Intersections of SLE paths

ICM 2014: Seoul, Korea

Recent Progress in Random Conformal Geometry

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Joint work with Jason Miller (MIT)

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Outline

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SLE (Schramm Loewner Evolution)

Random fractal curves in $D \subset \mathbb{C}$ from a to b. Candidates for the scaling limit of discrete Statistical Physics models.

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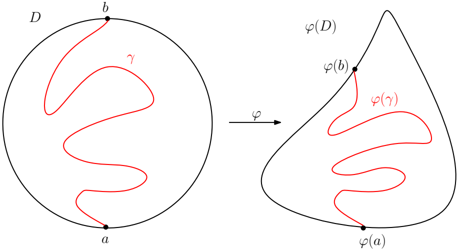
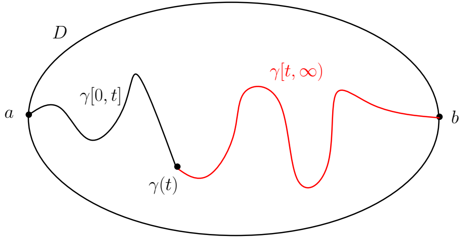


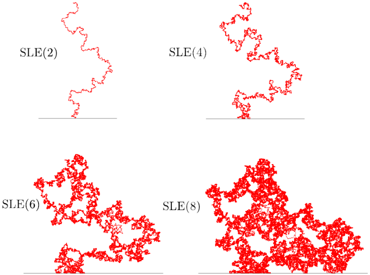
Conformal invariance :
If γ is in D from a to b, and $\varphi : D \to \varphi(D)$ conformal map, then $\varphi(\gamma) \overset{d}{\sim}$ the one in $\varphi(D)$ from $\varphi(a)$ to $\varphi(b)$.



Domain Markov property :
the conditional law of $\gamma[t, \infty)$ given $\gamma[0, t]$ $\overset{d}{\sim}$ the one in $D \setminus \gamma[0, t]$ from $\gamma(t)$ to b.

Examples of SLE

One parameter family of growing processes SLE_κ for $\kappa \geq 0$.
Simple, $\kappa \in [0, 4]$; Self-touching, $\kappa \in (4, 8)$; Space-filling, $\kappa \geq 8$.



Thanks to Tom Kennedy

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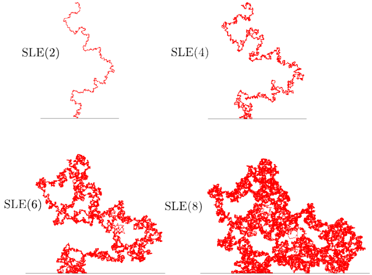
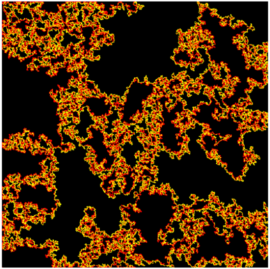
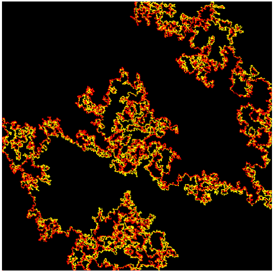


Thanks to Tom Kennedy

- $\kappa = 2$: LERW (Lawler, Schramm, Werner)
- $\kappa = 3$: Critical Ising (Smirnov, Chelkak et al.)
- $\kappa = 4$: Level line of GFF (Schramm, Sheffield, Miller)
- $\kappa = 6$: Percolation (Smirnov, Camia, Newman)
- $\kappa = 8$: UST (Lawler, Schramm, Werner)

SLE double point and cut point dimensions





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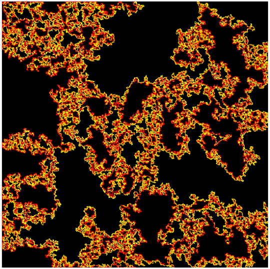

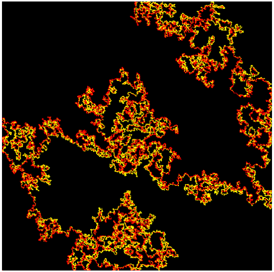

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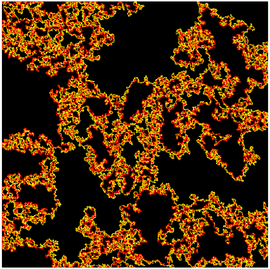
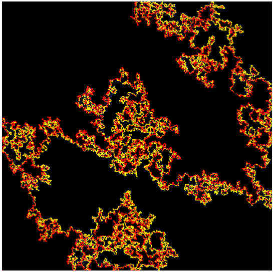
Proposition : [Miller, W.]
The Hausdorff dimension of the double points of SLE_κ is, almost surely,

$$1 + \frac{\kappa}{8} - \frac{6}{\kappa} \quad \text{for} \quad \kappa \in (4, 8)$$

$$1 + \frac{2}{\kappa} \quad \text{for} \quad \kappa \geq 8$$

SLE double point and cut point dimensions





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Proposition : [Miller, W.]
The Hausdorff dimension of the cut points of SLE_κ is, almost surely,

$$3 - \frac{3\kappa}{8} \quad \text{for} \quad \kappa \in (4, 8)$$

Consistence with previous results

$\kappa \in (4, 8)$
Double points

$$1 + \frac{\kappa}{8} - \frac{6}{\kappa}$$

Cut points

$$3 - \frac{3\kappa}{8}$$

Consistence with previous results

$\kappa \in (4, 8)$
Double points

$$1 + \frac{\kappa}{8} - \frac{6}{\kappa}$$

Cut points

$$3 - \frac{3\kappa}{8}$$

- **Critical percolation :** $\kappa = 6$
  double point dimension : $\frac{3}{4}$,
  predicted by Duplantier in 1987
  cut point dimension : $\frac{3}{4}$,
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- **Brownian excursion :**
  cut point dimension : $\frac{3}{4}$,
  proved by Lawler, Schramm, Werner in 2001
- **FK model :** $\kappa \in (4, 8)$
  double point dimension and cut point dimension,
  predicted by Duplantier in 1989 and 2004 respectively.

Relation with other dimensions

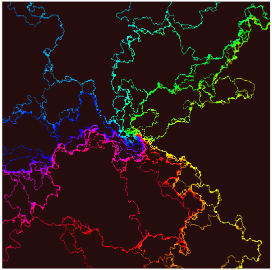
Beffara

Miller and Wu

SLE_κ	$\kappa \in (0,4]$	$\kappa \in (4,8)$	$\kappa \geq 8$
Trace	$1+\frac{\kappa}{8}$	$1+\frac{\kappa}{8}$	2
Double point	$\emptyset$	$1+\frac{\kappa}{8}-\frac{6}{\kappa}$	$1+\frac{2}{\kappa}$
Triple point	$\emptyset$	$\emptyset$	countable
Cut point	$1+\frac{\kappa}{8}$	$3-\frac{3\kappa}{8}$	$\emptyset$
Boundary point	$\emptyset$	$2-\frac{8}{\kappa}$	1

Alberts and Sheffield

Key in the proof



Key in the proof :
one-point estimate : martingale.
two-point estimate : coupling between SLE and GFF, work by Sheffield and Miller

- Imaginary Geometry I, II, III, IV

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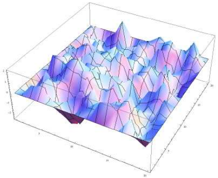
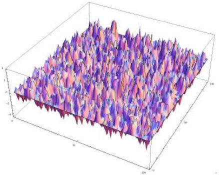
GFF (Gaussian Free Field)





Thanks to Miller, Sheffield

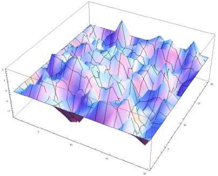
DGFF with mean zero : a measure h on functions $\rho : D \to \mathbb{R}$ and $\rho = 0$ on ∂D with density

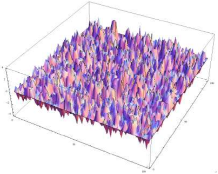
$$\frac{1}{\mathcal{Z}} \exp(-\frac{1}{2} \sum_{x \sim y} (\rho(x) - \rho(y))^2)$$

for $D \subset \mathbb{Z}^2$.

- For each vertex x, $h(x)$ Gaussian r.v.
- Covariance : Green's function for SRW
- Mean value : zero.

GFF (Gaussian Free Field)





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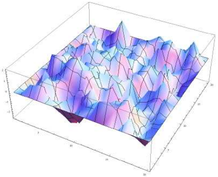
for $D \subset \mathbb{Z}^2$.

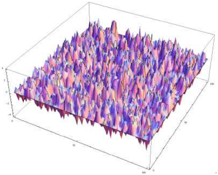
- For each vertex x, $h(x)$ Gaussian r.v.
- Covariance : Green's function for SRW
- Mean value : zero.

DGFF with mean h_∂ : DGFF with mean zero plus a harmonic function h_∂.

- For each vertex x, $h(x)$ Gaussian r.v.
- Covariance : Green's function for SRW
- Mean value : $h_\partial(x)$

GFF (Gaussian Free Field)





Thanks to Miller, Sheffield

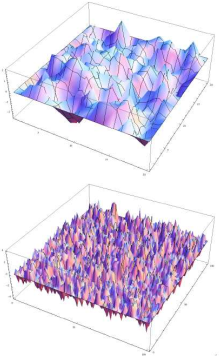
DGFF $\to$ GFF h

- (h, ρ) Gaussian r.v.
- Covariance :

$$cov((h, \rho_1), (h, \rho_2)) = \iint dxdy G_D(x,y)\rho_1(x)\rho_2(y).$$

- Mean value :
  $\mathbb{E}((h, \rho)) = (h_\partial, \rho)$.

GFF (Gaussian Free Field)



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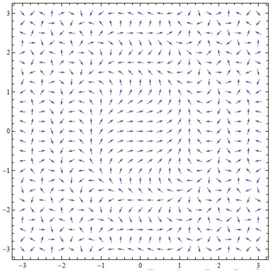
$$cov((h,\rho_1),(h,\rho_2)) = \iint dxdy G_D(x,y)\rho_1(x)\rho_2(y).$$

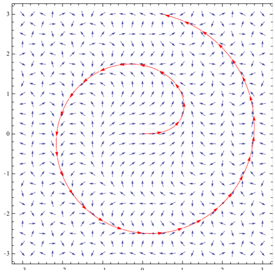
- Mean value :
  $\mathbb{E}((h,\rho)) = (h_\partial, \rho).$
- **Conformal invariance**
  Domain Markov Property

Flow lines of GFF



- h smooth, $\chi > 0$ constant. Vector field $e^{ih/\chi}$

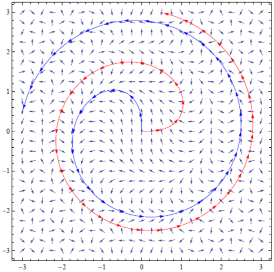
Flow lines of GFF



- h smooth, $\chi > 0$ constant. Vector field $e^{ih/\chi}$
- Flow line of the field :

$$\frac{d}{dt}\eta(t) = e^{ih(\eta(t))/\chi}$$

Flow lines of GFF



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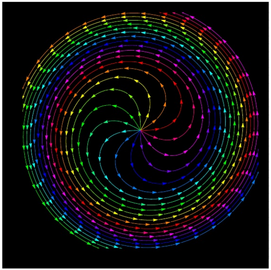
- Flow line of the field with angle θ : $h + \theta\chi$

Flow lines of GFF



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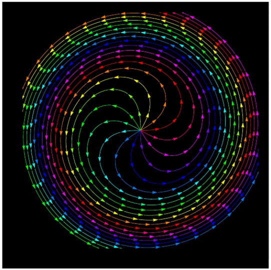
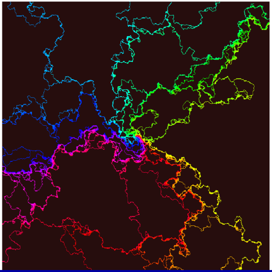
- Flow line of the field with angle θ : $h + \theta\chi$
- Property : monotonicity.

Flow lines of GFF





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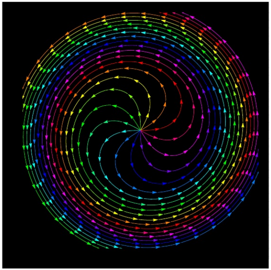
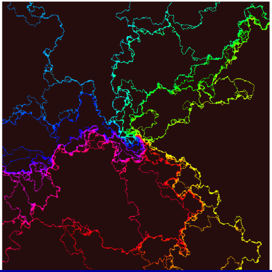
Flow lines of GFF





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- Flow line of the field with angle θ : $h + \theta\chi$
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- h GFF, "Vector field" $e^{ih/\chi}$
- Flow lines of the field are SLE_κ curves

$$\kappa \in (0,4), \quad \chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$$

Interactions of flow lines $\kappa \in (0,4), \chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$

Flow lines of $e^{ih/\chi}$ with angles θ_1 and θ_2 : η_1 and η_2

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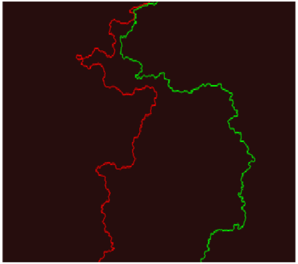

$\theta_1 > \theta_2$:
η_1 stays to the left of η_2, but may have intersection

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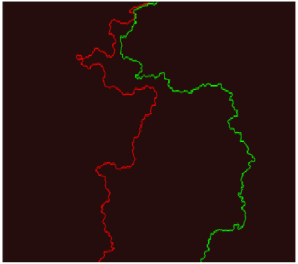

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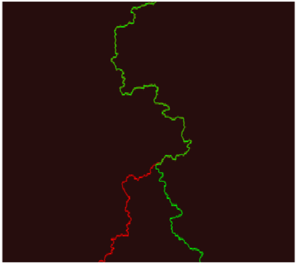

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η_1 merges with η_2 upon intersecting and never separates

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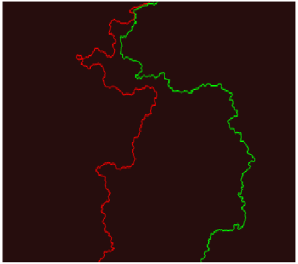

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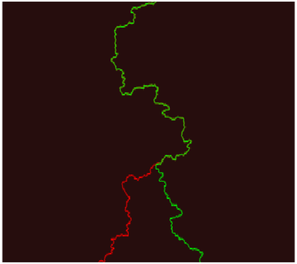

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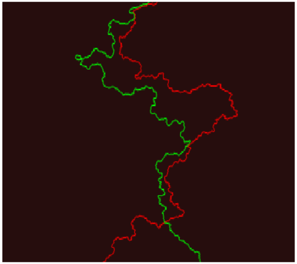
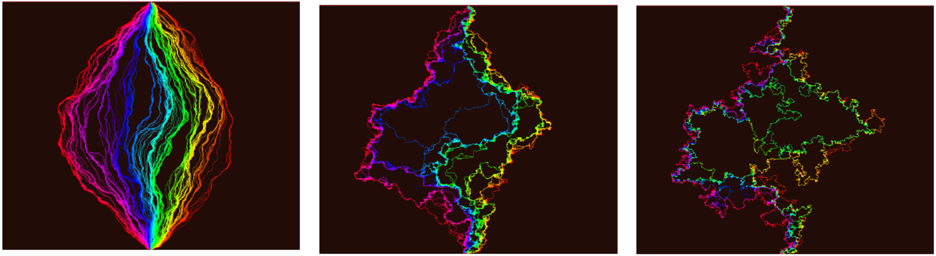


$\theta_1 < \theta_2$:
η_1 crosses η_2 upon intersecting and never crosses back

Simulations of the flow lines of GFF

$$\kappa \in (0,4), \quad \chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}, \quad \exp(ih/\chi)$$



$\kappa = 1/8$ $\kappa = 1$ $\kappa = 2$

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Intersection of flow line and the boundary

Proposition [Miller and W.]
$\eta \sim \text{SLE}_\kappa(\rho)$, $\kappa \in (0,4)$, $\rho \in (-2, \frac{\kappa}{2} - 2)$,

$$\dim_H(\eta \cap \mathbb{R}) = 1 - \frac{1}{\kappa}(\rho+2)\left(\rho + 4 - \frac{\kappa}{2}\right), \quad a.s.$$

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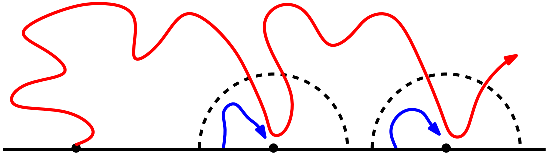

- one-point estimate : martingale.
- two-point estimate : Interaction of flow lines.

Intersection of two flow lines

Proposition [Miller and W.]

$\theta_1 < \theta_2$, $\eta_1 \sim$ angle θ_1, $\eta_2 \sim$ angle θ_2, $\rho = (\theta_2 - \theta_1)\chi/\lambda - 2$

$$\dim_H(\eta_1 \cap \eta_2 \cap \mathbb{H}) = 2 - \frac{1}{2\kappa}\left(\rho + \frac{\kappa}{2} + 2\right)\left(\rho - \frac{\kappa}{2} + 6\right), \quad \textit{a.s.}$$

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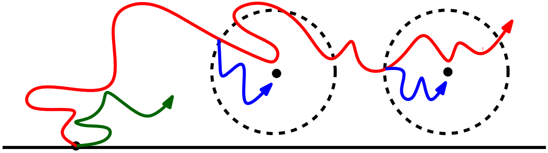

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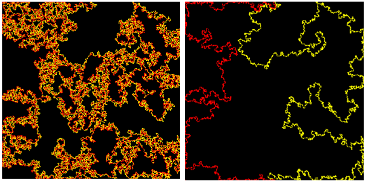
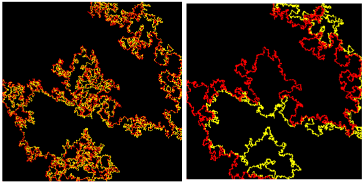
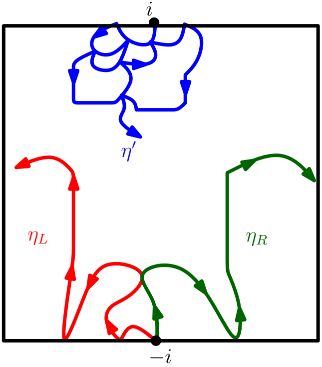
Cut point dimension–Duality







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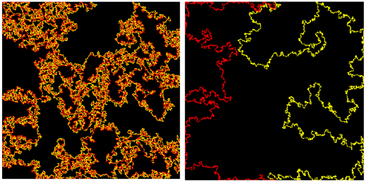
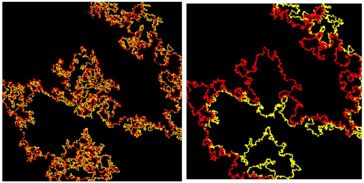
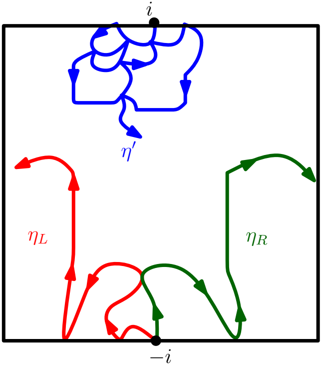
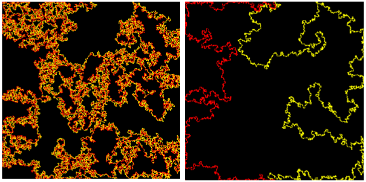


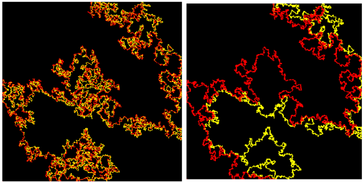


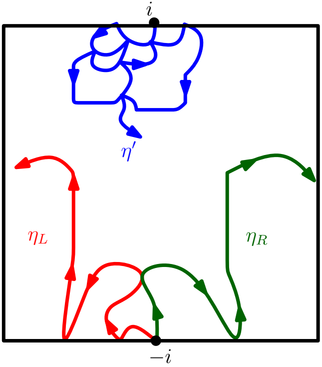


- $\eta' \sim \mathrm{SLE}_{\kappa'}$ from i to $-i$.
  $\kappa' \in (4, 8)$, $\kappa = 16/\kappa' \in (2, 4)$

Cut point dimension–Duality

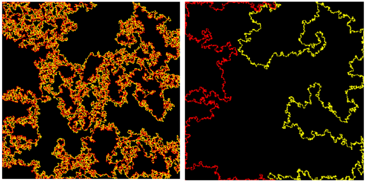


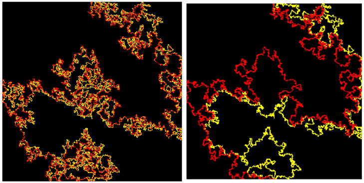


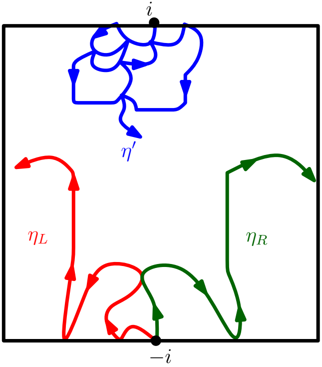


- $\eta' \sim \mathrm{SLE}_{\kappa'}$ from i to $-i$.
  $\kappa' \in (4,8)$, $\kappa = 16/\kappa' \in (2,4)$
- $\eta_L \sim$ left boundary of η'
  $\sim$ flow line with angle $\pi/2$

Cut point dimension–Duality







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- $\eta_R \sim$ right boundary of η'
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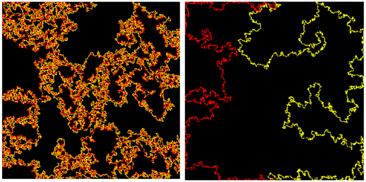
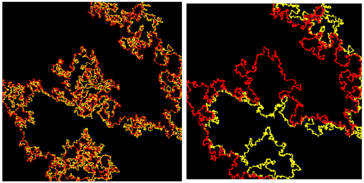
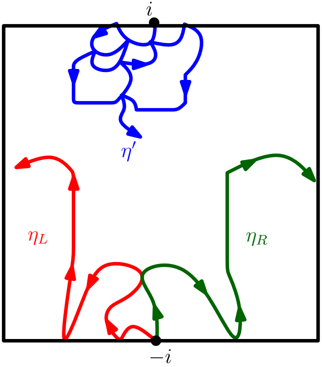
Cut point dimension–Duality







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- $\eta_L \sim$ left boundary of η'
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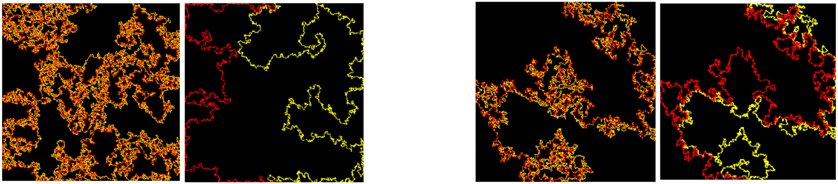
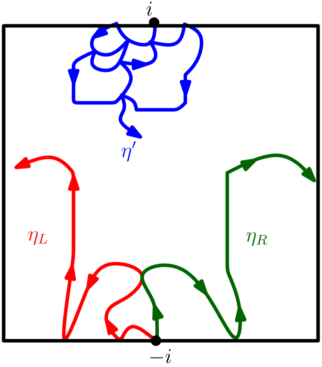
- The cut point set of η' is $\eta_L \cap \eta_R$.

Cut point dimension–Duality





- $\eta' \sim \mathrm{SLE}_{\kappa'}$ from i to $-i$.
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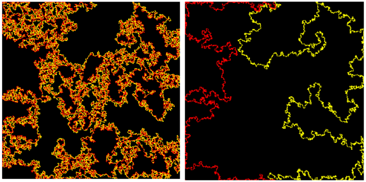
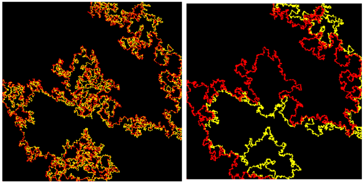
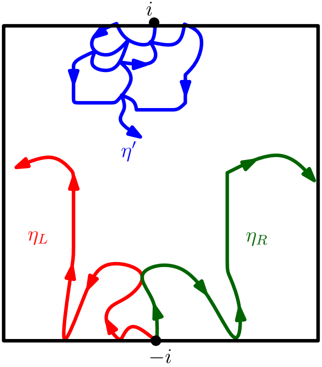
- The cut point set of η' is $\eta_L \cap \eta_R$.
- The angle difference is π

Cut point dimension–Duality







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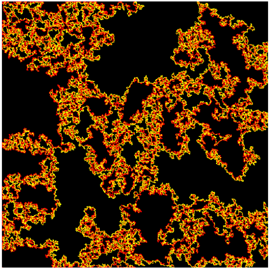
  $\sim$ flow line with angle $-\pi/2$
- The cut point set of η' is $\eta_L \cap \eta_R$.
- The angle difference is π
  $\rightarrow$ cut point dimension.

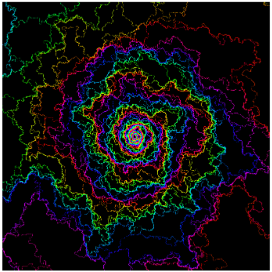
Miscellanies

- Dimension for the double points for $\kappa > 4$.

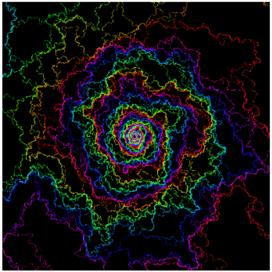


Miscellanies

- Dimension for the double points for $\kappa > 4$.
- Radial $\text{SLE}_\kappa(\rho)$, $\kappa \in (0, 4)$, $\rho \in (-2, \kappa/2 - 2)$

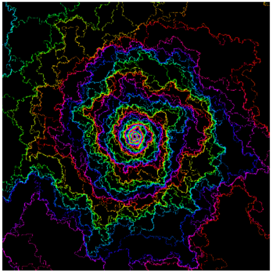


Miscellanies



- Dimension for the double points for $\kappa > 4$.
- Radial SLE$_\kappa(\rho)$, $\kappa \in (0, 4)$, $\rho \in (-2, \kappa/2 - 2)$
  B_j : the points on the boundary that the curve hits j times.

Miscellanies



- Dimension for the double points for $\kappa > 4$.
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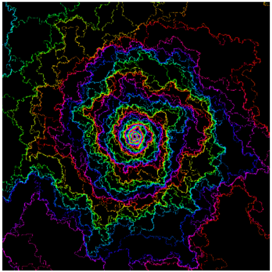
  B_j : the points on the boundary that the curve hits j times.
  ✓ $\dim_H(B_j)$

Miscellanies



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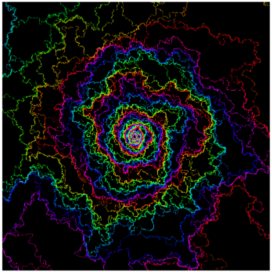
  B_j : the points on the boundary that the curve hits j times.
  ✓ $\dim_H(B_j)$
  I_j : the points in the interior that the curve hits j times.

Miscellanies



- Dimension for the double points for $\kappa > 4$.
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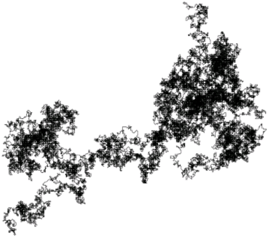
  B_j : the points on the boundary that the curve hits j times.
  ✓ $\dim_H(B_j)$
  I_j : the points in the interior that the curve hits j times.
  ✓ $\dim_H(I_j)$

Miscellanies



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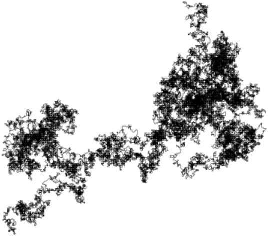
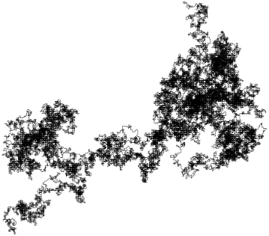
  I_j : the points in the interior that the curve hits j times.
  ✓ $\dim_H(I_j)$
- K : Conformal restriction sample with exponent β

Miscellanies

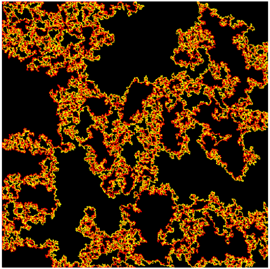


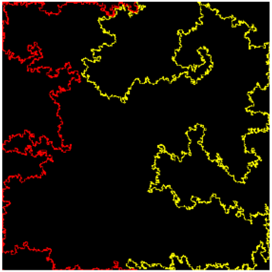
- Dimension for the double points for $\kappa > 4$.
- Radial SLE$_\kappa(\rho)$, $\kappa \in (0,4), \rho \in (-2, \kappa/2 - 2)$
  B_j : the points on the boundary that the curve hits j times.
  ✓ $\dim_H(B_j)$
  I_j : the points in the interior that the curve hits j times.
  ✓ $\dim_H(I_j)$
- K : Conformal restriction sample with exponent β
  $C(K)$: the cut points of K

Miscellanies



- Dimension for the double points for $\kappa > 4$.
- Radial SLE$_\kappa(\rho)$, $\kappa \in (0,4), \rho \in (-2, \kappa/2-2)$
  B_j : the points on the boundary that the curve hits j times.
  ✓ $\dim_H(B_j)$
  I_j : the points in the interior that the curve hits j times.
  ✓ $\dim_H(I_j)$
- K : Conformal restriction sample with exponent β
  $C(K)$: the cut points of K
  ✓ $\dim_H(C(K))$





Thanks !