Multiple SLE, Bessel processes and Random Matrices

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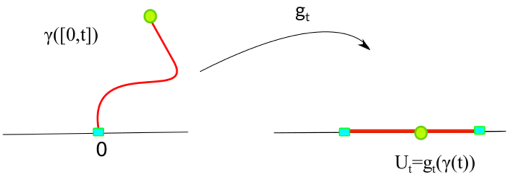
The Loewner Differential Equation and SLE

- For a non-self crossing curve $\gamma(t) : [0, \infty) \to \bar{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(\infty) = \infty$, we consider the simply connected domain $\mathbb{H} \setminus \gamma([0, t])$.



- The forward Loewner Differential Equation

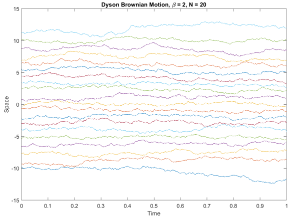
$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

- Oded Schramm: $U_t = \sqrt{\kappa} B_t$, with $\kappa \geq 0$.
- **SLE** curves: $\gamma(t) := \lim_{y \to 0+} \hat{g}_t^{-1}(iy)$. (Rohde-Schramm, Lawler-Schramm-Werner).

Multiple SLE model

- **Multiple SLE:** Alberts, Ang, Bauer, Beffara, Bernard, Binder, Byun, Cardy, Del Monaco, Dubédat, Duplantier, Hotta, Izyurov, Kang, Karrila, Katori, Koshida, Kozdron, Kytölä, Lawler, Lenells, Makarov, Miller, Olsiewski Healey, Peltola, Schleissinger, Sun, Viklund, Wang, Wu, Yu, Zhan, ...
- Random Matrices appeared already in the very nice talk of Sheffield.
- **Brownian Motion** $\to$ **Dyson Brownian Motion**(DBM), $\beta = 8/\kappa$.

$$d\lambda_t^i = \frac{\sqrt{2}}{\sqrt{N\beta}} dB_t^i + \frac{1}{N} \sum_{j \neq i} \frac{dt}{\lambda_t^j - \lambda_t^i}, i = 1, \cdots, N.$$



Multiple SLE

- **Multiple SLE** simultaneous growth

$$\partial_t g_t^N(z) = \frac{1}{N}\sum_{j=1}^{N} \frac{2}{g_t^N(z) - \lambda_t^j}.$$

- Feature of DBM: **Very fast** convergence to **local equilibrium**. *Locally*, the information from the initial conditions is *lost very fast*.

- This property was used by Erdős, Yau, and collaborators in the three-step-strategy proof of **universality** in Random Matrices. Initial progress was done by Johansson.

RMT and SLE

Theorem (Chen-M.)

For $N = 2$ drivers, perturbations of initial value/ parameter $\kappa \in (0,4]$, convergence a.s /with high probability in the Carathéodory sense.

Theorem (Campbell-Luh-M.)

Let $\beta = 1$ or $\beta = 2$, and let us consider Dyson Brownian motion beginning at the origin. Let K_T be the multiple SLE hull at time $T > 0$. Then, for any $\varepsilon > 0$, for the multiple SLE maps for N curves, we have that

$$\sup_{t \in [0,T], z \in G} \left| g_t^N(z) - g_t^\infty(z) \right| = O\left(\frac{1}{N^{1/3-\varepsilon}}\right),$$

with overwhelming probability, for a given $G \subset \mathbb{H} \backslash K_T$.

- E holds with o.p. if, for every $p > 0, \mathbb{P}(E) \geq 1 - O\left(N^{-p}\right)$.
- Uses modern RMT techniques such as Stieltjes transforms, self-consistent equations, etc.

Idea of the proof of the first result

- For $\kappa, \kappa^* \in (0, 4]$, we consider

$$\partial_t g_t(z) = \frac{1}{g_t(z) - \lambda_t^1} + \frac{1}{g_t(z) - \lambda_t^2}$$

  and

$$\partial_t g_t^*(z) = \frac{1}{g_t^*(z) - \lambda_t^{1*}} + \frac{1}{g_t^*(z) - \lambda_t^{2*}}.$$

- The (rescaled) gaps between the drivers are **Bessel processes** of dimensions $d = 1 + \frac{8}{\kappa}$, and $d^* = 1 + \frac{8}{\kappa^*}$.
- One can show that

$$\mathbb{P}\left(\|g_t(z) - g_t^*(z)\|_{[0,T]\times G} > \varphi\left(\kappa^* - \kappa\right)\right) < \zeta\left(\kappa^* - \kappa\right),$$

  for some explicit functions $\varphi(\cdot)$ and $\zeta(\cdot)$ such that $\lim_{x\to 0} \varphi(x) = 0$, and $\lim_{x\to 0} \zeta(x) = 0$.

Idea of the proof of the second result

First, Del Monaco and Schleissinger:

$$\frac{\partial}{\partial t} g_t^{\infty}(z) = M_t^{\infty}(g_t(z)), \quad g_0(z) = z,$$

where M_t^{∞} is a solution to the complex Burgers equation

$$\begin{cases} \frac{\partial M_t^{\infty}(z)}{\partial t} = -2M_t^{\infty}(z)\frac{\partial M_t^{\infty}(z)}{\partial z}, t > 0 \\ M_0^{\infty}(z) = \int_{\mathbb{R}} \frac{2}{z-x} d\mu_0(x) \end{cases}$$

- Let $M_t^N(\bullet) = \frac{1}{N}\sum_{j=1}^{N} \frac{2}{\bullet - \lambda_t^j}$.
- Let us consider the time interval $[0, 1]$ and a uniform partition with $t_k = \frac{k}{n}, k = 0, 1, \ldots, n$. Let $t \in (t_1, t_2)$.
- The proof is based on controlling $\sup_{\bullet \in G} \left| M_t^{\infty}(\bullet) - M_t^N(\bullet) \right| \leq \sup_{\bullet \in G} \left| M_t^{\infty}(\bullet) - M_{t_1}^{\infty}(\bullet) \right| + \sup_{\bullet \in G} \left| M_{t_1}^{\infty}(\bullet) - M_{t_1}^N(\bullet) \right| + \sup_{\bullet \in G} \left| M_{t_1}^N(\bullet) - M_t^N(\bullet) \right|$, for $G \subset \mathbb{H}$.

Idea of the proof of the second result

- Let $\beta = 1$. We have that $M_t^N(\bullet) = \frac{-2}{N} \operatorname{tr} \left(A_t - \bullet \cdot I\right)^{-1}$, where $A_t = \sqrt{t}A$, with A a GOE.
- For M_t^∞, we have $M_t^\infty(\bullet) - S^N\left(z - 2tM_t^\infty(\bullet)\right) = 0$, with $S^N(\bullet) = -\frac{2}{N} \operatorname{tr} Q(\bullet)$, where $Q(\bullet)$ is a certain resolvent matrix.
- By a union bound, we have that, for any $\epsilon > 0$

$$\mathbb{P}\left(\bigcup_{t_k} \left| M_{t_k}^\infty(\bullet) - M_{t_k}^N(\bullet) \right| = \Omega\left(\frac{C}{N^{1/3-\epsilon}}\right)\right)$$
$$\leq \sum_{k=1}^{n} \mathbb{P}\left(\left| M_{t_k}^\infty(\bullet) - M_{t_k}^N(\bullet) \right| = \Omega\left(\frac{C}{N^{1/3-\epsilon}}\right)\right)$$
$$\leq ne^{-CN}.$$

- Note $g = \Omega(f) \leftrightarrow f = O(g)$.

Future directions

Medium term goals: Analysis and Geometry of the Multiple SLE curves; Study of the fixed N, asymptotic case, etc.
Approximations schemes (**working on this over the summer REU/G**).

- Interplay between **RMT/DBM** (gaps, etc.) and **Loewner theory**.

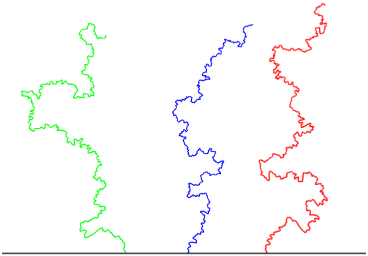


Figure: Multiple SLE for N=3. Credits to K. Luh

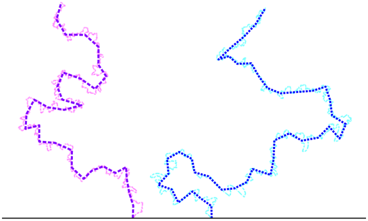


Figure: Approximation Scheme. Credits to K. Luh

Thank you very much KIAS and the organizers for the great support and thank you all for your attention!