There are at most M pairwise disjoint simply connected subsets $U_1, \ldots, U_m$ of $A \setminus X$ which disconnect the inside/outside of A.

(II) For each i there exists a positive and finite Borel measure μ_i on $I_i = \partial U_i \cap \partial U_{i+1}$ and $a \ll d_i$, $a \ll 2 - d_i$ such that

(a) $\mu_i(I_i) \geq M^{-1} 2^{-d_i k}$ and for every $Y \subseteq I_i$ Borel $\mu_i(Y) \leq M \mathrm{diam}(Y)^{d_i - a}$.



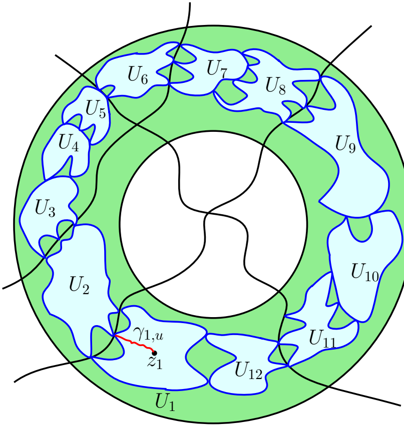
Conformal removability condition

Suppose for each $k \in \mathbf{N}$ we can cover X by annuli of size $\asymp 2^{-k}$ so that the following hold for some $M \in \mathbf{N}$.

(I) There are at most M pairwise disjoint simply connected subsets $U_1, \ldots, U_m$ of $A \setminus X$ which disconnect the inside/outside of A.

(II) For each i there exists a positive and finite Borel measure μ_i on $I_i = \partial U_i \cap \partial U_{i+1}$ and $a \ll d_i$, $a \ll 2 - d_i$ such that

- (a) $\mu_i(I_i) \geq M^{-1} 2^{-d_i k}$ and for every $Y \subseteq I_i$ Borel $\mu_i(Y) \leq M \mathrm{diam}(Y)^{d_i - a}$.
- (b) There exists $z \in U_i$ such that for at least "μ_i-high fraction" $u \in I_i$, $\mathrm{dist}(\gamma_{i,u}(t), \partial U_i) \lesssim 2^{-k} t^{-1/a}$ where $\gamma_{i,u}$ is the hyperbolic geodesic in U_i from z_i to u.



Conformal removability condition

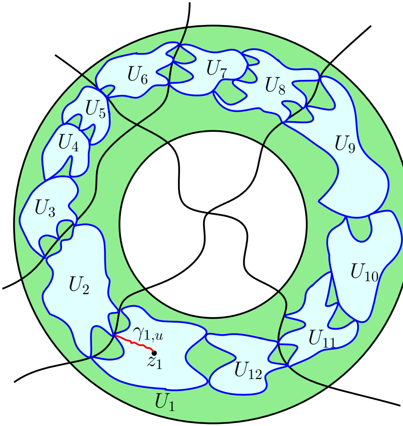
Suppose for each $k \in \mathbf{N}$ we can cover X by annuli of size $\asymp 2^{-k}$ so that the following hold for some $M \in \mathbf{N}$.

(I) There are at most M pairwise disjoint simply connected subsets $U_1, \ldots, U_m$ of $A \setminus X$ which disconnect the inside/outside of A.

(II) For each i there exists a positive and finite Borel measure μ_i on $I_i = \partial U_i \cap \partial U_{i+1}$ and $a \ll d_i$, $a \ll 2 - d_i$ such that

  (a) $\mu_i(I_i) \geq M^{-1} 2^{-d_i k}$ and for every $Y \subseteq I_i$ Borel $\mu_i(Y) \leq M \mathrm{diam}(Y)^{d_i - a}$.

  (b) There exists $z \in U_i$ such that for at least "μ_i-high fraction" $u \in I_i$, $\mathrm{dist}(\gamma_{i,u}(t), \partial U_i) \lesssim 2^{-k} t^{-1/a}$ where $\gamma_{i,u}$ is the hyperbolic geodesic in U_i from z_i to u.



$\to \exists \gamma$ disconnecting the inside/outside of A so that

$$\mathrm{diam}(f(\gamma)) \lesssim 2^{(3a-1)k} + 2^{(1-a)k} \int_{A \setminus X} |f'(u)|^2 du \qquad \text{holds.}$$

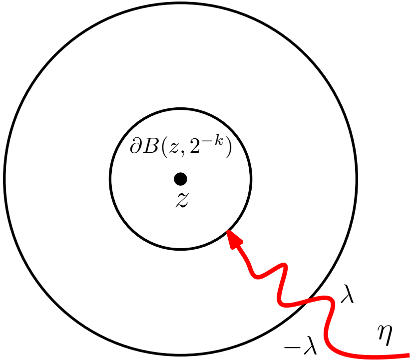
Strategy to apply theorem to SLE

Need to specify:

- the annuli A
- the components of $A \setminus \eta$ (complement of an enlargement of the SLE η)
- the measures on the component boundary intersections ("natural measures")

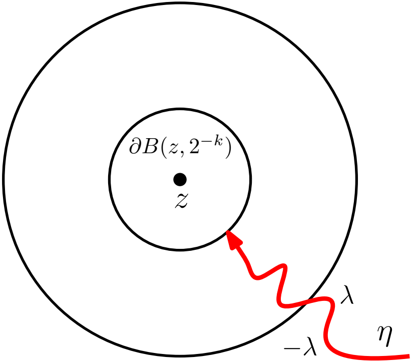
Use the GFF to show that "good" annuli where the conditions hold cover all of space and at all scales.

SLE_4



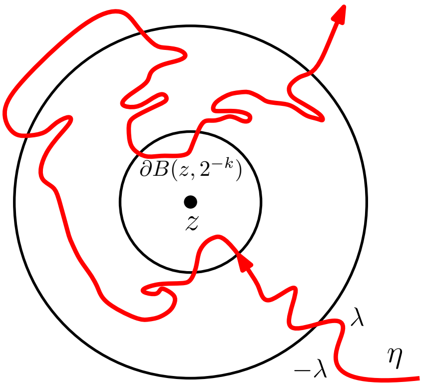
- View an SLE_4 in $\mathbf{H}$ from 0 to ∞ a the zero level line of a GFF h

SLE_4



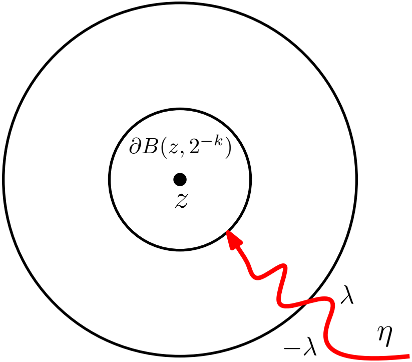
- View an SLE_4 in **H** from 0 to ∞ a the zero level line of a GFF h
- Fix $z \in \mathbf{H}$. Run η up until hitting $\partial B(z, 2^{-k})$

SLE_4



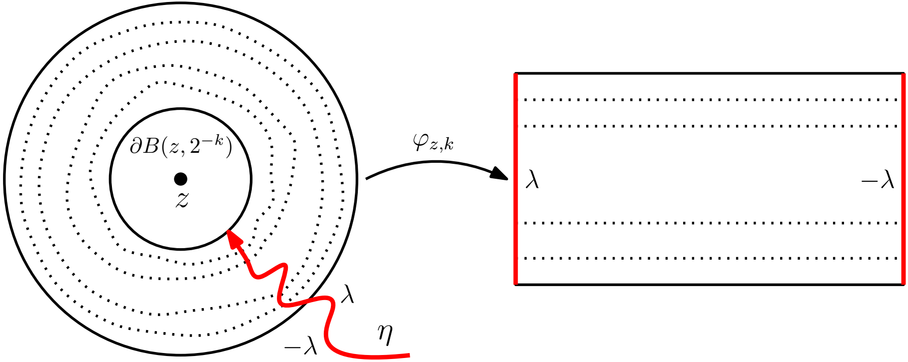
- View an SLE_4 in $\mathbf{H}$ from 0 to ∞ a the zero level line of a GFF h
- Fix $z \in \mathbf{H}$. Run η up until hitting $\partial B(z, 2^{-k})$
- η may make several crossings across $A_{z,k} = B(z, 2^{-k+1}) \setminus B(z, 2^{-k})$ which are correlated with crossings across other annuli centered at z

SLE_4



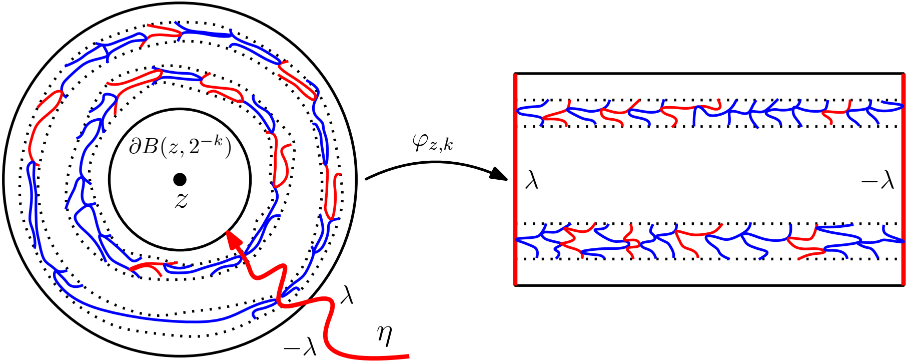
- View an SLE_4 in $\mathbf{H}$ from 0 to ∞ a the zero level line of a GFF h
- Fix $z \in \mathbf{H}$. Run η up until hitting $\partial B(z, 2^{-k})$
- η may make several crossings across $A_{z,k} = B(z, 2^{-k+1}) \setminus B(z, 2^{-k})$ which are correlated with crossings across other annuli centered at z
- Explore level lines in $A_{z,k}$ which are local to $h|_{A_{z,k}}$ and contain all such crossings of η

SLE_4



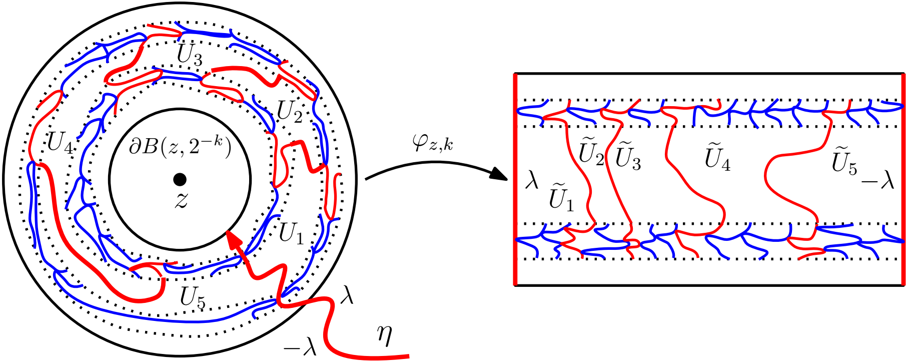
- View an SLE_4 in $\mathbf{H}$ from 0 to ∞ a the zero level line of a GFF h
- Fix $z \in \mathbf{H}$. Run η up until hitting $\partial B(z, 2^{-k})$
- η may make several crossings across $A_{z,k} = B(z, 2^{-k+1}) \setminus B(z, 2^{-k})$ which are correlated with crossings across other annuli centered at z
- Explore level lines in $A_{z,k}$ which are local to $h|_{A_{z,k}}$ and contain all such crossings of η
- Map to rectangle since need law on crossings not to depend on first crossing

SLE_4



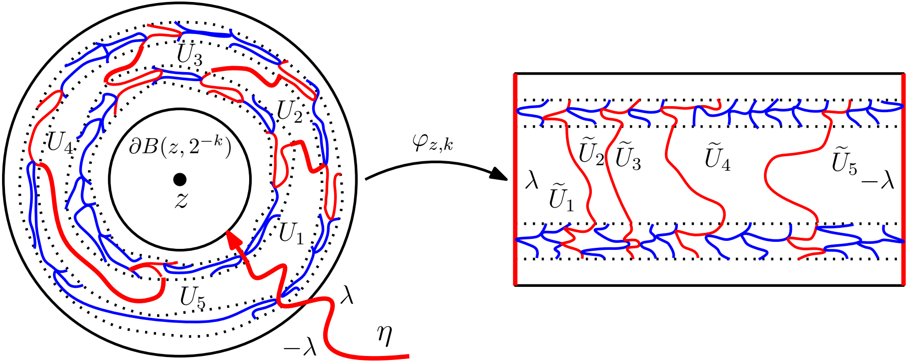
- View an SLE_4 in $\mathbf{H}$ from 0 to ∞ a the zero level line of a GFF h
- Fix $z \in \mathbf{H}$. Run η up until hitting $\partial B(z, 2^{-k})$
- η may make several crossings across $A_{z,k} = B(z, 2^{-k+1}) \setminus B(z, 2^{-k})$ which are correlated with crossings across other annuli centered at z
- Explore level lines in $A_{z,k}$ which are local to $h|_{A_{z,k}}$ and contain all such crossings of η
- Map to rectangle since need law on crossings not to depend on first crossing

SLE_4



- View an SLE_4 in $\mathbf{H}$ from 0 to ∞ a the zero level line of a GFF h
- Fix $z \in \mathbf{H}$. Run η up until hitting $\partial B(z, 2^{-k})$
- η may make several crossings across $A_{z,k} = B(z, 2^{-k+1}) \setminus B(z, 2^{-k})$ which are correlated with crossings across other annuli centered at z
- Explore level lines in $A_{z,k}$ which are local to $h|_{A_{z,k}}$ and contain all such crossings of η
- Map to rectangle since need law on crossings not to depend on first crossing

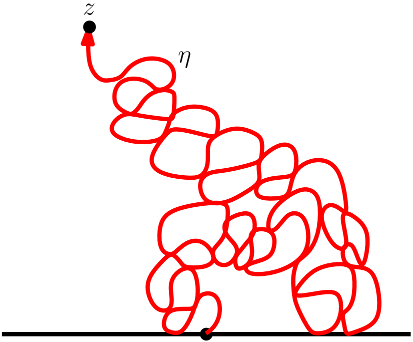
SLE_4



- View an SLE_4 in $\mathbf{H}$ from 0 to ∞ a the zero level line of a GFF h
- Fix $z \in \mathbf{H}$. Run η up until hitting $\partial B(z, 2^{-k})$
- η may make several crossings across $A_{z,k} = B(z, 2^{-k+1}) \setminus B(z, 2^{-k})$ which are correlated with crossings across other annuli centered at z
- Explore level lines in $A_{z,k}$ which are local to $h|_{A_{z,k}}$ and contain all such crossings of η
- Map to rectangle since need law on crossings not to depend on first crossing
- Measure on $\partial U_i \cap \partial U_{i+1}$ given by the natural parameterization

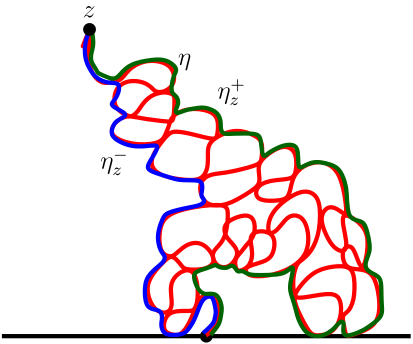
SLE_κ for $\kappa \in (4,8)$

- View SLE_κ in **H** from 0 to ∞ as the counterflow line η of a GFF h



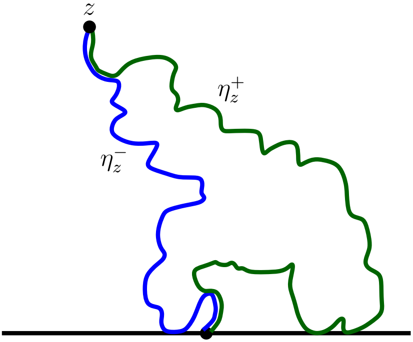
SLE_κ for $\kappa \in (4, 8)$

- View SLE_κ in **H** from 0 to ∞ as the counterflow line η of a GFF h
- SLE duality: left/right boundaries of η stopped upon hitting z are the flow lines $\eta_z^\pm$ of h with angles $\pm\pi/2$



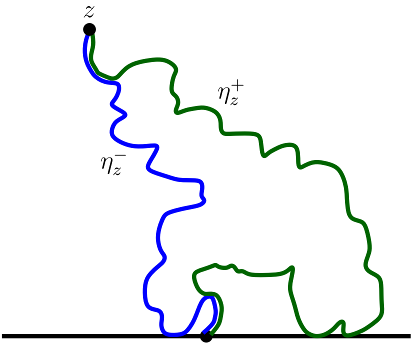
SLE_κ for $\kappa \in (4,8)$

- View SLE_κ in **H** from 0 to ∞ as the counterflow line η of a GFF h
- SLE duality: left/right boundaries of η stopped upon hitting z are the flow lines $\eta_z^\pm$ of h with angles $\pm\pi/2$
- Conditional law of η given $\eta_z^\pm$ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$



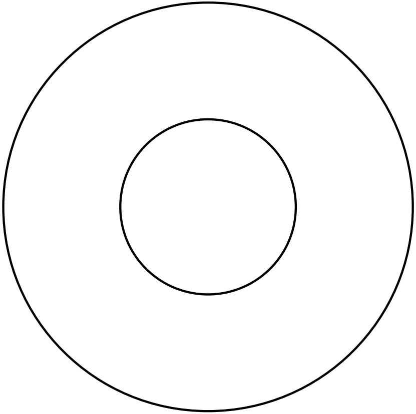
SLE_κ for $\kappa \in (4,8)$

- View SLE_κ in **H** from 0 to ∞ as the counterflow line η of a GFF h
- SLE duality: left/right boundaries of η stopped upon hitting z are the flow lines $\eta_z^\pm$ of h with angles $\pm\pi/2$
- Conditional law of η given $\eta_z^\pm$ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4;\frac{\kappa}{2}-4)$
- Can localize behavior of η in an annulus $A_{z,k} = B(z,2^{-k}) \setminus B(z,2^{-k-1})$ using GFF flow lines in $A_{z,k}$



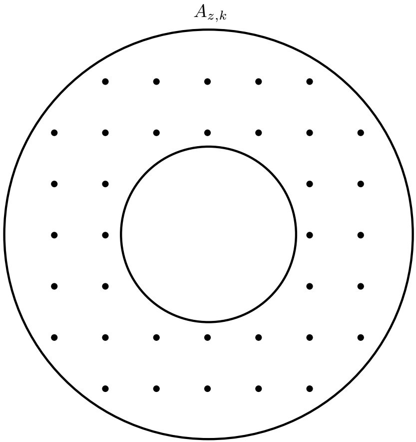
SLE_κ for $\kappa \in (4,8)$

- View SLE_κ in $\mathbf{H}$ from 0 to ∞ as the counterflow line η of a GFF h
- SLE duality: left/right boundaries of η stopped upon hitting z are the flow lines $\eta_z^\pm$ of h with angles $\pm\pi/2$
- Conditional law of η given $\eta_z^\pm$ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$
- Can localize behavior of η in an annulus $A_{z,k} = B(z, 2^{-k}) \setminus B(z, 2^{-k-1})$ using GFF flow lines in $A_{z,k}$



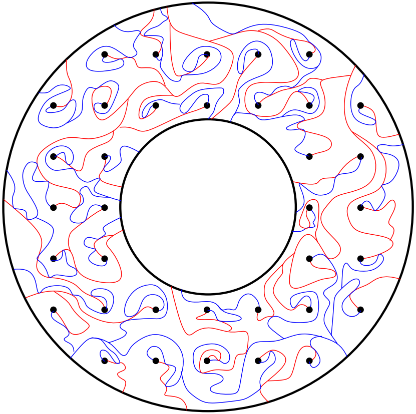
SLE_κ for $\kappa \in (4,8)$

- View SLE_κ in $\mathbf{H}$ from 0 to ∞ as the counterflow line η of a GFF h
- SLE duality: left/right boundaries of η stopped upon hitting z are the flow lines $\eta_z^\pm$ of h with angles $\pm\pi/2$
- Conditional law of η given $\eta_z^\pm$ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$
- Can localize behavior of η in an annulus $A_{z,k} = B(z,2^{-k}) \setminus B(z,2^{-k-1})$ using GFF flow lines in $A_{z,k}$
- Start flow lines $\eta_w^\pm$ in a fine grid of points so likely to disconnect inside/outside



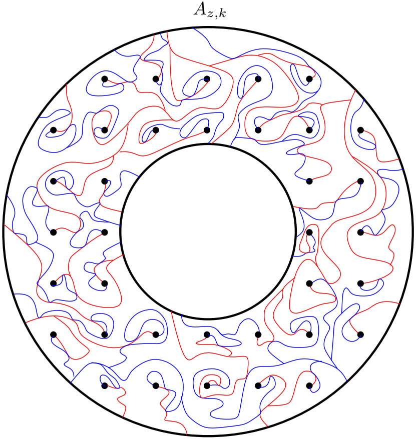
SLE_κ for $\kappa \in (4, 8)$

- View SLE_κ in $\mathbf{H}$ from 0 to ∞ as the counterflow line η of a GFF h
- SLE duality: left/right boundaries of η stopped upon hitting z are the flow lines $\eta_z^\pm$ of h with angles $\pm\pi/2$
- Conditional law of η given $\eta_z^\pm$ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2} - 4; \frac{\kappa}{2} - 4)$
- Can localize behavior of η in an annulus $A_{z,k} = B(z, 2^{-k}) \setminus B(z, 2^{-k-1})$ using GFF flow lines in $A_{z,k}$
- Start flow lines $\eta_w^\pm$ in a fine grid of points so likely to disconnect inside/outside



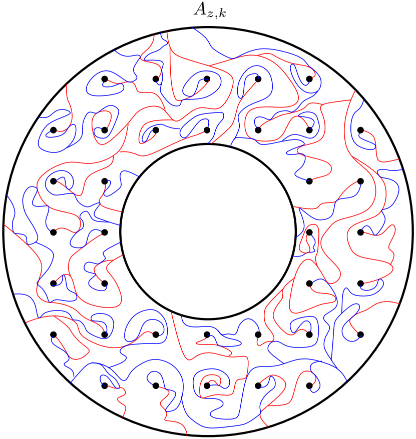
SLE_κ for $\kappa \in (4, 8)$

- View SLE_κ in $\mathbf{H}$ from 0 to ∞ as the counterflow line η of a GFF h
- SLE duality: left/right boundaries of η stopped upon hitting z are the flow lines $\eta_z^\pm$ of h with angles $\pm\pi/2$
- Conditional law of η given $\eta_z^\pm$ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2} - 4; \frac{\kappa}{2} - 4)$
- Can localize behavior of η in an annulus $A_{z,k} = B(z, 2^{-k}) \setminus B(z, 2^{-k-1})$ using GFF flow lines in $A_{z,k}$
- Start flow lines $\eta_w^\pm$ in a fine grid of points so likely to disconnect inside/outside
- Add in counterflow lines to components disconnected from $\partial A_{z,k}$



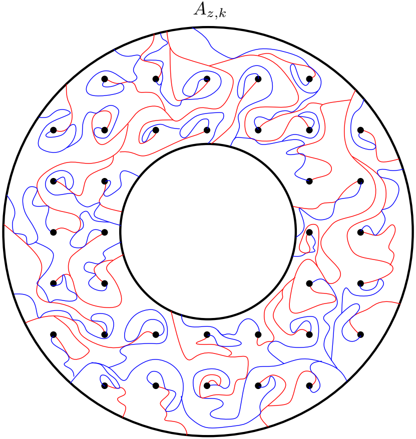
SLE_κ for $\kappa \in (4,8)$

- View SLE_κ in $\mathbf{H}$ from 0 to ∞ as the counterflow line η of a GFF h
- SLE duality: left/right boundaries of η stopped upon hitting z are the flow lines $\eta_z^\pm$ of h with angles $\pm\pi/2$
- Conditional law of η given $\eta_z^\pm$ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$
- Can localize behavior of η in an annulus $A_{z,k} = B(z,2^{-k}) \setminus B(z,2^{-k-1})$ using GFF flow lines in $A_{z,k}$
- Start flow lines $\eta_w^\pm$ in a fine grid of points so likely to disconnect inside/outside
- Add in counterflow lines to components disconnected from $\partial A_{z,k}$
- The U_i's are given by a finite chain of components which disconnects inside/outside of $A_{z,k}$



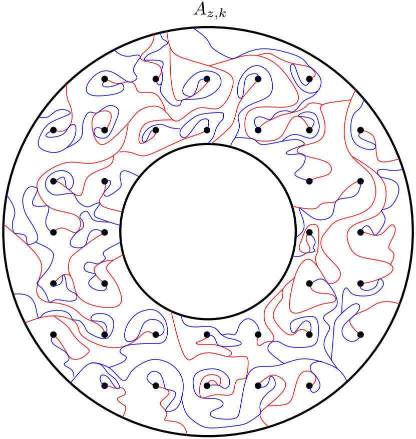
SLE_κ for $\kappa \in (4,8)$

- View SLE_κ in $\mathbf{H}$ from 0 to ∞ as the counterflow line η of a GFF h
- SLE duality: left/right boundaries of η stopped upon hitting z are the flow lines $\eta_z^\pm$ of h with angles $\pm\pi/2$
- Conditional law of η given $\eta_z^\pm$ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$
- Can localize behavior of η in an annulus $A_{z,k} = B(z,2^{-k}) \setminus B(z,2^{-k-1})$ using GFF flow lines in $A_{z,k}$
- Start flow lines $\eta_w^\pm$ in a fine grid of points so likely to disconnect inside/outside
- Add in counterflow lines to components disconnected from $\partial A_{z,k}$
- The U_i's are given by a finite chain of components which disconnects inside/outside of $A_{z,k}$
- Measure on $\partial U_i \cap \partial U_{i+1}$ given by the "natural measure" on cut points



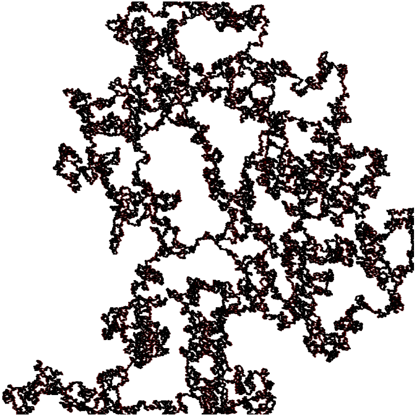
SLE_κ for $\kappa \in (4,8)$

- View SLE_κ in $\mathbf{H}$ from 0 to ∞ as the counterflow line η of a GFF h
- SLE duality: left/right boundaries of η stopped upon hitting z are the flow lines $\eta_z^\pm$ of h with angles $\pm\pi/2$
- Conditional law of η given $\eta_z^\pm$ is an $\mathrm{SLE}_\kappa(\frac{\kappa}{2}-4; \frac{\kappa}{2}-4)$
- Can localize behavior of η in an annulus $A_{z,k} = B(z, 2^{-k}) \setminus B(z, 2^{-k-1})$ using GFF flow lines in $A_{z,k}$
- Start flow lines $\eta_w^\pm$ in a fine grid of points so likely to disconnect inside/outside
- Add in counterflow lines to components disconnected from $\partial A_{z,k}$
- The U_i's are given by a finite chain of components which disconnects inside/outside of $A_{z,k}$
- Measure on $\partial U_i \cap \partial U_{i+1}$ given by the "natural measure" on cut points
- Need existence and strong estimates



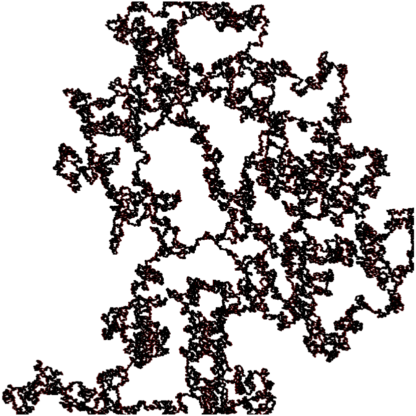
Natural measure on SLE cut points

- "Natural parameterization" of SLE_κ is the time-parameterization which conjecturally describes the scaling limit of an interface from a discrete model parameterized by the number of edges crossed.



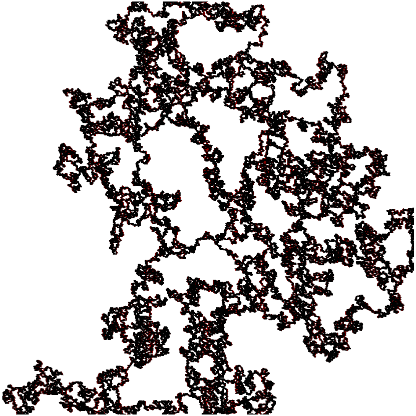
Natural measure on SLE cut points

- "Natural parameterization" of SLE_κ is the time-parameterization which conjecturally describes the scaling limit of an interface from a discrete model parameterized by the number of edges crossed.
- Constructed by **Lawler-Sheffield** indirectly using the Doob-Meyer decomposition



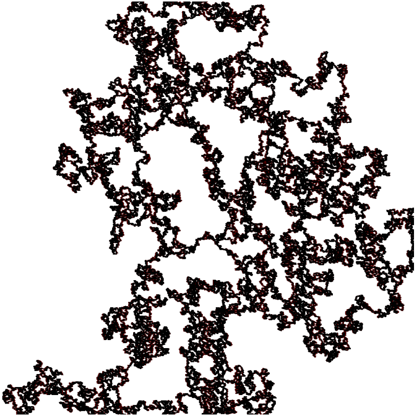
Natural measure on SLE cut points

- "Natural parameterization" of SLE_κ is the time-parameterization which conjecturally describes the scaling limit of an interface from a discrete model parameterized by the number of edges crossed.
- Constructed by **Lawler-Sheffield** indirectly using the Doob-Meyer decomposition
- Later shown by **Lawler-Rezaei** to be the Minkowski content measure: μ^{SLE}



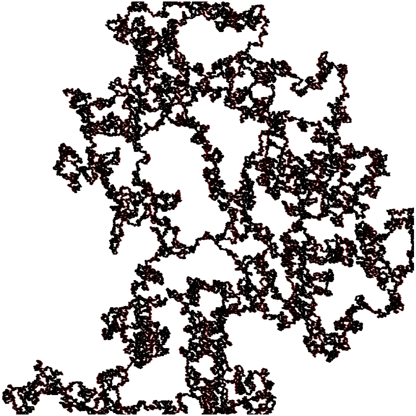
Natural measure on SLE cut points

- "Natural parameterization" of SLE_κ is the time-parameterization which conjecturally describes the scaling limit of an interface from a discrete model parameterized by the number of edges crossed.
- Constructed by **Lawler-Sheffield** indirectly using the Doob-Meyer decomposition
- Later shown by **Lawler-Rezaei** to be the Minkowski content measure: μ^{SLE}
- Strong estimates proved by **Rezaei-Zhan**: $\mu^{\mathrm{SLE}}(X) \lesssim \mathrm{diam}(X)^{d_{\mathrm{SLE}}+o(1)}$ as $\mathrm{diam}(X) \to 0$



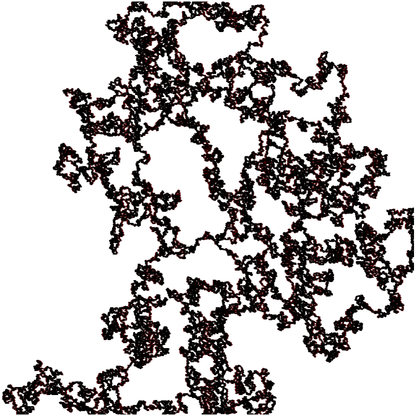
Natural measure on SLE cut points

- "Natural parameterization" of SLE_κ is the time-parameterization which conjecturally describes the scaling limit of an interface from a discrete model parameterized by the number of edges crossed.
- Constructed by **Lawler-Sheffield** indirectly using the Doob-Meyer decomposition
- Later shown by **Lawler-Rezaei** to be the Minkowski content measure: μ^{SLE}
- Strong estimates proved by **Rezaei-Zhan**: $\mu^{\mathrm{SLE}}(X) \lesssim \mathrm{diam}(X)^{d_{\mathrm{SLE}}+o(1)}$ as $\mathrm{diam}(X) \to 0$
- Constructed by **Benoist** using the GFF



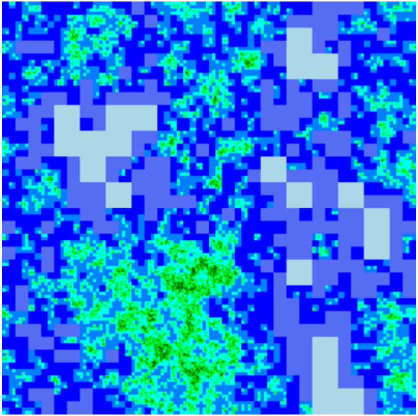
Natural measure on SLE cut points

- "Natural parameterization" of SLE_κ is the time-parameterization which conjecturally describes the scaling limit of an interface from a discrete model parameterized by the number of edges crossed.
- Constructed by **Lawler-Sheffield** indirectly using the Doob-Meyer decomposition
- Later shown by **Lawler-Rezaei** to be the Minkowski content measure: μ^{SLE}
- Strong estimates proved by **Rezaei-Zhan**: $\mu^{\mathrm{SLE}}(X) \lesssim \mathrm{diam}(X)^{d_{\mathrm{SLE}}+o(1)}$ as $\mathrm{diam}(X) \to 0$
- Constructed by **Benoist** using the GFF
- Put an SLE_κ for $\kappa \in (4,8)$ on an LQG surface



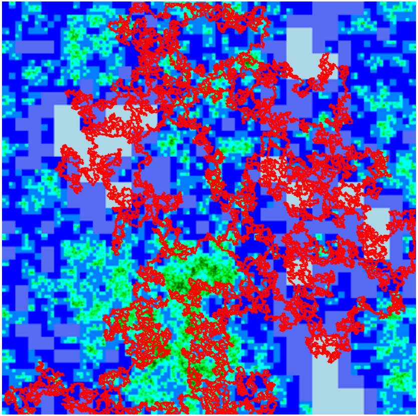
Natural measure on SLE cut points

- "Natural parameterization" of SLE_κ is the time-parameterization which conjecturally describes the scaling limit of an interface from a discrete model parameterized by the number of edges crossed.
- Constructed by **Lawler-Sheffield** indirectly using the Doob-Meyer decomposition
- Later shown by **Lawler-Rezaei** to be the Minkowski content measure: μ^{SLE}
- Strong estimates proved by **Rezaei-Zhan**: $\mu^{\mathrm{SLE}}(X) \lesssim \mathrm{diam}(X)^{d_{\mathrm{SLE}}+o(1)}$ as $\mathrm{diam}(X) \to 0$
- Constructed by **Benoist** using the GFF
- Put an SLE_κ for $\kappa \in (4,8)$ on an LQG surface



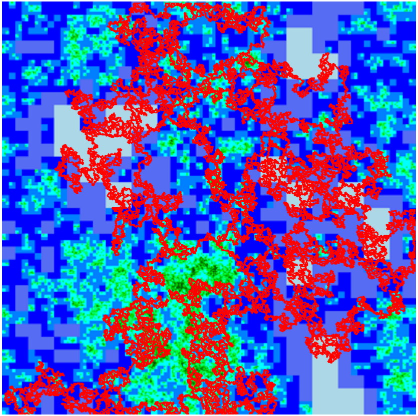
Natural measure on SLE cut points

- "Natural parameterization" of SLE_κ is the time-parameterization which conjecturally describes the scaling limit of an interface from a discrete model parameterized by the number of edges crossed.
- Constructed by **Lawler-Sheffield** indirectly using the Doob-Meyer decomposition
- Later shown by **Lawler-Rezaei** to be the Minkowski content measure: μ^{SLE}
- Strong estimates proved by **Rezaei-Zhan**: $\mu^{\mathrm{SLE}}(X) \lesssim \mathrm{diam}(X)^{d_{\mathrm{SLE}}+o(1)}$ as $\mathrm{diam}(X) \to 0$
- Constructed by **Benoist** using the GFF
- Put an SLE_κ for $\kappa \in (4, 8)$ on an LQG surface



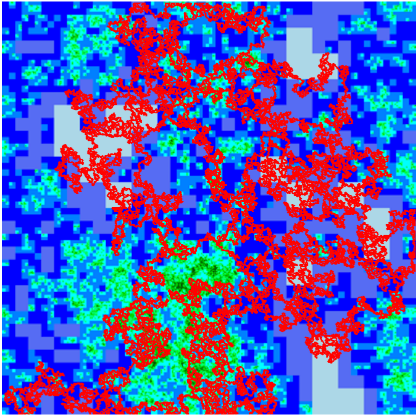
Natural measure on SLE cut points

- "Natural parameterization" of SLE_κ is the time-parameterization which conjecturally describes the scaling limit of an interface from a discrete model parameterized by the number of edges crossed.
- Constructed by **Lawler-Sheffield** indirectly using the Doob-Meyer decomposition
- Later shown by **Lawler-Rezaei** to be the Minkowski content measure: μ^{SLE}
- Strong estimates proved by **Rezaei-Zhan**: $\mu^{\mathrm{SLE}}(X) \lesssim \mathrm{diam}(X)^{d_{\mathrm{SLE}}+o(1)}$ as $\mathrm{diam}(X) \to 0$
- Constructed by **Benoist** using the GFF
- Put an SLE_κ for $\kappa \in (4,8)$ on an LQG surface
- **Duplantier-M.-Sheffield**: there is a natural LQG measure μ_h^{cut} on the SLE_κ cut points



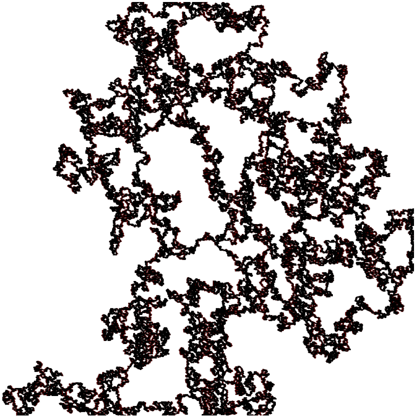
Natural measure on SLE cut points

- "Natural parameterization" of SLE_κ is the time-parameterization which conjecturally describes the scaling limit of an interface from a discrete model parameterized by the number of edges crossed.
- Constructed by **Lawler-Sheffield** indirectly using the Doob-Meyer decomposition
- Later shown by **Lawler-Rezaei** to be the Minkowski content measure: μ^{SLE}
- Strong estimates proved by **Rezaei-Zhan**: $\mu^{\mathrm{SLE}}(X) \lesssim \mathrm{diam}(X)^{d_{\mathrm{SLE}}+o(1)}$ as $\mathrm{diam}(X) \to 0$
- Constructed by **Benoist** using the GFF
- Put an SLE_κ for $\kappa \in (4,8)$ on an LQG surface
- **Duplantier-M.-Sheffield**: there is a natural LQG measure μ_h^{cut} on the SLE_κ cut points
- Natural cut point measure is $\mu^{\mathrm{cut}} = \mathbf{E}[\mu_h^{\mathrm{cut}} \mid \eta]$



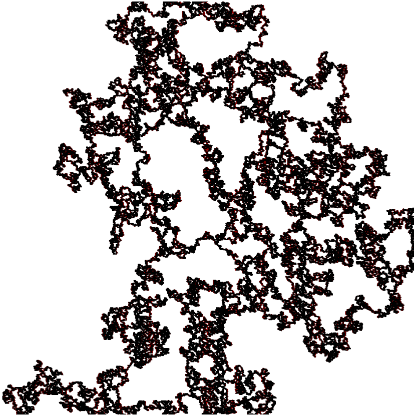
Natural measure on SLE cut points

- "Natural parameterization" of SLE_κ is the time-parameterization which conjecturally describes the scaling limit of an interface from a discrete model parameterized by the number of edges crossed.
- Constructed by **Lawler-Sheffield** indirectly using the Doob-Meyer decomposition
- Later shown by **Lawler-Rezaei** to be the Minkowski content measure: μ^{SLE}
- Strong estimates proved by **Rezaei-Zhan**: $\mu^{\mathrm{SLE}}(X) \lesssim \mathrm{diam}(X)^{d_{\mathrm{SLE}}+o(1)}$ as $\mathrm{diam}(X) \to 0$
- Constructed by **Benoist** using the GFF
- Put an SLE_κ for $\kappa \in (4,8)$ on an LQG surface
- **Duplantier-M.-Sheffield**: there is a natural LQG measure μ_h^{cut} on the SLE_κ cut points
- Natural cut point measure is $\mu^{\mathrm{cut}} = \mathbf{E}[\mu_h^{\mathrm{cut}} \mid \eta]$



Natural measure on SLE cut points

- "Natural parameterization" of SLE_κ is the time-parameterization which conjecturally describes the scaling limit of an interface from a discrete model parameterized by the number of edges crossed.
- Constructed by **Lawler-Sheffield** indirectly using the Doob-Meyer decomposition
- Later shown by **Lawler-Rezaei** to be the Minkowski content measure: μ^{SLE}
- Strong estimates proved by **Rezaei-Zhan**: $\mu^{\mathrm{SLE}}(X) \lesssim \mathrm{diam}(X)^{d_{\mathrm{SLE}}+o(1)}$ as $\mathrm{diam}(X) \to 0$
- Constructed by **Benoist** using the GFF
- Put an SLE_κ for $\kappa \in (4,8)$ on an LQG surface
- **Duplantier-M.-Sheffield**: there is a natural LQG measure μ_h^{cut} on the SLE_κ cut points
- Natural cut point measure is $\mu^{\mathrm{cut}} = \mathbf{E}[\mu_h^{\mathrm{cut}} \,|\, \eta]$
- **Kavvadias-M.-Schoug** Strong estimates: $\mu^{\mathrm{cut}}(X) \lesssim \mathrm{diam}(X)^{d_{\mathrm{cut}}+o(1)}$ as $\mathrm{diam}(X) \to 0$



Final words

- **New results**: SLE_4, SLE_κ for $\kappa \in (4,8)$ removable whenever graph of complementary components is connected
- General strategy seems to give that any set in the SLE world is removable provided the graph of complementary components is connected
- **Question**: is the graph of complementary components of an SLE_κ connected *for all* $\kappa \in (4,8)$?

Thanks!