Sharp asymptotics for arm events in critical planar percolation

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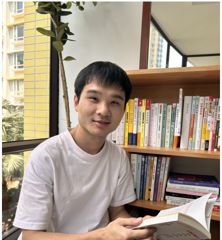
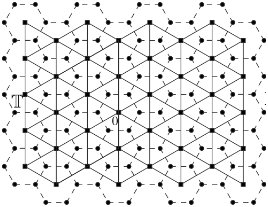
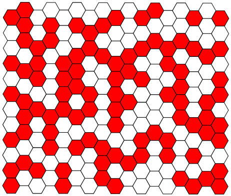
joint work with Hang Du (PKU→MIT), Yifan Gao (CityUHK) and Zijie Zhuang (UPenn)



Critical planar percolation on the triangular lattice

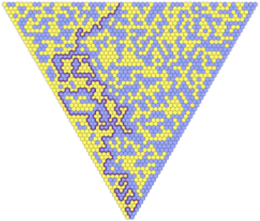
- Great breakthrough in the last 20 years on **planar critical** percolation (thanks to the **conformal invariance**), under the setup of site percolation on the triangular lattice $\mathbb{T}$, in particular in the understanding of **scaling limit** and **critical exponents**.
- For illustration, paint the hexagons in the dual lattice $\mathbb{T}^*$ instead.
- $p_c = 1/2$ in this case.

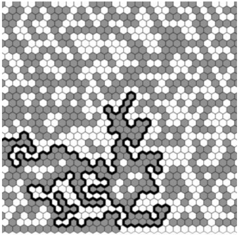




A non-exhaustive list of progresses in understanding the limit of critical planar percolation

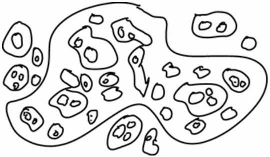
- [Schramm '00] Introduction of **Schramm-Loewner Evolution** as the conjectural scaling limit of percolation interface (and many more critical 2D models);
- [Smirnov '01] Conformal invariance of the crossing probability (aka **Cardy's formula**) and scaling limit of the interface as SLE_6;
- [Smirnov-Werner '01][Lawler-Schramm-Werner '01] Identification of the value of various **arm exponents**;

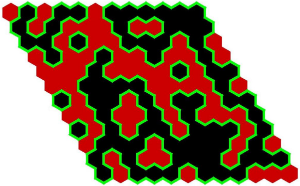




A non-exhaustive list of progresses in understanding the limit of critical planar percolation

- [Camia-Newman ’06] The full scaling limit of interfaces and characterization as **Conformal Loop Ensemble**;





- [Garban-Pete-Schramm ’10, ’13, ’18] Construction of scaling limits of **near-critical** and **dynamical** percolation via scaling limit of pivotal measures in the critical percolation;
- ······
- [Holden-L.-Sun ’22] Scaling limit in **natural parametrization** of percolation exploration process and pivotal points.

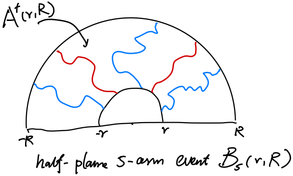
Arm events

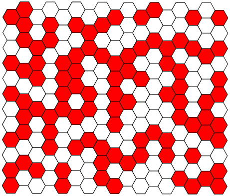
Definition

- *An* **arm** *is a self-avoiding path of nearest-neighbor hexagons of the same color;*
- $A^+(r,R)$: *half-annulus of inner- and outer-radius* r *and* R;
- *The* **half-plane** j*-arm event:*

$$\mathcal{B}_j(r,R) := \{\exists j \text{ disjoint arms of alternating colors crossing } A^+(r,R)\};$$

(One can also consider other color sequences. A color switching trick shows that it produces an event of the same probability.)





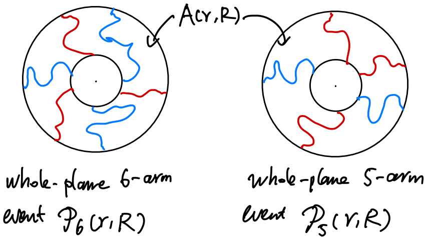
More arm events

Definition

- *$A(r, R)$: annulus of inner- and outer-radius r and R;*
- *The* **whole-plane** *(polychromatic) j-arm event:*

$$\mathcal{P}_j(r, R) := \Big\{ \exists j \text{ disjoint arms NOT all of the same color (except } j = 1) \text{ crossing } A(r, R) \Big\}.$$

(In the whole-plane case there is a fundamental difference between **polychromatic** and **monochromatic** arm events, except in the case of $j = 1$.)



Arm events and percolation

Arm events are central objects of interest for the study of critical (and near-critical) planar percolation.

- Whole-plane events:
  - one-arm: the cluster containing the origin;
  - two-arm: the interface, aka the exploration process;
  - four-arm event: pivotal points, correlation length, near-critical percolation, dynamical percolation;
  - ⋯⋯
- Half-plane events:
  - one-arm: the cluster touching a specific point on the boundary;
  - two-arm: the hitting of the exploration process on the boundary;
  - ⋯⋯

Arm exponents via the knowledge of SLE_6

Theorem (Werner-Smirnov '01)

Half-plane *exponents: for any* $j \geq 1$,

$$\mathbf{P}[\mathcal{B}_j(r,R)] = R^{-j(j+1)/6+o(1)}.$$

Whole-plane *exponents: for any* $j > 1$,

$$\mathbf{P}[\mathcal{P}_j(r,R)] = R^{-(j^2-1)/12+o(1)}.$$

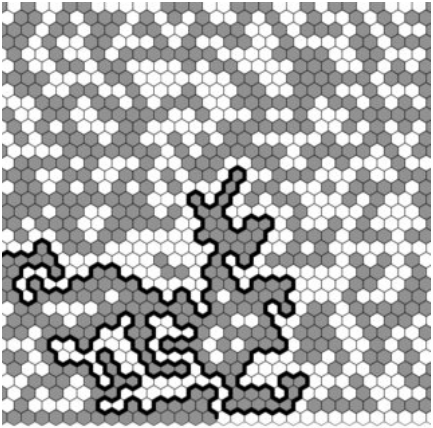
Theorem (Lawler-Schramm-Werner '01)

$$\mathbf{P}[\mathcal{P}_1(r,R)] = R^{-5/48+o(1)}.$$

The quest for improvement of arm probabilities

As long as there is "$o(1)$" in the exponent of asympototics for arm probability, one is not entirely satisfied, especially with scaling limit results involving microscopic quantities.

- Planar 2-arm exponent: $1/4$;
- Length of interface in $O(N)$ scale: $N^{7/4+o(1)}$.



The quest for improvement of arm probabilities

Recall the setup: critical planar site percolation on the triangular lattice in a domain (Ω_n, a_n, b_n) with Dobrushin boundary condition.

Let γ_η be the percolation exploration process in mesh size η such that each edge is traversed in

$$\xi_\eta := \eta^2 / \mathbf{P}[\mathcal{P}_2(1, \eta^{-1})]$$

units of time (so that the total duration of γ_η is $O(1)$).
Let γ be the chordal SLE_6 curve in Ω from a to b under natural parametrization (i.e., parametrized via Minkowski content).

Theorem (Garban-Pete-Schramm '13, Lawler-Rezaei '15, Holden-L.-Sun '22)

As $\eta \to 0$, $\gamma_\eta \xrightarrow{\mathrm{d}} \gamma$ under sup-norm topology.

However, in an "ideal" version of the theorem above, one should replace ξ_η by $\xi'_\eta := \eta^{7/4} = \eta^2 / \eta^{1/4}$.

The quest for improvement of arm probabilities

In the proceedings of ICM 2006, Oded Schramm proposed the following

Problem (3.1)

Improve the estimates $R^{-5/48+o(1)}$ *and* $R^{-5/4+o(1)}$ *mentioned above*[a] *(as well as other similar estimates) to more precise formulas. It would be especially nice to obtain estimates that are sharp up to multiplicative constants.*

[a] i.e. arm probability asymptotics for $j = 1$ and $j = 4$ in the whole-plane case

Cited from "Conformally invariant scaling limits: an overview and a collection of problems" by Oded Schramm, *ICM proceedings*, 2006.

A small step forward

Some easy up-to-constants estimates: in the half-plane case,

$$\mathbf{P}[\mathcal{B}_2(1,R)] \asymp R^{-1} \quad \text{and} \quad \mathbf{P}[\mathcal{B}_3(1,R)] \asymp R^{-2};$$

in the whole-plane case,

$$\mathbf{P}[\mathcal{P}_5(1,R)] \asymp R^{-2}.$$

Improvements for other arm probability asymptotics are much more difficult.

- Mendelson, Nachmias and Watson obtained a rate of convergence for the Cardy's formula[1], and improved half-plane 1-arm asymptotics:

Theorem (Mendelson-Nachmias-Watson '14)

$$\mathbf{P}[\mathcal{B}_1(1,R)] = e^{O(\sqrt{\log\log R})} R^{-1/3} = (\log R)^{O(1/\sqrt{\log\log R})} R^{-1/3}.$$

[1] Also independently obtained in [Binder-Chayes-Lei '15].

Sharp asymptotics for half-plane arm probabilities

In the half-plane case, we are now able to give sharp asymptotics for arm probabilities.

Theorem (Du-Gao-L.-Zhuang '22)

For any $j \geq 1$, for any $r \geq r_0(j)$, $\exists C = C(r)$, $c = c(r)$ s.t.

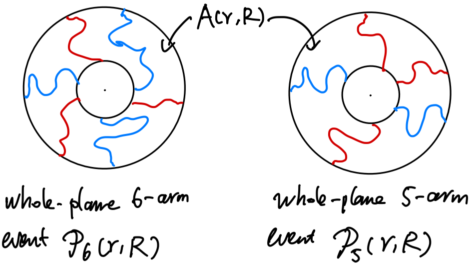
$$\mathbf{P}[\mathcal{B}_j(r,R)] = CR^{-j(j+1)/6}\big(1 + O(R^{-c})\big).$$

In particular, one can take $r_0(j) = 1$ for $j = 1, 2, 3$.

- The raison-d'être of the requirement that $r \geq r_0$ is to ensure $\mathbf{P}[\mathcal{B}_j(r,R)] > 0$.

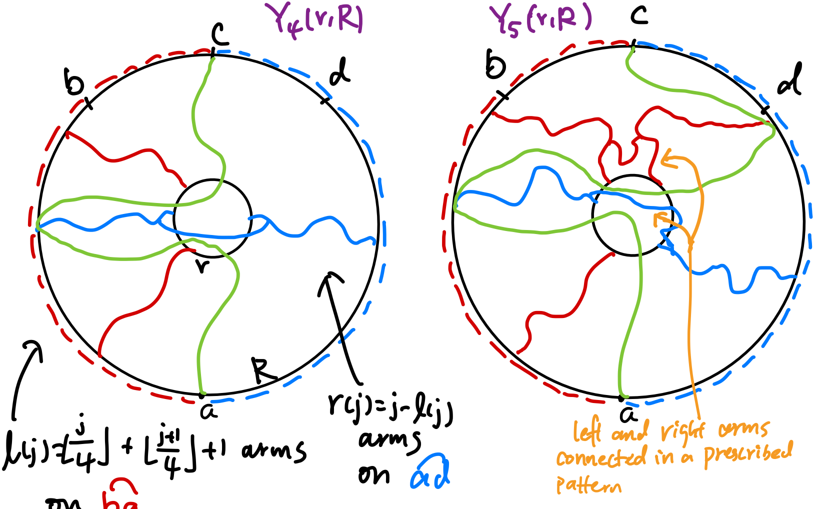
Up-to-constants asymptotics in the whole-plane case

- The whole-plane arm events are more complicated. We are only able to obtain sharp asymptotic probability in the case of $j > 1$.
- We need to consider a variant $\mathcal{Y}_j(r, R)$ with additional constraints on arms and come back to **alternating** arm event $\mathcal{A}_j(r, R)$ (the most useful case) and the classical $\mathcal{P}_j(r, R)$ with (slightly) weaker asymptotics.



The whole-plane equivalent for $\mathcal{H}_j$

The following event $\mathcal{Y}_j(r, R)$ can be described by the exploration process:



Theorem (Du-Gao-L.-Zhuang '22)

For any $j > 1$, for $r \geq r_0(j)$,

$$\mathbf{P}[\mathcal{Y}_j(r, R)] = C(r)R^{-(j^2-1)/12}(1 + O(R^{-c(r)})).$$

As a result, for alternating arm event $A_j(r, R)$,

$$\mathbf{P}[\mathcal{A}_j(r, R)] = C'(r)R^{-(j^2-1)/12}(1 + o(1)).$$

For classical arm events,

$$\mathbf{P}[\mathcal{P}_j(r, R)] \asymp R^{-(j^2-1)/12}.$$

In particular, one can take $r_0(j) = 1$ for $j = 2, \cdots, 6$.

The need of a rate of convergence for discrete processes to SLE

(Still cited from Schramm, ibid.)

- *The difficulty in getting more precise estimates is not in the analysis of SLE. Rather, it is due to the passage between the discrete and continuous setting. Consequently, the above problem seems to be related to the following*

Problem (3.2)

Obtain reasonable estimates for the speed of convergence of the discrete processes which are known to converge to SLE.

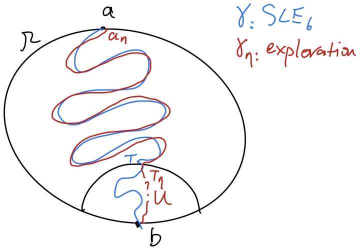
Rate of convergence for random discrete models to their SLE counterparts

- Through the study of convergence rate of planar loop-erased random walk towards SLE_2, [Viklund '15] developed a framework for obtaining power-law convergence rate of random discrete models towards SLE via the **power-law convergence rate** of the driving function (of the Loewner evolution) and additional geometric information (Aizenman-Burchard type criteria).
- Binder and Richards[2] verified that mechanism also works for **percolation**, the **harmonic explorer** and **FK-Ising**. In particular, they obtained a power-law convergence rate for the exploration process of planar critical percolation up to some stopping time.

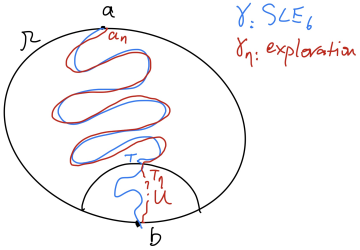
[2] Richards's thesis available at http://blog.math.toronto.edu/GraduateBlog/files/2021/02/LarissaRichardsThesis.pdf

Power-law rate of convergence for exploration process

- Take a Jordan domain Ω and two boundary points a, b.
- For $\eta > 0$, let $(\Omega_\eta, a_\eta, b_\eta)$ be the η-discretization of Ω by $\eta\mathbb{T}^*$, along with the discrete approximation of the marked points.
- Consider critical face percolation on $\eta\mathbb{T}^*$ with Dobrushin boundary condition. Let γ_η be the exploration process from a_η to b_η and a chordal SLE_6 γ in Ω from a to b.
- Given open $U \subset \Omega$, such that $a \notin U$ and $b \in U$, let T_η (resp. T) be the first time that γ_η (resp. γ) enters U_η (resp. U).



Power-law rate of convergence for exploration process



Theorem (Binder-Richards '21)

For any $\eta > 0$, there is $u = u(\Omega, a, b, U) > 0$ and a coupling $\mathbf{P}$ of γ_η and γ such that

$$\mathbf{P}\left[d\left(\gamma_\eta|_{[0,T_\eta]}, \gamma|_{[0,T]}\right) > \eta^u\right] < O(\eta^u)\,,$$

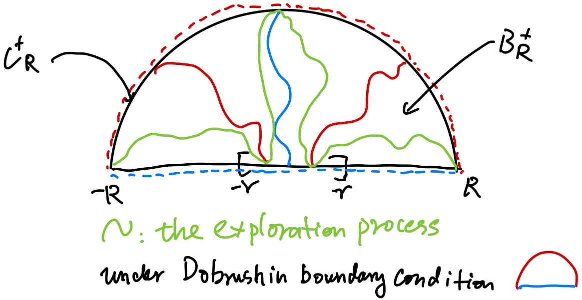
where d is the up-to-reparametrization metric between two curves.

Strategy of the proof, Step 1

We focus on the “clean” half-plane case.

- Consider variants of arm events that are tailored for the application of [Binder-Richards ’21]:

$$\mathcal{H}_j(r, R) := \{\exists\ j \text{ disjoint arms of alternating colors from } [-r, r] \times 0 \text{ to } C_R^+ \text{ in } B_R^+\}.$$

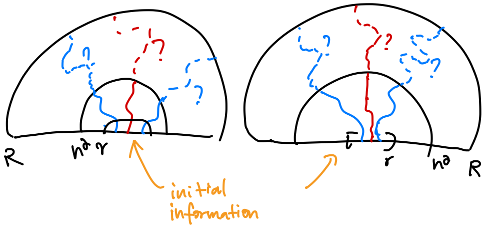


Note that this is an event that can be described by the exploration process.

Strategy of the proof, Step 2

- Use couplings of conditioned percolation configurations to relate the classical arm events to the variant defined above: for $\alpha \in (0,1)$, with universal constants,

  $$\mathbf{P}[\mathcal{B}_j(r,R)|\mathcal{B}_j(r,R^\alpha)] = \mathbf{P}[\mathcal{H}_j(r,R)|\mathcal{H}_j(r,R^\alpha)]\big(1+O(R^{-c})\big),$$



  and to establish a proportion between microscopic and mesoscopic arm probabilities: for $\alpha \in (0,1)$,

  $$\frac{\mathbf{P}[\mathcal{H}_j(r,mR)]}{\mathbf{P}[\mathcal{H}_j(r,R)]} = \frac{\mathbf{P}[\mathcal{H}_j(R^\alpha,mR)]}{\mathbf{P}[\mathcal{H}_j(R^\alpha,R)]}\big(1+O(R^{-c})\big).$$

Strategy of the proof, Step 3

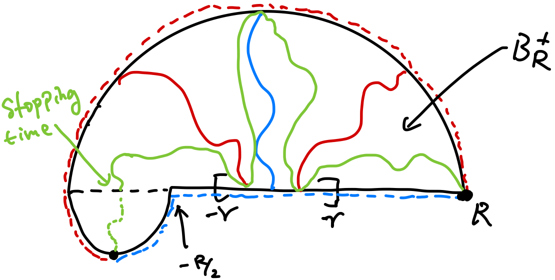
- Apply [Binder-Richards] to obtain a comparison across scales at mesoscopic level:

Proposition

There exists $c_1 > 0$ such that for all $a \in (1 - c_1, 1)$ and $m \in (1.1, 10)$

$$\mathbf{P}[\mathcal{H}_j(n^\alpha, R)] = \mathbf{P}[\mathcal{H}_j(mR^\alpha, mR)]\big(1 + O(^{-c})\big)$$

with universal constants.



Strategy of the proof, Steps 4 and 5

- Combine various asymptotic “identities” of proportions to conclude with the following one:

$$\frac{\mathbf{P}[\mathcal{B}_j(r, m^2 R)]}{\mathbf{P}[\mathcal{B}_j(r, mR)]} = \frac{\mathbf{P}[\mathcal{B}_j(r, mR)]}{\mathbf{P}[\mathcal{B}_j(r, R)]}\big(1 + O(R^{-c})\big)$$

  with uniform constants for $m \in (1.1, 10)$.
- Finally, use a “functional equation” trick to obtain the desired result from the asymptotic proportion above.

Similar strategy also appears in [L.-Shiraishi ’19] to deal with sharp one-point function for 3-dimensional loop-erased random walk.

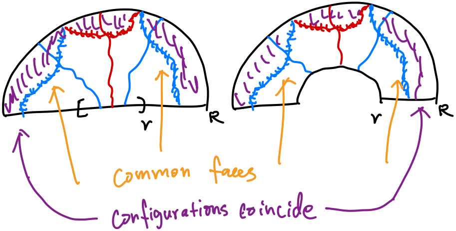
Coupling of configurations conditioned on arm events

In Step 2, we use coupling to establish estimates such as

$$\mathbf{P}[\mathcal{B}_j(r,R)|\mathcal{B}_j(r,R^\alpha)] = \mathbf{P}[\mathcal{H}_j(r,R)|\mathcal{H}_j(r,R^\alpha)]\big(1+O(R^{-c})\big),$$

Proposition

For any $j \geq 2$, there exists $\delta = \delta(j) > 0$ such that for any large r and R, there is a coupling Q of the conditional laws $\mathbf{P}[\cdot \mid \mathcal{H}_j(r,R)]$ and $\mathbf{P}[\cdot \mid \mathcal{B}_j(r,R)]$ such that if we sample (ω, ω') according to Q, then with probability at least $1 - (\frac{r}{R})^\delta$, there exists a common configuration of j inner faces Θ^* around C_R^+ and ω coincides with ω' outside these faces.



A list of open questions

- Can similar strategy be applied to other models to obtain sharp asymptotics of arm probabilities, in particular the critical **FK-Ising** model (partial progress) and the harmonic explorer (seems difficult)?
- Can one improve asympotics for $\mathbf{P}[\mathcal{P}_1(1,R)]$, the whole-plane one-arm probability?
- $\cdots\cdots$

Preprint available at `arXiv:2205.15901`

Thank you!

Half-plane super-strong separation lemma

A crucial step in establishing the coupling is to show that interfaces between arms exhibit a "separation phenomenon" with uniformly positive probability in each scale.

Let Γ be a set of interfaces from C_u^+ to C_v^+, $u < v$. Suppose that Γ contains $j \geq 1$ interfaces and let $e^1, \cdots, e^j$ be end-edges of the interfaces in Γ on C_v^+ (in counterclockwise order). Then Γ has an **exterior quality**

$$Q_{\mathrm{ex}}(\Gamma) := Q_{\mathrm{ex}}^v(\Gamma) = \frac{1}{v}\,\mathrm{d}(v, e^1) \wedge \mathrm{d}(e^1, e^2) \wedge \cdots \wedge \mathrm{d}(e^j, -v).$$

Proposition (Super-strong separation lemma)

For any $j \geq 2$, there exist $M = M(j) > 1$ and $c = c(j) > 0$ such that for any $r_0 < r < u < Mu \leq R$, and any $B_u^+(\omega)$,

$$\mathbf{P}[Q_{ex}(\Gamma) > j^{-1} \mid \mathcal{B}_j(r, R), B_u^+(\omega)] > c$$

where Γ is the set of interfaces crossing $A^+(u, R)$.