Conformal Field Theory for Multiple SLE

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Kang-Makarov Conformal Field Theory

- Similar to physicists but with more geometry and complex analysis
- GFF is a convenient tool for representing **correlation functions**
- *Two-point example:*

$$G(z_1, z_2) = \mathbb{E}\Phi(z_1)\Phi(z_2) - \mathbb{E}\Phi(z_1)\mathbb{E}\Phi(z_2)$$

- *Multi-point example:*

$$\mathbb{E}[\Phi_{\boldsymbol{\beta}}(z_1)\bar{\partial}\Phi_{\boldsymbol{\beta}}(z_2)(\partial\Phi_{\boldsymbol{\beta}}(z_3) \odot \Phi_{\boldsymbol{\beta}}(z_4))e^{\odot i(\Phi_{\boldsymbol{\beta}}(z_5) - \Phi_{\boldsymbol{\beta}}(z_6))}]$$

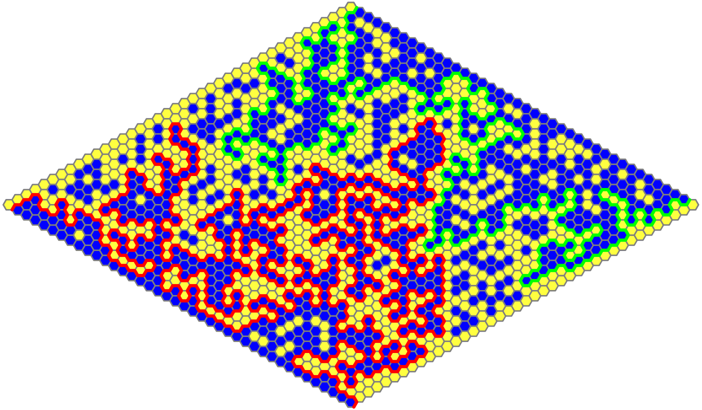
- Calculated using **Wick theorems** or **operator product expansion**
- Calculus for perturbative theory: **Ward equations**, **stress tensor**

Kang-Makarov Conformal Field Theory

- **Inspirations:**
  - Cardy
  - Bauer-Bernard, Bauer-Bernard-Kytölä
  - Rushkin-Bettelheim-Gruzberg-Wiegmann
- **Important aspect:** correlation functions and associated calculus are done in a *coordinate free* way
  - Unified language for CFT on Riemann surfaces
  - Or on annuli or multiply connected domains [Byun-Kang-Tak]
  - Can check and guess formulas via **dimension calculus**
- **Penultimate result:** an infinite family of correlation functions are martingale observables for an associated SLE process

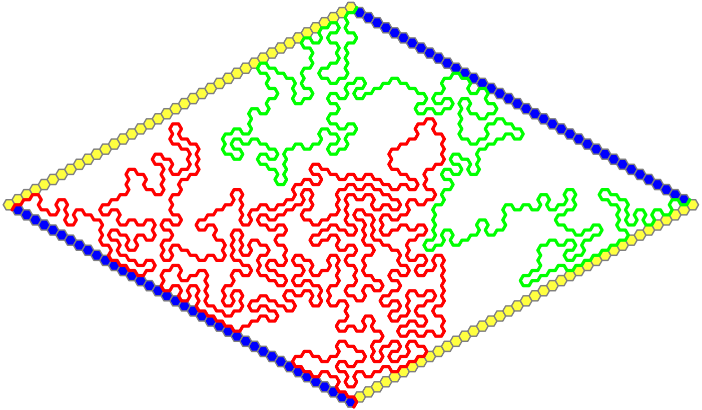
Multiple SLE

Multiple SLE



Two percolation interfaces in a domain with alternating boundary conditions

Multiple SLE



Two percolation interfaces in a domain with alternating boundary conditions

Multiple SLE

Global constructions
Weight the product measure of individual SLE curves by an appropriate factor. Iterated re-sampling methods.
[Lawler-Kozdron, Lawler, Peltola-Wu, Beffara-Peltola-Wu, . . .]

Local construction via Loewner flows
Grow multiple curves by adding singularities to the Loewner equation, vector of singularities is a multi-dimensional diffusion.
[Dubédat, Zhan, Bernard-Bauer-Kytölä, Kytölä-Peltola, . . .]

Conformal weldings of LQG surfaces
[Ang-Pu-Sun]

Main Results

Main Results of AKM

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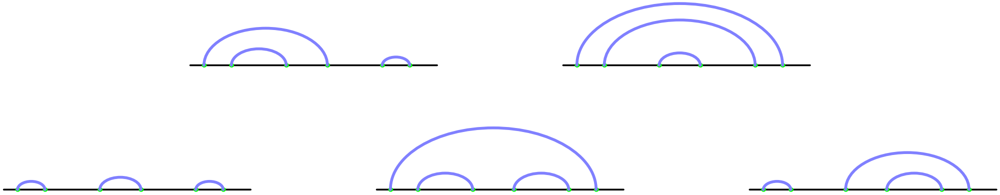
- Modification of Dubédat scheme for generating solutions to **BPZ equations** via method of screening (Coulomb gas integrals, Dotsenko-Fateev integrals)
  - Simplification of Flores-Kleban proof of linear independence of solutions
- Proof of **Dubédat commutation relations** via CFT methods (generalized Cardy equations + commutation of Lie derivatives)
- Infinite family of **screened martingale observables** for systems of multiple SLEs

Multiple SLE via Loewner

Loewner Construction of Multiple SLE

- **Goal:** In $\mathbb{H}$ with $x_1 < x_2 < \ldots < x_{2n}$, $x_i \in \mathbb{R}$, use Loewner flow to construct n SLE curves that connect x_i in a non-crossing way
  - *Pure law:* n curves almost surely connect according to fixed pattern
  - *Mixture:* Scaling limit of a discrete model with alternating BCs



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Dubédat's guiding principles

- Induced law on curves should not depend on which conformal coordinates of $\mathbb{H}$ are used to describe Loewner flow
- Induced law on curves should be invariant with respect to the order in which they are grown

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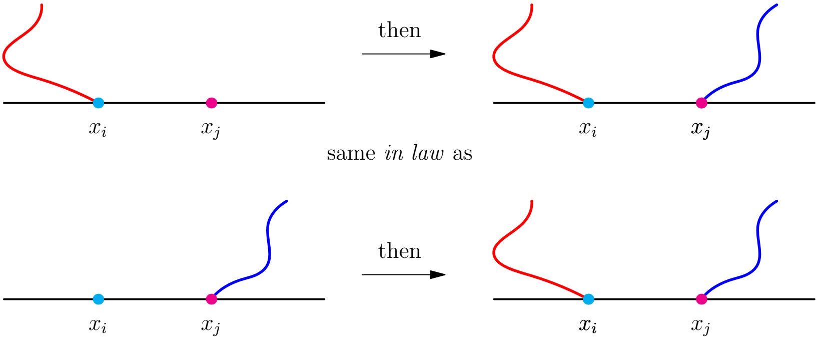

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  - *Pure law:* n curves almost surely connect according to fixed pattern
  - *Mixture:* Scaling limit of a discrete model with alternating BCs
- **Task:** Find functions $Z = Z(x_1, \ldots, x_{2n})$ satisfying

BPZ Equations

$$\frac{\kappa}{2}(\partial_{x_i} + \bar{\partial}_{x_i})^2 Z + \sum_{j \neq i}^{2n} \frac{2}{x_j - x_i}(\partial_{x_i} + \bar{\partial}_{x_i}) Z - \frac{6-\kappa}{\kappa} \frac{Z}{(x_i - x_j)^2} = 0, \quad i = 1, \ldots, 2n.$$

+ 3 additional PDEs called *conformal Ward identities*

BPZ Equations via Commutation Relations

- Ansatz: dynamics for a single curve

$$\partial_t g_t^{(i)}(z) = \frac{2}{g_t^{(i)}(z) - \xi_t^{(i)}}, \quad \xi_t^{(k)} = g_t^{(i)}(\xi_0^{(k)}), \quad \xi_0^{(i)} = x_i \in \mathbb{R}$$

$$d\xi_t^{(i)} = \sqrt{\kappa}\, dB_t^{(i)} + \kappa(\partial_{\mathbf{x_i}} \log \mathbf{Z})(\xi_t^{(1)}, \ldots, \xi_t^{(2n)})\, dt$$

- Multiple SLE is a **Girsanov reweighting** of independent SLE
- $\mathfrak{G}_{x_i}$ = the generator of $\xi_t = (\xi_t^{(1)}, \ldots, \xi_t^{(2n)})$ under the above dynamics
- Guiding principles + argument involving generators + algebra $\implies Z$ must satisfy BPZ + conformal Ward

BPZ Equations in Geometric Language

BPZ Equations in Coordinates

$$\overbrace{\left(\frac{\kappa}{2}(\partial_{x_i}+\bar{\partial}_{x_i})^2+\sum_{j\neq i}^{2n}\frac{2}{x_j-x_i}(\partial_{x_i}+\bar{\partial}_{x_i})-\frac{1-6/\kappa}{(x_i-x_j)^2}\right)}^{BPZ_i} Z=0,\quad i=1,\dots,2n.$$

BPZ Equations in Geometric Language

$Z=Z(x_1,\dots,x_{2n})$ is a $\left[\frac{6-\kappa}{2\kappa},0\right]$-differential in each x_i and

$$\left(\frac{\kappa}{2}\left(\partial_{x_i}+\bar{\partial}_{x_i}\right)^2-2\mathcal{L}_{k_{x_i}}\right)Z=0,\quad i=1,\dots,2n$$

Lie Derivative and Conformal Fields

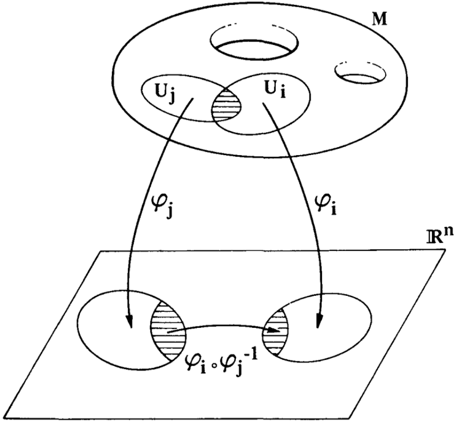
- $\mathcal{L}_v =$ Lie derivative operator with respect to vector field v
- Acts on conformal fields and returns conformal fields
- Conformal field $=$ function equipped with a transformation law under conformal changes of coordinates
- **Example:** $f = f(z)$ is a $[\lambda, 0]$-differential if

$$f = (h')^{\lambda} \tilde{f} \circ h, \quad h = \tilde{\phi} \circ \phi^{-1}$$

- Vector fields are $[-1, 0]$-differentials
- Lie derivative of a differential:

$$\mathcal{L}_v f = (v\partial + \lambda v')f$$

Lie Derivative and Conformal Fields



Credit: https://www.researchgate.net/profile/Vadim-Belov

Coordinate Free BPZ Equations

BPZ Equations in Geometric Language

$Z = Z(x_1, \ldots, x_{2n})$ is a $\left[\frac{6-\kappa}{2\kappa}, 0\right]$-differential in each x_i and

$$\left(\frac{\kappa}{2}\left(\partial_{x_i} + \overline{\partial}_{x_i}\right)^2 - 2\mathcal{L}_{k_{x_i}}\right) Z = 0, \quad i = 1, \ldots, 2n$$

k_ζ is the Loewner vector field defined by

$$k_\zeta(z) = \frac{1}{\zeta - z}$$

Solving BPZ: Method of Screening

Dubédat proposal: method of screening

- Also **Coulomb gas integrals**, **Dotsenko-Fateev integrals**, **Euler integrals**
- Dubédat introduces an *auxiliary function*

$$\varphi(x_1, \ldots, x_{2n}, \zeta_1, \ldots, \zeta_{n-1}) = \varphi(\boldsymbol{x}, \boldsymbol{\zeta})$$

which is a product of terms $x_i - x_j$, $\zeta_k - \zeta_\ell$, and $x_i - \zeta_k$, raised to κ dependent powers

- By explicit computation he shows

$$\mathfrak{BPZ}_i \varphi = \sum_{j=1}^{n-1} \partial_{\zeta_j} (f_{i,j} \varphi)$$

for explicit (and lengthy) rational functions $f_{i,j} = f_{i,j}(\boldsymbol{x}, \boldsymbol{\zeta})$

Dubédat Solutions to BPZ: method of screening

$$\mathfrak{BPZ}_i \varphi = \sum_{j=1}^{n-1} \partial_{\zeta_j} (f_{i,j} \varphi)$$

- Integrate $\zeta_1, \ldots, \zeta_{n-1}$ along **closed cycles** $\mathcal{C} = \mathcal{C}_1 \times \ldots \times \mathcal{C}_{n-1}$ to obtain

$$\mathfrak{BPZ}_i \oint_{\mathcal{C}} \varphi \, d\boldsymbol{\zeta} = \oint_{\mathcal{C}} \sum_{j=1}^{n-1} \partial_{\zeta_j} (f_{i,j} \varphi) \, d\boldsymbol{\zeta} = 0$$

- Hence $Z(\boldsymbol{x}) = \oint_{\mathcal{C}} \varphi(\boldsymbol{x}, \boldsymbol{\zeta}) \, d\boldsymbol{\zeta}$ is a solution
- **Summary:** Introduce *more* variables and integrate them out to solve BPZ

Dubédat Solutions to BPZ: method of screening

$$
\begin{aligned}
\varphi(\boldsymbol{x}, \zeta) = & \prod_{i=1}^{2N-1} (x_{2N} - x_i)^{1-6/\kappa} \prod_{j=1}^{N-1} (x_{2N} - \zeta_j)^{12/\kappa - 2} \\
& \cdot \prod_{1 \leqslant i < i' < 2N} (x_i - x_{i'})^{2/\kappa} \prod_{1 \leqslant j < j' < N} (\zeta_j - \zeta_{j'})^{8/\kappa} \\
& \cdot \prod_{\substack{1 \leqslant i < 2N \\ 1 \leqslant j < N}} (x_i - \zeta_j)^{-4/\kappa}
\end{aligned}
$$

Vertex Exponentials and CGCFs

- CGCF = **Coulomb gas correlation function**

$$\mathbb{E}\exp\left\{\sum_i \sigma_i \Phi(z_i)\right\} = e^{\sum_i \sigma_i \mathbb{E}\Phi(z_i)} \cdot \exp\left\{\sum_{i<j} \sigma_i \sigma_j \operatorname{Cov}(\Phi(z_i), \Phi(z_j))\right\}$$

$$\cdot \exp\left\{\sum_i \frac{\sigma_i^2}{2} \operatorname{Var}(\Phi(z_i))\right\}$$

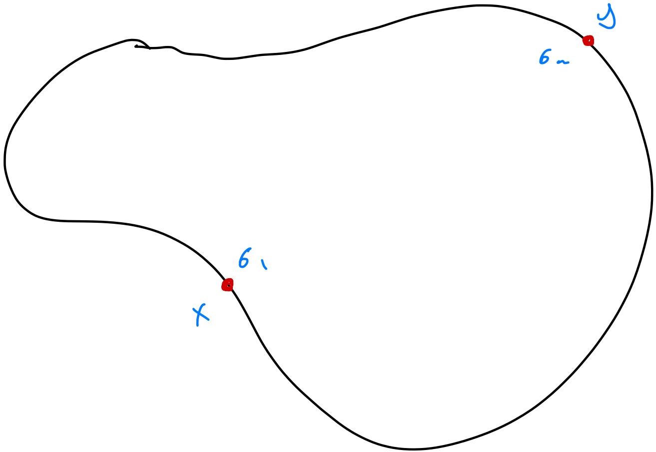
- Simply define

$$C\left[\sum_i \sigma_i \cdot z_i\right] = \prod_{i<j} (z_i - z_j)^{\sigma_i \sigma_j}$$

- $\prod_{i<j}(z_i - z_j)^{\sigma_i\sigma_j} \exp\{\odot \sum_i \sigma_i \Phi(z_i)\}$ satisfies **Ward's equation,** $\exp\{\odot \sum_i \sigma_i \Phi(z_i)\}$ does not

CGCFs in Pictures

$(x,y)^{\sigma_1,\sigma_2}$



Divisors

- CGCFs are characterized in terms of divisors
- Finite atomic measures on a Riemann surface

$$\boldsymbol{\beta} = \sum_i \boldsymbol{\sigma}_i \delta_{z_i} = \sum_i \boldsymbol{\sigma}_i \cdot z_i$$

- Write $|\boldsymbol{\beta}| = \sum_i \beta_i$
- β_i can be complex valued
- Terminology borrowed from algebraic geometry
- In the full theory, there is a second divisor for the anti-holomorphic part of the free field

CGCFs are Differentials

- Let $\beta = \sum_i \sigma_i \cdot z_i$ be a divisor on the Riemann sphere and $|\beta| = 2b$, $b \in \mathbb{R}$
- Define $C[\beta]$ in the identity chart of $\mathbb{C}$ by

$$C[\beta] \,||\, \mathrm{id} = \prod_{i<j} (z_j - z_i)^{\sigma_i \sigma_j}$$

- **Definition:** At each z_i, $C[\beta]$ is a $[\lambda_b(\sigma_i), 0]$-differential, where

$$\lambda_b(x) = \tfrac{1}{2} x^2 - xb$$

Theorem: Möbius Invariance of CGCFs

For any divisor β on the Riemann sphere

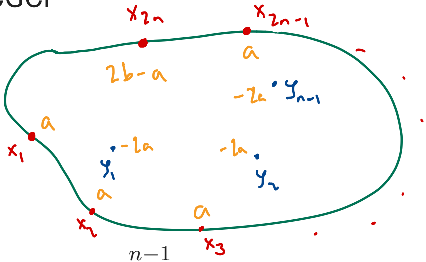
$$\mathcal{L}_{z^k} C[\beta] = 0, \quad k = 0, 1, 2$$

Numerology

- Each divisor $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| = 2b$ can describe SLE(κ) where

$$b = \sqrt{\kappa/8} - \sqrt{2/\kappa}, \qquad a = \sqrt{2/\kappa}$$

- Dubédat's $\varphi(\boldsymbol{x}, \boldsymbol{\zeta})$ is the CGCF



$(2n-1)a$
$+ 2b - a$
$+ (n-1)(-2a)$
$= 2b$

- The relation

$$\mathfrak{BPZ}_i \varphi = \sum_{j=1}^{n-1} \partial_{\zeta_j} (f_{i,j} \varphi)$$

is its BPZ equation at any x_i

BPZ Equations for CGCFs

- Points x with $\beta(x) = a$ are starting points of "locally" SLE(κ) curves
- At points x with $\beta(x) = a$, **level two degeneracy** occurs
- Leads to PDEs for CGCFs

BPZ Equations for CGCFs

$$\left(\frac{\kappa}{2}(\partial_x + \overline{\partial}_x)^2 - 2\mathcal{L}_{k_x}\right) C[\beta] = 0, \quad \text{wherever } \beta(x) = a$$

- Special case of more general principle

Explanation of Dubédat Relation

- Lie derivative of $C[\beta_{\text{Dub}}]$ involves a *Leibniz rule*

$$\mathcal{L}_v C[\beta_{\text{Dub}}] = \sum_{j=1}^{2n} \overbrace{\mathcal{L}_v(x_j)C[\beta_{\text{Dub}}]}^{BPZ_i} + \sum_{k=1}^{n-1} \overbrace{\mathcal{L}_v(\zeta_k)C[\beta_{\text{Dub}}]}^{\partial_{\zeta_k}(vC[\beta_{\text{Dub}}])}$$

- $\beta(\zeta_k) = -2a$ and $\lambda_b(-2a) = 1 \implies C[\beta_{\text{Dub}}]$ is a $[1,0]$-differential at ζ_k!
- Lie derivative of a $[1,0]$-differential:

$$\mathcal{L}_v f = (v\partial + 1v')f = \partial(vf)$$

- Relation for Dubédat φ is forced by geometry, not just algebra
- $[1,0]$-differential at $\zeta_k \implies$ integrals will also be coordinate free

General BPZ Equations

- Follows from **Ward equations + operator product expansions**
- Key step in proof that correlation functions are martingale observables

General BPZ Equations of Kang-Makarov

$$C[\beta]\mathbb{E}(T_\beta * Y)(x)\mathcal{X} = \mathbb{E}Y(x)\mathcal{L}_{k_x}(C[\beta]\mathcal{X})$$

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Flores-Kleban

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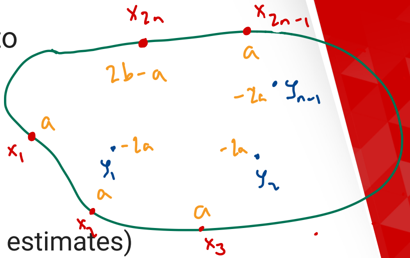
- $2b - a$ charge at x_{2n} is peculiar
- Causes technical difficulties in the Flores-Kleban construction of solutions Z
- **Flores-Kleban result:** Dimension of solution space to

$$\mathfrak{BPZ}_i Z = 0, \quad i = 1, \ldots, 2n$$

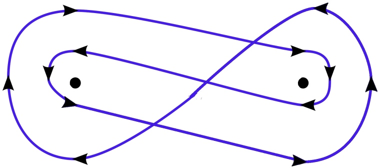
is exactly $C_n = \frac{1}{n+1}\binom{2n}{n}$

- *Upper bound:* PDE based methods (Green's function estimates)
- *Lower bound:* Explicitly construct C_n linearly independent solutions

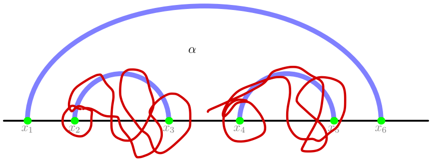


Flores-Kleban Argument

- Contours follow chords in the link pattern
- $\mathcal{C}_i$ is the **Pochhammer contour** surrounding endpoints of chord
- No contour for the chord connected to x_{2n}



Pochhammer

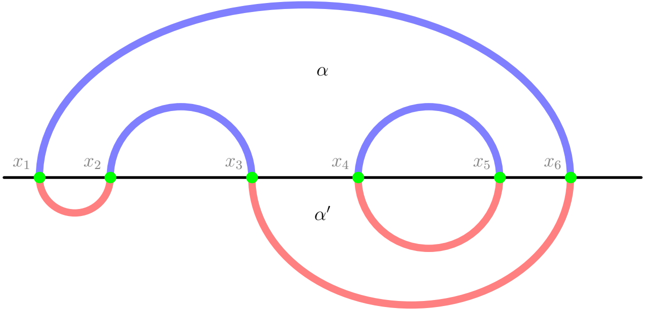


$$Z_{\mathrm{FK},\alpha}(\boldsymbol{x}) = \oint_{\mathcal{C}_1} \dots \oint_{\mathcal{C}_{n-1}} C[\boldsymbol{\beta}_{\mathrm{Dub}}]\, d\zeta_{n-1} \dots d\zeta_1$$

Flores-Kleban Argument

- Solutions characterized by boundary behavior as $x_{i+1} \to x_i$
- **Merging operators:**

$$\mathcal{L}_{\alpha'} Z = \lim_{x_{\iota_{2n}}, x_{\iota_{2n-1}} \to y_n} \cdots \lim_{x_{\iota_2}, x_{\iota_1} \to y_1} (x_{\iota_{2n}} - x_{\iota_{2n-1}})^{6/\kappa - 1} \cdots (x_{\iota_2} - x_{\iota_1})^{6/\kappa - 1} Z$$



Flores-Kleban Argument

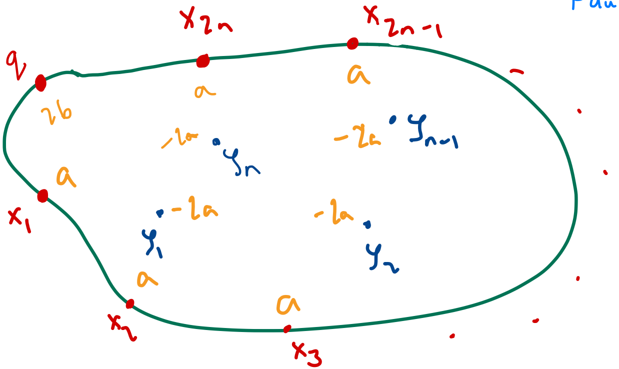
- $Z_{\mathrm{FK},\alpha}$ are linearly independent $\iff$ invertibility of the matrix

$$(\mathcal{L}_{\alpha'} Z_{\mathrm{FK},\alpha})_{\alpha,\alpha'}$$

- Flores-Kleban compute all entries, show it equals the **meander matrix**
- *Many* special cases for the Dubédat $2b - a$ charge
- We find a modification of φ to help with this

AKM Divisor

- **AKM:** add one more ζ_k and one more marked point



$$\beta_{dub} + (2a - 2b)\cdot x_{2n} - 2a \cdot y_n \boxed{+ 2b \cdot q}$$

AKM Screened Solutions: non-dependence on q

$$\mathcal{L}_v C[\boldsymbol{\beta}_{\mathrm{AKM}}] = \overbrace{\sum_{j=1}^{2n} \mathcal{L}_v(x_j) C[\boldsymbol{\beta}_{\mathrm{AKM}}]}^{\mathrm{BPZ}_i} + \overbrace{\sum_{k=1}^{n} \mathcal{L}_v(\zeta_k) C[\boldsymbol{\beta}_{\mathrm{AKM}}]}^{\partial_{\zeta_k}(vC)} + \overbrace{\mathcal{L}_v(q) C[\boldsymbol{\beta}_{\mathrm{AKM}}]}^{v\,\partial_q C}$$

- $\lambda_b(2b) = 0 \implies C[\boldsymbol{\beta}_{\mathrm{AKM}}]$ is a $[0,0]$-differential at q

Lemma

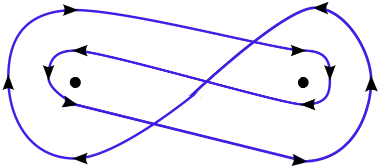
$$\partial_q C[\boldsymbol{\beta}_{\mathrm{AKM}}] = \sum_{k=1}^{n} \partial_{\zeta_k} C[\boldsymbol{\beta}_{\mathrm{AKM},k}]$$

for $\boldsymbol{\beta}_{\mathrm{AKM},k}$ obtained by swapping charges in $\boldsymbol{\beta}_{\mathrm{AKM}}$

- Implies a non-trivial relation for hypergeometric integrals/functions
- Is an artifact of CFT, not just algebra

AKM Screened Solutions

- Contours follow chords in the link pattern
- $\mathcal{C}_i$ is the **Pochhammer contour** surrounding endpoints of chord
- All n contours are used



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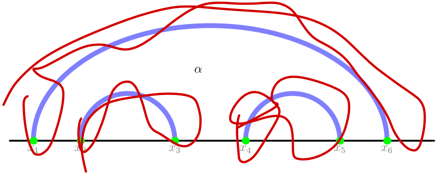
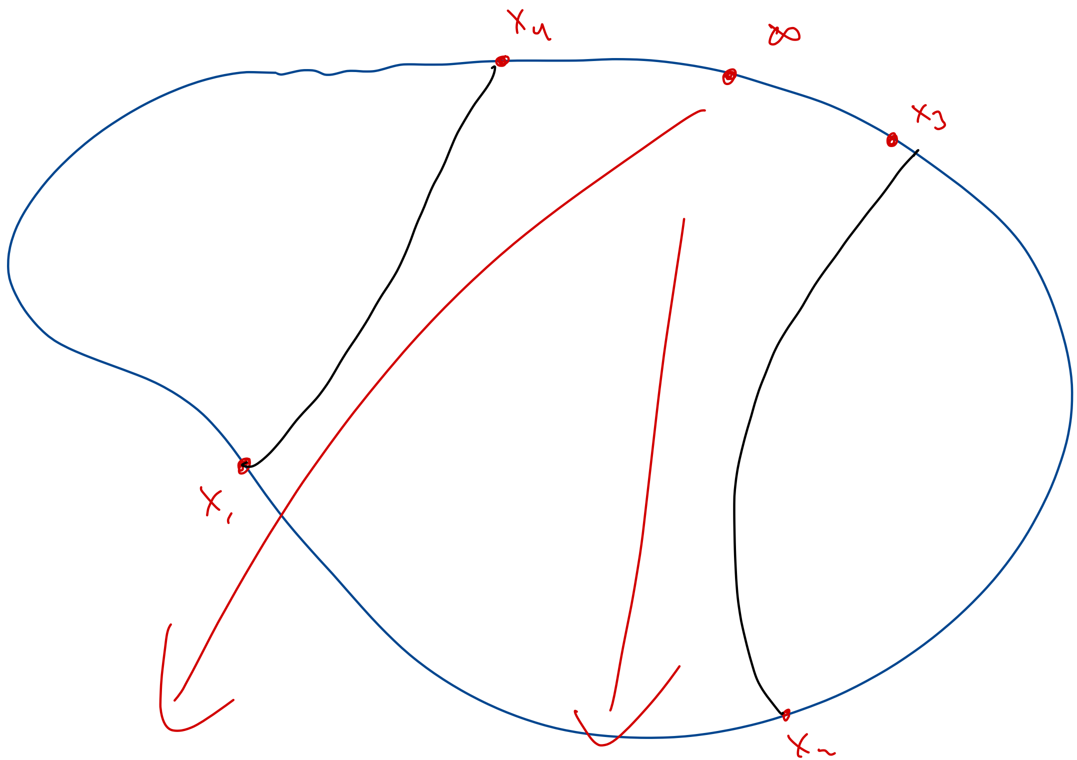


$$Z_{\mathrm{AKM},\alpha}(\boldsymbol{x}, q) = \oint_{\mathcal{C}_1} \cdots \oint_{\mathcal{C}_n} C[\boldsymbol{\beta}_{\mathrm{AKM}}]\, d\zeta_n \ldots d\zeta_1$$

An extra marked point is necessary to distinguish between patterns



Commutation Relations via CFT

BPZ Equations via Commutation Relations

- Ansatz: dynamics for a single curve

$$\partial_t g_t^{(i)}(z) = \frac{2}{g_t^{(i)}(z) - \xi_t^{(i)}}, \quad \xi_t^{(k)} = g_t^{(i)}(\xi_0^{(k)}), \quad \xi_0^{(i)} = x_i \in \mathbb{R}$$

$$d\xi_t^{(i)} = \sqrt{\kappa}\, dB_t^{(i)} + \kappa \psi_i(\xi_t^{(1)}, \ldots, \xi_t^{(2n)})\, dt$$

- $\mathfrak{G}_{x_i}$ = the generator of $\xi_t = (\xi_t^{(1)}, \ldots, \xi_t^{(2n)})$ under the above dynamics

Commutation Relations for Multiple SLE

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- $\mathfrak{G}_{x_i}$ = the generator of $\xi_t = (\xi_t^{(1)}, \ldots, \xi_t^{(2n)})$ under the above dynamics

Dubédat's Commutation Relations

$$[\mathfrak{G}_{x_i}, \mathfrak{G}_{x_j}] = \frac{4}{(x_i - x_j)^2}(\mathfrak{G}_{x_j} - \mathfrak{G}_{x_i})$$

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Commutation Relations for Fock Space Fields

Fock space fields are functionals of the Gaussian free field $\Phi : D \to \mathbb{R}$

Expected values are *deterministic* functions $D^n \backslash \mathrm{diag} \to \mathbb{C}$

Examples

$$(z_1, z_2) \mapsto \mathbb{E}[\Phi(z_1)\Phi(z_2)], \quad z \mapsto \mathbb{E}[(\partial\Phi * \partial\Phi)(z)], \quad z \mapsto \int_x^y \mathbb{E}[\Phi(z)e^{-\odot 2a\Phi(\zeta)}]\, d\zeta$$

Commutation Relations for Fock Space Fields

Fock space fields are functionals of the Gaussian free field $\Phi : D \to \mathbb{R}$

Expected values are *deterministic* functions $D^n \backslash \mathrm{diag} \to \mathbb{C}$

Examples

$$(z_1, z_2) \mapsto \mathbb{E}[\Phi(z_1)\Phi(z_2)], \quad z \mapsto \mathbb{E}[(\partial\Phi * \partial\Phi)(z)], \quad z \mapsto \int_x^y \mathbb{E}[\Phi(z)e^{-\odot 2a\Phi(\zeta)}]\, d\zeta$$

Commutation Relations for Fock Space Fields (AKM)

$$[\mathfrak{G}_{x_i}, \mathfrak{G}_{x_j}]\mathbb{E}\mathcal{X} = \frac{4}{(x_i - x_j)^2}(\mathfrak{G}_{x_i} - \mathfrak{G}_{x_j})\mathbb{E}\mathcal{X}$$

for $\mathcal{X} = X_1(z_1)\dots X_N(z_N)$ with $X_1, \dots, X_N$ arbitrary Fock space fields and $z_1, \dots, z_N$ distinct points in D

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Generalized Cardy Equations

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- AKM proof based on tools from **complex differential geometry** and:

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- $[\mathfrak{G}_x, \mathfrak{G}_y]$ converts to $[\mathcal{L}_{k_x}, \mathcal{L}_{k_y}] = \mathcal{L}_{[k_x,k_y]}$