Quasisymmetric images of Brownian graph: minimal dimension

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Random Conformal Geometry and Related Fields

June 8, 2023

based on joint works with Hrant Hakobyan and Wenbo Li

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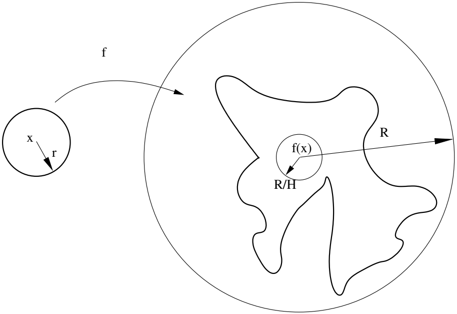
- The inverse of a quasisymmetric map is also quasisymmetric.

Quasisymmetry



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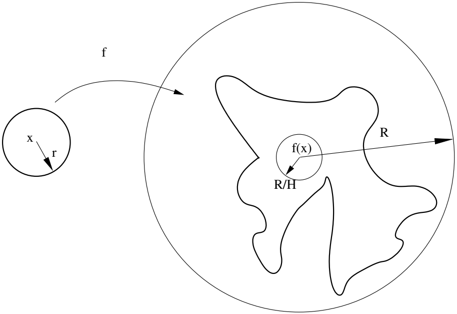

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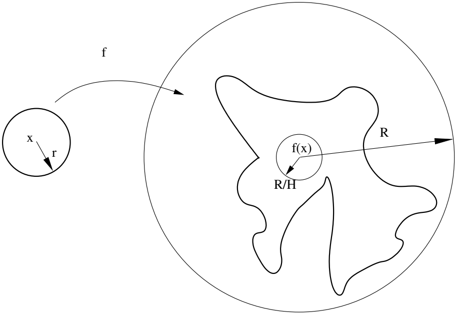

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This is weakly quasisymmetric map. Equivalent to quasisymmetry for "nice" X and Y.

Quasisymmetry and Quasiconformality

- A homeomorphism $f : X \to Y$ is K-Quasiconformal/QC if for every $x \in X$

$$Dil_f(x) = \limsup_{r \to 0} \frac{\max_{d_X(x,y) \le r} d_Y(f(x), f(y))}{\min_{d_X(x,z) \ge r} d_Y(f(x), f(z))} \le K$$

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- Every quasisymmetric map is K-QC with $K = \eta(1)$.
- If X, Y are domains in $\mathbb{R}^d$, $d \geq 2$, then every K-QC map is locally QS.

Dimension can always be increased

- If $\dim X > 0$ and $0 < \varepsilon < 1$ then the Snowflake map

$$id : (X, d) \to (X, d^{\varepsilon})$$

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Theorem. (Bishop, '99)

If $E \subset \mathbb{R}^n$, $\dim E > 0$, and $\varepsilon > 0$ then there is a QS map $f : \mathbb{R}^n \to \mathbb{R}^n$ s.t.

$$\dim f(E) > n - \varepsilon.$$

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- Topological dimension of a metric space X

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- X is minimal if

$$\dim_{\mathrm{C}} X = \dim_H X.$$

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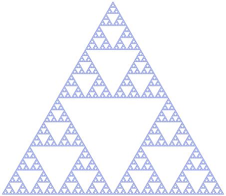
- Sierpiński Gasket $SG, \dim(SG) = \frac{\log 3}{\log 2}, \dim_C(SG) = 1.$(Tyson, Wu, '06)



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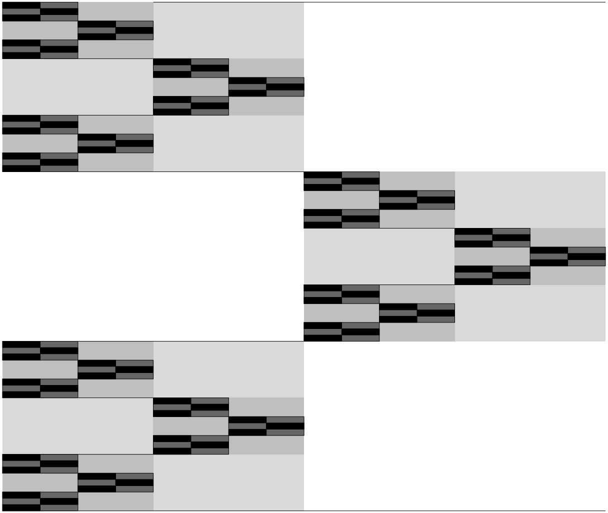
Theorem. (Bishop, Tyson'01)

For any $\alpha \geq 1$, there exists a $X \subset \mathbb{R}^d$ such that $\dim(X) = \dim_C(X) = \alpha$.

McMullen set



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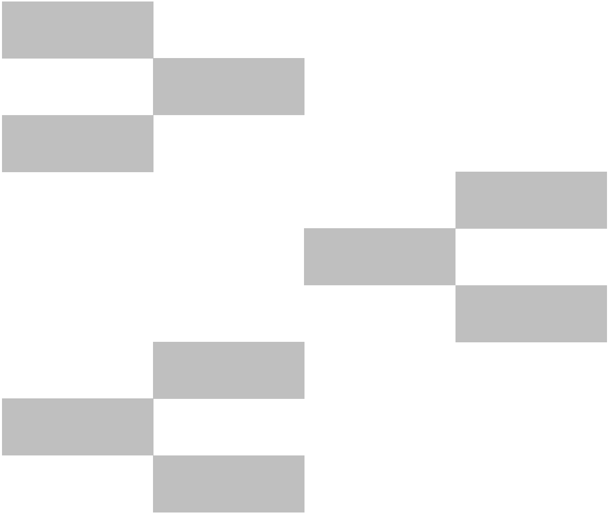

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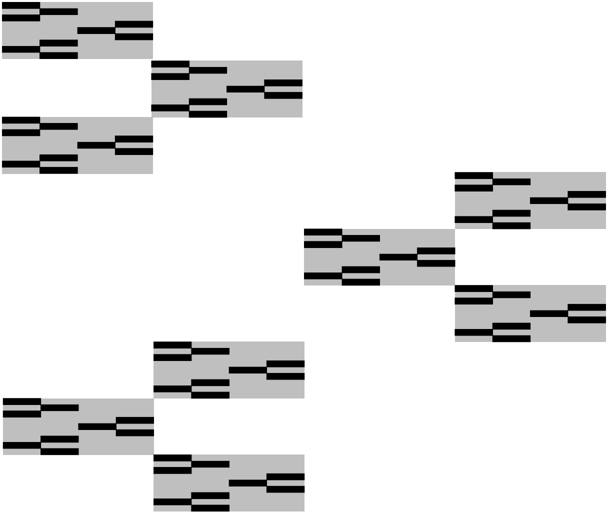

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- $\dim_{\mathrm{H}}(E_D) = \log_m \left(\sum_{j=1}^{m} r_j^{\log_n m}\right)$ (McMullen, '86)
- Except for the case $r_1 = r_2 = \cdots = r_m$,

$$1 + \mathrm{Vert}(E_D) < \dim_{\mathrm{H}} E_D < \dim_M E_D.$$

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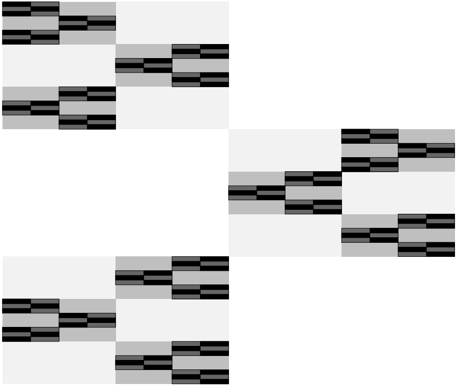
- Different ways to shuffle the columns have the same probability and are independent for different rectangles. The intersection is a fractal percolation cluster $\tilde{E}_D$.
- Almost surely, $\tilde{E}_D$ satisfies

$$1 + Vert(\tilde{E}_D) = \dim_H \tilde{E}_D = \dim_M \tilde{E}_D.$$



In this example we a.s. get

$$\dim_H \tilde{E} = 1 + \log_3 \frac{3}{2}.$$

Here the second term is the a.s. dimension of a random Cantor set in the line.

What is the conformal dimension of SLE_κ trace?

SLE

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Conjecture.

SLE_κ trace is a.s. minimal.

$$\dim_C(SLE_\kappa) = \min(1 + \kappa/8, 2)$$

Conformal dimension of McMullen sets and fractal percolation clusters

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Theorem. (B., Hakobyan, Li'22)

Let $B(t)$ be standard one-dimensional Brownian motion, $\Gamma_B = \{(t, B(t))\}$ be its graph. Then almost surely it is minimal

$$\dim_C \Gamma_B = \dim_H \Gamma_B = \frac{3}{2}.$$

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$$\mathrm{mod}_p(\Gamma, X, \mu) > 0 \implies \mathrm{mod}_q(\Gamma, X, \mu) > 0$$

An application: lower bound on Conformal dimension

Theorem. (Tyson/Bishop-Tyson, '01)

Let metric measure space (X, d, μ) satisfies the following conditions

- *It is doubling: there exists a constant C so that*

$$\mu(B(x, 2r)) \leq C\mu(B(x, r))$$

  for all balls $B(x, r) \subset X$.
- *μ is q-smooth: for all $B(x, r) \subset X$ $\mu(B(x, r)) \leq Cr^q$.*
- *There is a curve family Γ in X s.t.*

$$\mathrm{mod}_p\Gamma > 0 \text{ for some } 1 < p \leq q.$$

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Corollary (Bishop-Tyson, '01)

$(0, 1) \times Y$ is minimal for every $Y \subset \mathbb{R}^n$ or if Y is Ahlfors regular.

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Fuglede modulus.

- Let (X, d, μ) be a metric measure space and $\Lambda = \{\lambda\}$ be a collection of measures such that every μ-measurable set is λ-measurable for all $\lambda \in \Lambda$.
- A Borel (or continuous) function $\rho : X \to [0, \infty]$ is admissible for Λ, or $\rho \wedge \Lambda$ if
$$\int \rho d\lambda \geq 1$$
for every $\lambda \in \Lambda$
- For $p \geq 1$ define p-modulus of Λ as
$$\mathrm{mod}_p(\Lambda, X, \mu) = \inf_{\rho \wedge \Lambda} \int_X \rho^p d\mu.$$
- As for curves, if $p < q$
$$\mathrm{mod}_p(\Lambda, X, \mu) > 0 \implies \mathrm{mod}_q(\Lambda, X, \mu) > 0$$

Main technical tool

Theorem. (Hakobyan'08)

Let (X, d, μ) be a doubling compact,metric measure space with q-smooth measure μ.
Assume that there exists a collection $\mathcal{E}$ of subsets of X with

$$\dim_C E \geq 1, \ \forall \ E \in \mathcal{E}.$$

Assume also that there exists a collection of measures $\Lambda = \{\lambda_E\}_{E\in\mathcal{E}}$ and $C > 0$, such that for all $E \in \mathcal{E}$

1. $\operatorname{supp} \lambda_E = E$.
2. *There is $C > 0$ such that for all $x \in E$ we have*
$$\lambda_E(B(x, r) \cap E) \geq C \cdot r.$$
3. $\mathrm{mod}_p(\mathbf{E}) > 0$ *for some $1 \leq p < q$*

Then $\dim_C(X) \geq q$.

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The condition 2 can be relaxed (BHL '22) to
$\forall x \in E: \ \dim_{loc} \mu(x) := \limsup \frac{\log \lambda_E(B(x,r))}{\log r} \leq 1.$

Constructing measure μ.

- Let $x = \sum_{j=1}^{\infty} \frac{x_j}{m^j}$ be the base m representation of x. Let $K := \{x \in [0,1] \ : \ \lim_{n\to\infty} \frac{\# x_j = k,\ j \le n}{n} = \frac{1}{m},\ k = 0, 1, \dots, m-1\}$.

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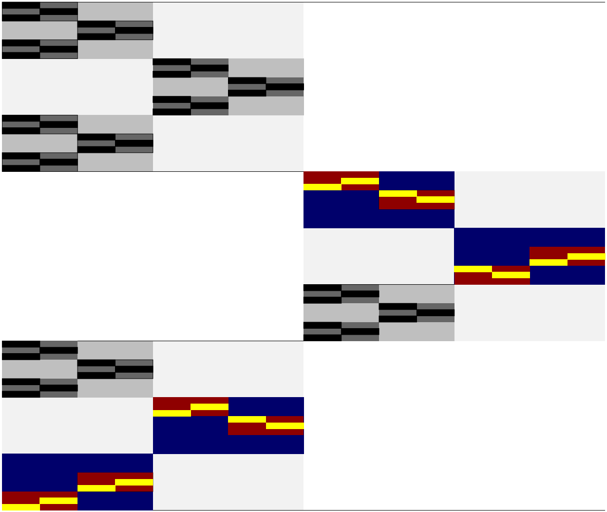
- $\dim X_0 = 1 + \log \frac{r_1 + r_2 + \ldots r_m}{m}$. Again: need it to be minimal.

Horizontal sets



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Lemma.

For any horizontal set E, $E \cap X_0$ is minimal.

- Take an admissible function $\rho(x, y)$ for Λ. Modify it to $\rho_1(x, y)$ by replacing $\rho(x, y)$ on any first-level rectangle by the values of ρ on the rectangle R in the same column with the smallest integral $\int_R \rho(x, y)\, d\mu(x, y)$.
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- Take $\rho_\infty(x,y)$ to be the weak limit of $\rho_k(x,y)$. Then $\rho_\infty(x,y) = \rho_\infty(x)$ does not depend on y and still admissible for Λ.

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Brownian motion and local time

Let B be a 1-dimensional Brownian motion.

- Let $a_n < 0 < b_n$ with $a_n \to 0$ and $b_n \to 0$. The local time of B is an increasing stochastic process defined on zeros of B in the following way:

$$L(t) = \lim_{n\to\infty} 2(b_n - a_n) D(a_n, b_n, t)$$

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where $D^n(a, t)$ is the number of downcrossings before time t of the n^{th}-dyadic interval containing a.
- Almost surely, L^a defines measures of dimension $\frac{1}{2}$ for all a. In particular, almost surely
$$\dim\{t : B(t) = a\} = \frac{1}{2} \text{ for every } a.$$

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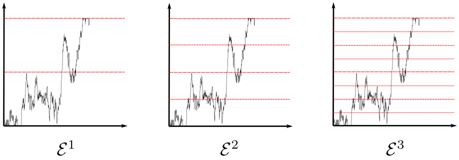

Figure: A standard linear Brownian motion and the collection of $E_j^{n,m}$ for 3 generations.

We construct a Cantor set E in the following way:

- Pick subset of Γ_B that lies between two adjacent hitting times of $Z_0, Z_{1/2}$ and another subset that lies between two adjacent hitting times of $Z_{1/2}, Z_1$ to be the first two elements of E.

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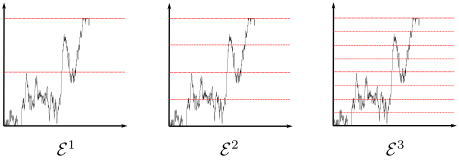

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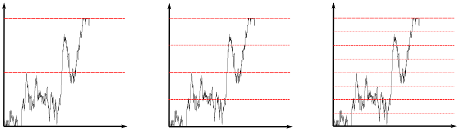

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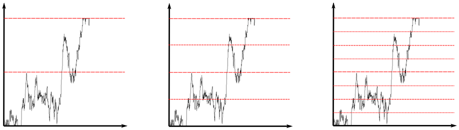

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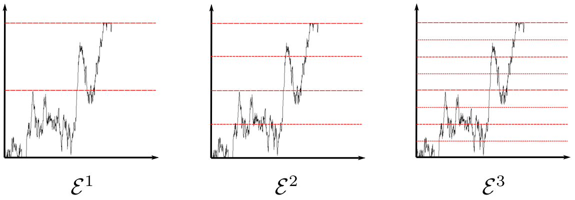
- Let $\mathcal{E}$ be the collection of all the Cantor sets constructed this way. Λ will be collection of vertical lengths on sets from $\mathcal{E}$.

- There exists a measure μ s.t. $\mu(B(x,r) \cap \Gamma(B)) \lesssim r^{\frac{3}{2}-\epsilon}$ for any $\epsilon > 0$.
- For any $E \in \mathcal{E}, \dim_C E \geq 1$.

The only thing left is to prove that $\mathrm{mod}_1(\Lambda) > 0$.



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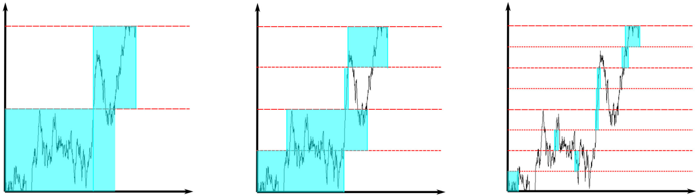


Figure: Constructing ρ_n.

- Let ρ be admissible for Λ. We replace ρ by an alternative admissible ρ_∞ with smaller mass.

The modulus is positive.

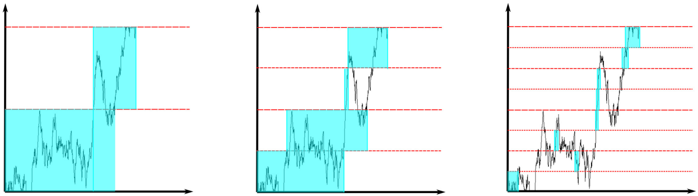
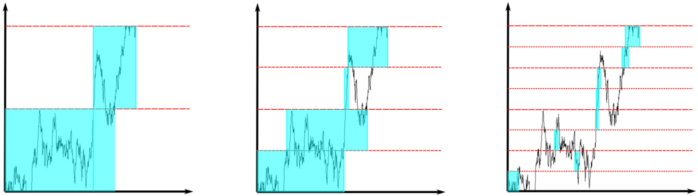


Figure: Constructing ρ_n.

- Let ρ be admissible for Λ. We replace ρ by an alternative admissible ρ_∞ with smaller mass.
- Let ρ_1 be a function on E_1 such that the integral of ρ_1 on every 1^{st}-generation element that achieves the minimal among all the 1^{st}-generations of the same level.

The modulus is positive.



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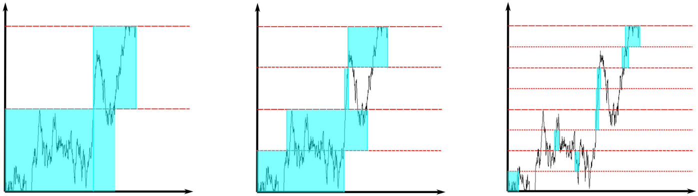
- Similarly, define ρ_n iteratively in the same way to cover all the n^{th}-generation.

The modulus is positive.



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- Similarly, define ρ_n iteratively in the same way to cover all the n^{th}-generation.
- Finally, we define $\rho_\infty := \liminf_{n\to\infty} \rho_n$.

The modulus is positive.

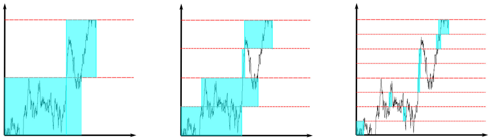


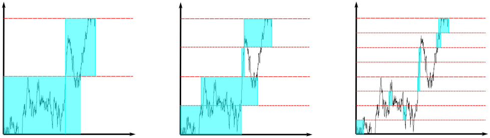
Figure: A minimal graph with 3 generations.

- For any $E \in \mathcal{E}$, one can find $F \in \mathcal{E}$ such that

$$\int_E \rho_\infty d\lambda_E \geq \int_F \rho\lambda_F \geq 1.$$

Thus ρ_∞ is admissible for Λ.

The modulus is positive.



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- $\int_X \rho d\mu \leq \int_X \rho_\infty d\mu$ where X is the space that $\mathcal{E}$ covers.

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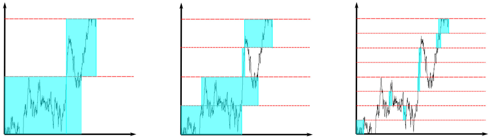

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- For any $E \in \mathcal{E}$, one can find $F \in \mathcal{E}$ such that

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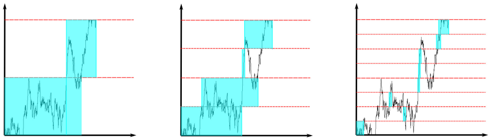
Thus ρ_∞ is admissible for Λ.

- $\int_X \rho d\mu \leq \int_X \rho_\infty d\mu$ where X is the space that $\mathcal{E}$ covers.
- It is sufficient to prove that $\int_X \rho_\infty d\mu > 0$.

The modulus is positive.



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- For any $E \in \mathcal{E}$, one can find $F \in \mathcal{E}$ such that

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  Thus ρ_∞ is admissible for Λ.
- $\int_X \rho d\mu \leq \int_X \rho_\infty d\mu$ where X is the space that $\mathcal{E}$ covers.
- It is sufficient to prove that $\int_X \rho_\infty d\mu > 0$.
- Recall that $\mu(B(x,r)) = \int_0^1 l_a(B(x,r) \cap Z_a) da$.
- $\int_X \rho_\infty d\mu \geq \left(\int_E \rho_\infty d\lambda_E\right) \inf_{a\in[0,1]} L^a(Z_a \cap X) \geq \inf_{a\in[0,1]} L^a(Z_a \cap X) > \delta > 0$.

Thank you!