Properties of critical Ising correlations: bosonization, BPZ equations, OPE to all orders

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Jeju

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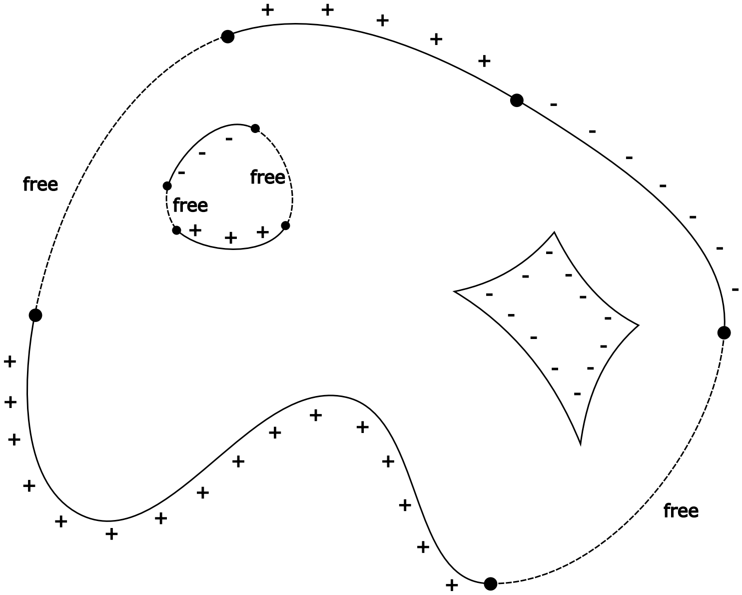
- for now, $\langle \mathcal{O}_{z_1} \ldots \mathcal{O}_{z_n} \rangle_\Omega$ is just a family of special (multi-valued) real analytic functions ($\langle \mathcal{O}_{z_1} \ldots \mathcal{O}_{z_n} \rangle_\Omega^2$ are single-valued).

Multiply-connected domain with boundary conditions



Scaling limits of correlations in the usual Ising model

Critical Ising model on a lattice approximation Ω^δ to Ω: assign spins at random to the vertices of Ω^δ:

$$\sigma : \text{Vertices}(\Omega^\delta) \to \{\pm 1\} \text{ random, } \mathbb{P}(\sigma) = \frac{1}{Z} e^{-\beta \mathcal{H}(\sigma)},$$

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For natural observables $\mathcal{O}^\delta_{z_i} \in \{\sigma_{z_i}, \mu_{z_i}, \varepsilon_{z_i}, \psi_{z_i}, \psi^\star_{z_i}\}$ in critical Ising model on Ω^δ we have

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- C_i *are (lattice-dependent) constants*
- $\tilde{\Delta}_i = \Delta_i + \Delta'_i$ *are scaling dimensions.*

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$\mathcal{O}_{z_i}$	Δ_i	Δ'_i
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μ_{z_i}	$\frac{1}{16}$	$\frac{1}{16}$
ε_{z_i}	$\frac{1}{2}$	$\frac{1}{2}$
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- $\psi_z = \sigma_{z-\frac{\delta}{2}}\mu_{z-\frac{i\delta}{2}} + i\sigma_{z+\frac{\delta}{2}}\mu_{z+\frac{i\delta}{2}}$ *(same thing as Smirnov–Hongler–... observable in contour representation)*

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- Conformal covariance:

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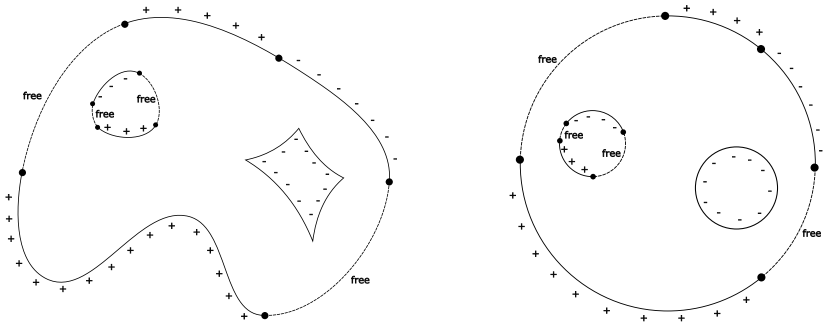

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These properties are enough to identify $\langle \mathcal{O}_{z_1} \dots \mathcal{O}_{z_n} \rangle_\Omega$ uniquely. The result is not very explicit.

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(G. Felder, BRST approach to minimal models, 1989)

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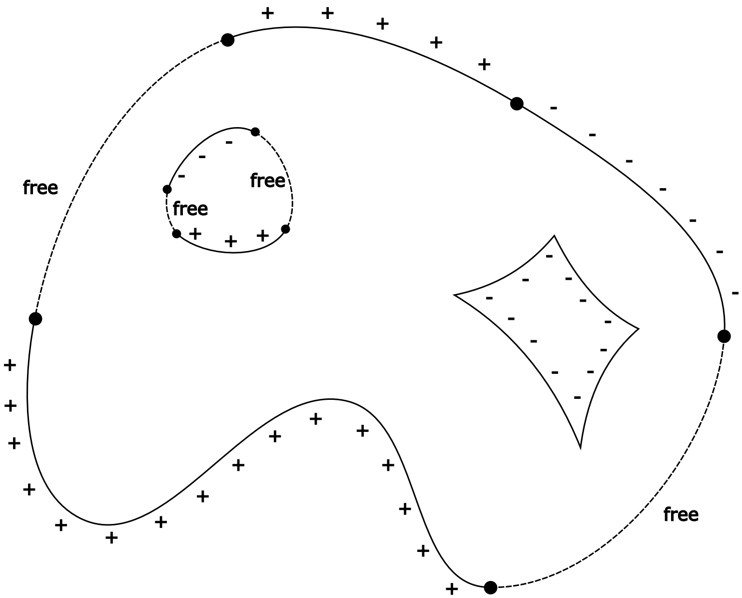

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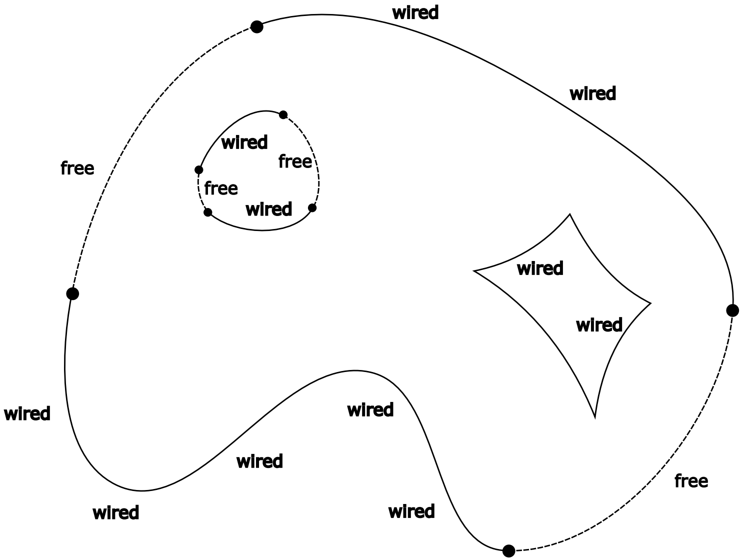

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- GFF φ is heuristically a "standard Gaussian on $H_0(\Omega)$", i.e., sampled with probability proportional to $e^{-\frac{1}{4\pi}\langle\nabla\varphi,\nabla\varphi\rangle}$".
- Also, $\langle\nabla\Phi,\nabla\Phi\rangle = \langle\nabla\varphi,\nabla\varphi\rangle + \langle\nabla\xi,\nabla\xi\rangle$ for $\varphi \in H_0(\Omega)$ and ξ harmonic.

Why "compactified"?

Consider $\tilde{\Phi} = \frac{1}{\sqrt{2}} e^{\sqrt{2}i\Phi} \in \frac{1}{\sqrt{2}}\mathbb{T}$ instead.

- All our correlations will be invariant under $\Phi \mapsto \Phi + \sqrt{2}\pi$, so naturally defined in terms of $\tilde{\Phi}$.
- The boundary conditions can be stated as:

$$\tilde{\Phi} \equiv \pm\frac{1}{\sqrt{2}} \text{ on wired/free part of } \partial\Omega,$$

$$\text{turns by } \pi \text{ and back on free arcs}$$

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- Therefore, $\tilde{\Phi}$ is a "random $\frac{1}{\sqrt{2}}\mathbb{T}$-valued function with probability proportional to $e^{-\frac{1}{4\pi}\langle\nabla\tilde{\Phi},\nabla\tilde{\Phi}\rangle}$" with boundary consitions as above.

Bosonization identity

Theorem (Bayraktaroglu, K. I., Virtanen, Webb, '23)

To each Ising field $\mathcal{O}_{z_i}$, *there corresponds a field* $\hat{\mathcal{O}}_{z_i}$ *in a bosonic theory, defined in terms of the compactified Gaussian free field* Φ, *such that*

$$\langle \mathcal{O}_{z_1} \dots \mathcal{O}_{z_n} \rangle_\Omega^2 = \langle \hat{\mathcal{O}}_{z_1} \dots \hat{\mathcal{O}}_{z_n} \rangle_\Omega .$$

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$\mathcal{O}_{z_i}$	$\hat{\mathcal{O}}_{z_i}$
σ_{z_i}	$\sqrt{2} : \cos\left(\frac{\sqrt{2}}{2}\Phi(z_i)\right) :$
μ_{z_i}	$\sqrt{2} : \sin\left(\frac{\sqrt{2}}{2}\Phi(z_i)\right) :$
ε_{z_i}	$-\frac{1}{2} : \lvert\nabla\Phi(z_i)\rvert^2 :$
ψ_{z_i}	$2\sqrt{2}\mathrm{i}\partial\Phi(z_i)$
$\psi^\star_{z_i}$	$-2\sqrt{2}\mathrm{i}\bar{\partial}\Phi(z_i)$

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- caveat about boundary conditions: free + *locally monochromatic* (spin constant on each connected component of $\mathbb{C} \setminus \Omega^\delta$).
- the right-hand side is as explicit as it gets (e.g., can be written in terms of harmonic measures of components via theta functions).

Example: freshman's dream

$$
\left(\frac{1}{(z_1-z_2)(z_3-z_4)}-\frac{1}{(z_1-z_3)(z_2-z_4)}+\frac{1}{(z_1-z_4)(z_2-z_3)}\right)^2
$$

$$
=\frac{1}{16}\langle\psi_{z_1}\psi_{z_2}\psi_{z_3}\psi_{z_4}\rangle_{\mathbb{H}}^2=4\langle\partial\varphi_{z_1}\partial\varphi_{z_2}\partial\varphi_{z_3}\partial\varphi_{z_4}\rangle_{\mathbb{H}}
$$

$$
=\frac{1}{(z_1-z_2)^2(z_3-z_4)^2}+\frac{1}{(z_1-z_3)^2(z_2-z_4)^2}+\frac{1}{(z_1-z_4)^2(z_2-z_3)^2}
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More generally

$$\mathrm{Pf}\left[\frac{1}{z_i-z_j}\right]^2=\mathrm{Hf}\left[\frac{1}{(z_i-z_j)^2}\right]$$

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More generally

$$\left(\sum_p(-1)^{s_{ij}}\prod_{\{i,j\}\in p}\frac{1}{z_i-z_j}\right)^2=\sum_p\prod_{\{i,j\}\in p}\frac{1}{(z_i-z_j)^2}.$$

Examples: spin correlations

$$\langle \sigma_{z_1} \dots \sigma_{z_n} \rangle_{\mathbb{H}}^2 = 2^{\frac{n}{2}} \langle : \cos \frac{1}{\sqrt{2}} \varphi(z_1) : \cdots : \cos \frac{1}{\sqrt{2}} \varphi(z_n) : \rangle_{\mathbb{H}}$$

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&= 2^{-\frac{n}{2}} \sum_{s \in \{\pm 1\}^n} \exp \left(\sum_{i \neq j} - \frac{s_i s_j}{4} \log \left| \frac{z_i - \overline{z}_j}{z_i - z_j} \right| - \sum_i \frac{1}{4} \log |z_i - \overline{z}_i| \right)
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$$= 2^{-\frac{n}{2}} \prod_i \frac{1}{(2 \mathrm{Im} z_i)^{\frac{1}{4}}} \sum_{s \in \{\pm 1\}^n} \prod_{i<j} \left| \frac{z_i - z_j}{z_i - \overline{z}_j} \right|^{\frac{s_i s_j}{2}} .$$

BPZ equations

Theorem (BPZ'84; K.I., Webb'23)

The Ising correlations satisfy the BPZ equations

$$\mathcal{L}\langle \mathcal{O}_{z_1} \dots \mathcal{O}_{z_n} \rangle_{\mathbb{H}} \equiv 0,$$

where

$$\mathcal{L} = \frac{3}{2(2\Delta_1 + 1)} \partial_{z_1}^2 + \frac{\overline{\partial}_{z_1}}{\overline{z}_1 - z_1} + \frac{\Delta_1'}{(\overline{z}_i - z_1)^2} + \sum_{i=2}^{n} \left(\frac{\partial_{z_i}}{z_i - z_1} + \frac{\overline{\partial}_{z_i}}{\overline{z}_i - z_1} + \frac{\Delta_i}{(z_i - z_1)^2} + \frac{\Delta_i'}{(\overline{z}_i - z_1)^2} \right).$$

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(Same methods give analogs for multiply-connected domains)

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What is the asymptotics of $\langle \mathcal{O}_{z_1} \dots \mathcal{O}_{z_n} \rangle_\Omega$ as $z_1 \to z_2$? For example:

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- We don't need anything else.

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(Holomorphic) stress-energy tensor T_z defined in terms of correlations:

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Virasoro algebra generators $\dots L_{-1}, L_0, L_1, \dots$

$$T_z \mathcal{O}_w = \sum_{k \in \mathbb{Z}} (z-w)^{-2-k} (L_k \mathcal{O})_w$$

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Virasoro algebra generators $\dots L_{-1}, L_0, L_1, \dots$; can iterate, e.g.,

$$(L_{-3}L_{-5}\sigma)_w = \frac{1}{(2\pi \mathfrak{i})^2} \oint_{z,w} \oint_w (\zeta - w)^{-2} (z-w)^{-4} T_\zeta T_z \sigma_w \, dz$$

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Virasoro generators $\dots L_{-1}, L_0, L_1$,satisfy Virasoro commutation relations:

$$[L_n; L_m] = (n-m)L_{n+m} + \frac{1}{24}\delta_{n,-m}(n^3-n).$$

- Hongler–Kytölä–Viklund'13 – '22 – ...: L_k can be defined in the discrete; hence $L_{-k_1} \dots L_{-k_l}\mathcal{O}$ are also scaling limits of lattice fields

OPE for Ising correlations

Theorem (BPZ'84; K.I., Webb'23)

Each pair of primary fields $\mathcal{O}_z \in \{\sigma_z, \mu_z, \varepsilon_z, \psi_z, \psi^\star_z\}$ *and* $\mathcal{O}_w \in \{\sigma_w, \mu_w, \varepsilon_w, \psi_w, \psi^\star_w\}$ *has an OPE of the form*

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where $n \in (\mathbb{N}+a) \cup (\mathbb{N}+b)$, $m \in (\mathbb{N}+c) \cup (\mathbb{N}+d)$, *for some* $a, b, c, d \in \mathbb{R}$.

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where $n \in (\mathbb{N}+a) \cup (\mathbb{N}+b)$, $m \in (\mathbb{N}+c) \cup (\mathbb{N}+d)$, *for some* $a, b, c, d \in \mathbb{R}$.
Moreover, the coefficients satisfy the recursion

$$\left[n(n-1) + \frac{2(2\Delta_2+1)}{3}(n-\Delta_1) \right] \mathcal{O}_w^{(n,m)} = \left(\frac{2(2\Delta_2+1)}{3} L_{-2} - L_{-1}^2 \right) \mathcal{O}_w^{(n-2,m)} + 2(n-1) L_{-1} \mathcal{O}_w^{(n-1,m)}.$$

OPE for Ising correlations

Theorem (BPZ'84; K.I., Webb'23)

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This recursion + conjugate one + beginning of the OPE known from Chelkak–Hongler–K.I.'21 determines the coefficients uniquely.

Example

$$\sigma_z \sigma_w = \sum_{n,m} (z-w)^n (\overline{z} - \overline{w})^m \mathcal{O}_w^{(n,m)}$$

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$$n(n-1) + \frac{3}{4}(n - \frac{1}{16}) = 0$$

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Plug $\Delta_1 = \Delta_2 = \frac{1}{16}$ into the characteristic equation

$$n = -\frac{1}{8} \text{ or } n = \frac{3}{8}.$$

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$$n, m \in \left(-\frac{1}{8} + \mathbb{Z}_{\geq 0} \right) \cup \left(\frac{3}{8} + \mathbb{Z}_{\geq 0} \right).$$

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$$\sigma_z \sigma_w = (z-w)^{-\frac{1}{8}} (\overline{z} - \overline{w})^{-\frac{1}{8}} + (z-w)^{\frac{3}{8}} (\overline{z} - \overline{w})^{\frac{3}{8}} \frac{\varepsilon_w}{2} + \dots$$

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\begin{aligned}
\sigma_z\sigma_w &= (z-w)^{-\frac{1}{8}}(\overline{z}-\overline{w})^{-\frac{1}{8}} + (z-w)^{\frac{3}{8}}(\overline{z}-\overline{w})^{\frac{3}{8}}\frac{\varepsilon_w}{2} \\
&- \frac{1}{3}(z-w)^{\frac{15}{8}}(\overline{z}-\overline{w})^{-\frac{1}{8}}(L_{-2}\mathbb{I})_w - \frac{1}{3}(z-w)^{-\frac{1}{8}}(\overline{z}-\overline{w})^{\frac{15}{8}}(\overline{L}_{-2}\mathbb{I})_w \\
&+ \frac{1}{4}(z-w)^{\frac{11}{8}}(\overline{z}-\overline{w})^{\frac{3}{8}}L_{-1}\varepsilon_w + +\frac{1}{4}(z-w)^{\frac{3}{8}}(\overline{z}-\overline{w})^{\frac{11}{8}}\overline{L}_{-1}\varepsilon_w + \ldots
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$$-\frac{1}{3}(z-w)^{\frac{15}{8}}(\overline{z}-\overline{w})^{-\frac{1}{8}}T_w - \frac{1}{3}(z-w)^{-\frac{1}{8}}(\overline{z}-\overline{w})^{\frac{15}{8}}\overline{T}_w$$
$$+\frac{1}{4}(z-w)^{\frac{11}{8}}(\overline{z}-\overline{w})^{\frac{3}{8}}\partial\varepsilon_w + +\frac{1}{4}(z-w)^{\frac{3}{8}}(\overline{z}-\overline{w})^{\frac{11}{8}}\overline{\partial}\varepsilon_w + \ldots$$

Bosonization identity: sketch of the proof

Theorem (Bayraktaroglu, K. I., Virtanen, Webb, '23)

$$\langle \mathcal{O}_{z_1} \dots \mathcal{O}_{z_n} \rangle_\Omega^2 = \langle \hat{\mathcal{O}}_{z_1} \dots \hat{\mathcal{O}}_{z_n} \rangle_\Omega .$$

$\mathcal{O}_{z_i}$	$\hat{\mathcal{O}}_{z_i}$
σ_{z_i}	$\sqrt{2} : \cos\left(\frac{\sqrt{2}}{2}\Phi(z_i)\right) :$
μ_{z_i}	$\sqrt{2} : \sin\left(\frac{\sqrt{2}}{2}\Phi(z_i)\right) :$
ε_{z_i}	$-\frac{1}{2} : \lvert\nabla\Phi(z_i)\rvert^2 :$
ψ_{z_i}	$2\sqrt{2}\mathrm{i}\partial\Phi(z_i)$
$\psi^\star_{z_i}$	$-2\sqrt{2}\mathrm{i}\bar{\partial}\Phi(z_i)$

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$$\psi_z \sigma_w = e^{\frac{i\pi}{4}} (z-w)^{-\frac{1}{2}} \cdot \mu_w + \dots;$$

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$\mathcal{O}_{z_i} \in \{\sigma_{z_i}, \mu_{z_I}\}$, turns out to be a limiting form of

Theorem (D. Hejhal, independently J. Fay, 1973)

Given a spin line bundle χ on a compact Riemann surface $\hat{\Omega}$, one has

$$\Lambda_{\hat{\Omega},\chi}(z,w)^2 = \beta(z,w) + \sum_{i,j} \frac{\partial_{z_i}\partial_{z_j}\theta[\chi](0)}{\theta[\chi](0)} u_i(z) u_j(w),$$

- $\Lambda_{\hat{\Omega},\chi}(z,w)$ *is the Szegö kernel of* χ*;*
- $\beta(z,w)$ *is the Abelian differential of the second kind;*
- $u_1(z), \dots, u_g(z)$ *are Abelian differentials of the first kind;*
- $\theta[\chi](0)$ *is the theta-function with characteristic* χ*.*

The surface $\hat{\Omega}$

The surface $\hat{\Omega}$ is the Shottky double of Ω with small handles attached at z_i and its "mirror image" (more handles at endpoints of free arcs)

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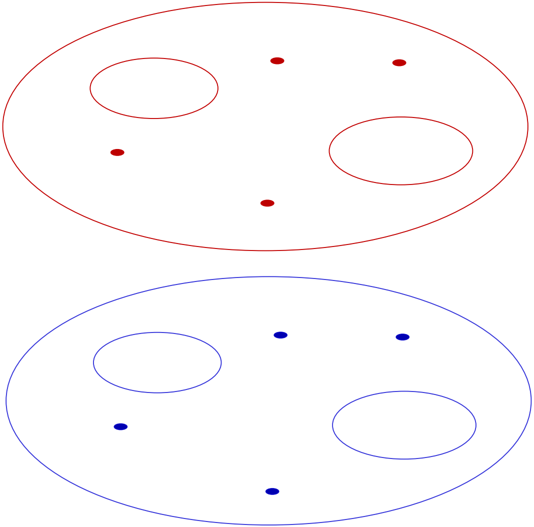

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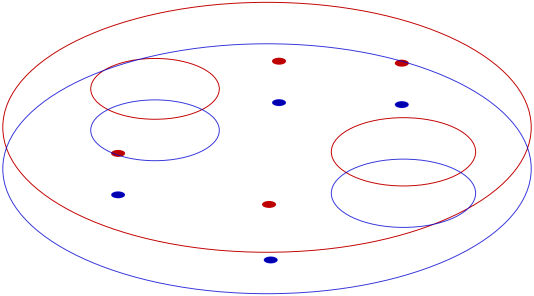

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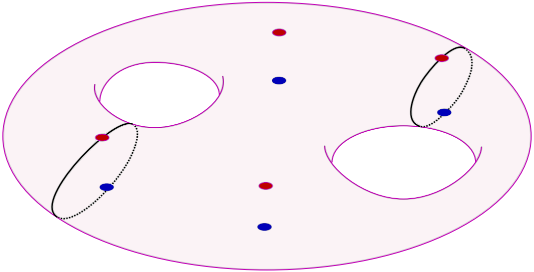

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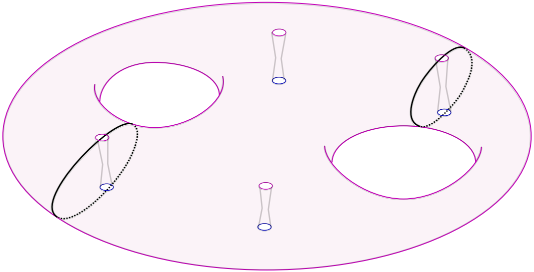

Thank you!