Multiplicative chaos measures from thick points of log-correlated fields

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Joint work with G. Lambert and C. Webb (arXiv:2209.06548)

Log-correlated Gaussian fields

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$$X_{S^1}(z) = \sqrt{2}\,\mathrm{Re}\Big(\sum_{k=1}^{\infty} \frac{Z_k}{\sqrt{k}} z^k\Big)$$

on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, where Z_k are i.i.d. standard complex Gaussians. In this case we have exactly $C_{X_{S^1}}(z,w) = \log|z-w|^{-1}$.

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- **X is not a pointwise defined function, but can be given sense as a random Schwartz distribution.**

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- Rigorous definition requires a limiting procedure: $\mu(dx) := \lim_{t\to\infty} \mu_t(x)\, dx := \lim_{t\to\infty} e^{\gamma X_t(x) - \frac{\gamma^2}{2}\mathbb{E}[X_t(x)^2]}\, dx$, where $X_t \to X$ is a smoothened version of X.

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- Appears e.g. in Liouville quantum gravity or as a limit of characteristic polynomials of random matrices.

Existence in the L^2-phase $\gamma \in (0, \sqrt{d})$

For a given test function f we may compute

$$\begin{aligned}\mathbb{E}[|\mu_t(f)|^2] &= \int f(x)f(y)\mathbb{E}[e^{\gamma X_t(x)+\gamma X_t(y)-\frac{\gamma^2}{2}\mathbb{E}[X_t(x)^2]-\frac{\gamma^2}{2}\mathbb{E}[X_t(y)^2]}]\,dx\,dy \\ &= \int f(x)f(y)e^{\gamma^2\mathbb{E}[X_t(x)X_t(y)]}\,dx\,dy \lesssim \int |x-y|^{-\gamma^2}\,dx\,dy.\end{aligned}$$

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- Thus $\mathbb{E}[|\mu_t(f)|^2]$ is bounded uniformly in t if $\gamma < \sqrt{d}$.
- If $\mu_t(f)$ is a martingale we automatically obtain convergence in $L^2(\Omega)$.

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- Note that by Girsanov

$$\mathbb{E}\left[e^{\gamma X_t(x) - \frac{\gamma^2}{2}t} \mathbf{1}_{A_{t_0,t}(x)^c}\right] = \mathbb{P}[X_s(x) + \gamma s \geq (\gamma + \varepsilon)s \text{ for some } s \in [t_0, t]]],$$

so that

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- These two estimates together show that $\int e^{\gamma X_t(x) - \frac{\gamma^2}{2}t}\, dx$ is uniformly integrable and we get a non-trivial limit.

Thick points

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- The set $T_\gamma = \{x : \limsup_{t\to\infty} \frac{X_t(x)}{t} = \gamma\}$ is known as the set of γ**-thick points** and μ gives full mass to T_γ.

GMC from level sets of thick points

- It turns out that one can construct the GMC measure μ directly from thick points:

$$\frac{\mathbf{1}\{x : X_t(x) > \gamma t\}}{\mathbb{P}[X_t(x) > \gamma t]}\, dx \to d\mu(x).$$

- Our goal: Prove this under general assumptions, with applications in random matrices in mind.
- Similar results have appeared before, e.g. Biskup–Louidor for discrete GFF, Jego for thick points of Brownian motion.

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Conjecture (Fyodorov–Keating)

$$\frac{\frac{1}{2\pi}\int_0^{2\pi} \mathbf{1}\{\log|p_N(\theta)| > \gamma \log N\} d\theta}{N^{-\gamma^2} \frac{1}{\sqrt{\pi \log N}} \frac{G(1+\gamma)^2}{2\gamma G(1+2\gamma)} \frac{1}{\Gamma(1-\gamma^2)}}$$

converges in distribution to a r.v. with density

$$\mathcal{P}_\gamma(x) = \gamma^{-2} x^{-1-\gamma^2} e^{-x^{-\gamma^{-2}}} \mathbf{1}\{x > 0\}.$$

Here G is the Barnes G-function,

$$G(1+z) = (2\pi)^{z/2} \exp\left(-\frac{z + z^2(1+\gamma_{EM})}{2}\right) \prod_{k=1}^{\infty} \left[\left(1 + \frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k} - z\right)\right].$$

From CUE to GMC

- Relationship to log-correlated fields: $\log |p_N(\theta)|$ corresponds to an approximation of the log-correlated field $X_{S^1}(e^{i\theta})$ with variance $\sim \log N$, and $|p_N(\theta)|^\gamma / \mathbb{E}[|p_N(\theta)|^\gamma]$ corresponds to an approximation of the GMC measure.

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- Convergence of $|p_N(\theta)|^\gamma/\mathbb{E}[|p_N(\theta)|^\gamma]$ to GMC was proven by Webb (L^2-phase) and Nikula–Saksman–Webb (L^1-phase).

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- The explicit formula for the density $\mathcal{P}_\gamma$ of the total mass was proven by Remy (LCFT inspired) and also follows from an alternate description of the limit by Chhaibi–Najnudel.

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- The explicit formula for the density $\mathcal{P}_\gamma$ of the total mass was proven by Remy (LCFT inspired) and also follows from an alternate description of the limit by Chhaibi–Najnudel.
- **Thus the conjecture follows if one can show that the level set measure $\mathbf{1}\{\log |p_N(\theta)| > \gamma \log N\}$ converges to GMC (after renormalisation).**

Barriers and "mesoscopically Gaussian" log-correlated fields

- Take an approximation X_N of a log-correlated field X, for instance the logarithm of a CUE characteristic polynomial $X_N(x) = -\operatorname{Re}\Big(\sum_{k\geq 1} \frac{\operatorname{Tr} U_N^k}{k/\sqrt{2}} e^{-ikx} \Big)$.

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- Try to apply barrier arguments to X_N by employing a second layer of approximations $X_{N,\delta} = X_N * \rho_\delta$, e.g. $X_{N,\delta}(x) = -\operatorname{Re}\Big(\sum_{k\leq 1/\delta} \frac{\operatorname{Tr} U_N^k}{k/\sqrt{2}} e^{-ikx} \Big)$, where $\delta \in \{e^{-k} : k \in [L_0, \log(N^{1-\eta})]\}$ stays in mesoscopic range for the barrier ($\eta > 0$ is some small enough constant).

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- Capture approximate Gaussianity on mesoscopic scales by suitable assumptions on the Laplace transform $\mathbb{E}[e^{\zeta_1 X_N(x_1)+\zeta_2 X_N(x_2)+\sum_{j=1}^q \xi_j X_{N,\delta_j}(z_j)}] = \Psi_{N,\delta}^{(\zeta_1,\zeta_2,\xi)}(x_1, x_2; z)\times$ $e^{\frac{\zeta_1^2+\zeta_2^2}{2}\log N+\zeta_1\zeta_2 C_X(x_1,x_2)+\zeta_1\int C_X(x_1,u)\mathrm{f}_{\delta,z}(u)\,du+\zeta_2\int C_X(x_2,u)\mathrm{f}_{\delta,z}(u)\,du+\frac{1}{2}\mathbb{E}\langle X,\mathrm{f}_{\delta,z}\rangle^2}$

  where $f_{\delta,z} = \sum_{j=1}^q \xi_j \rho_{\delta_j,z_j}$.

The main theorem

Let us define

$$dv_N(x) = \frac{\mathbf{1}\{X_N(x) \geq \gamma \log N + u\}}{\mathbb{P}[X_N(x) \geq \gamma \log N]} dx, \quad d\mu_N(x) = \frac{e^{\gamma X_N(x)}}{\mathbb{E}[e^{\gamma X_N(x)}]} dx.$$

where u is some arbitrary continuous function (e.g. $u = 0$).

Theorem (Sketch)

Assume that $\Psi_{N,\delta}^{(\zeta_1,\zeta_2,\xi)}(x_1, x_2; z)$ *satisfies certain decorrelation properties, roughly meaning that* $\Psi_{N,\delta}^{(\zeta_1,\zeta_2,\xi)}(x_1, x_2; z) = \Psi(\zeta_1, x_1)\Psi(\zeta_2, x_2)(1 + o(1))$ *for* $|x_1 - x_2|, \delta_1, \ldots, \delta_q > N^{-1+\eta}$ *for some* $\eta > 0$. *Then*

$$\lim_{N\to\infty} \mathbb{E}_N[|v_N(f) - \mu_N(e^{-\gamma u} f)|] = 0.$$

In particular, if μ_N *converges to GMC measure* μ, *so does* v_N.

Applications

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- Checking the conditions for CUE is more involved:
    - Using the Heine–Szegö identity one can compute the Laplace transform as a certain Toeplitz determinant.
    - Asymptotics of such a determinant have been previously analysed by Deift–Its–Krasovsky based on orthogonal polynomials and Riemann–Hilbert problems, but we need some technical tweaks to their approach.

Thanks!