Lattice representations of the Virasoro algebra

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Recent Progress in Random Conformal Geometry
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Joint work with C. Hongler (EPFL) and K. Kytölä (Helsinki).

Plan:

- Background + motivation
- The model
- Statement
- Main ingredients of the proof

Several figures borrowed!

Attempts to understand concretely and mathematically (probabilistically) one aspect of a standard (typically non-rigorous) description of conformally invariant continuum limits of critical lattice models.

Critical percolation, Ising model, loop-erased random walk, self-avoiding walks, other O(n) models, etc...

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Physicists [BPZ '84, ...]: model using certain 2D field theories with conformal invariance built in: Conformal Field Theories (CFTs).

Limiting model described by correlation functions of a collection of local "fields" / "operators" indexed by points. Nice transformation rules with respect to conformal maps. Formal (Laurent) series expansions of products, etc.

Several algebraic structures are important in this approach.

The Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_0(m+n).$$

A Lie algebra that appears when studying how certain fields are affected by infinitesimal transformations.

The CFTs form a one-parameter family indexed by the central charge $c \in \mathbb{R}$.

CFTs corresponding to lattice models (typically) turn out to be particularly simple: Finite number of "primary" fields which are conformally covariant, and which generate all the other fields of the model.

Representation theory arguments produce linear relations between L_n:s paired with suitable fields; can be written as linear differential equations for the correlation functions.

→ predictions for correlation function formulas, various critical exponents, classifications, etc, etc.

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A lot of progress, in particular in two directions...

- SLE interpretations of CFT statements (in the continuum); $c(\kappa) = (3\kappa - 8)(6 - \kappa)/2\kappa$, restriction, loops,...;
- Discrete complex analysis (prove CFT formulas hold in the scaling limit; transfer matrix)
- ...

(LSW, Friedrich-Werner, Bauer-Bernard, Kang-Makarov, Dubedat, Kenyon, Smirnov, Hongler, -Kytölä, ...)

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- Is there a direct connection between the discrete and continuum structures that allow for "exact solvability"?

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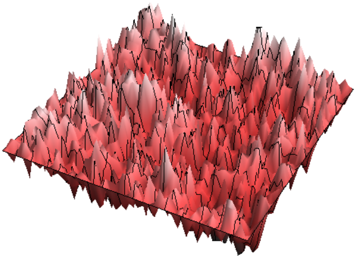
- E.g., are the algebraic structures that are important in CFT also present on the discrete level?
- Is there a direct connection between the discrete and continuum structures that allow for "exact solvability"?

Preferably with simple and concrete interpretations, constructions, etc. Also to clarify the source of these structures. We are looking for exact results.

Model: discrete Gaussian field on $\mathbb{Z}^2$. Random surface model.



We work with $\phi(z)$, a "pinned" mean 0 real Gaussian field indexed by the vertices of $\mathbb{Z}^2$ such that $\phi(0) = 0$. The correlation functions are:

$$\left\langle \phi(z)\phi(w) \right\rangle = G(z,0) + G(0,w) - G(z,w).$$

Here $G(z,w) = G(w,z) := a(z-w)$, where a is the free Green's function (or "potential kernel") for random walk on $\mathbb{Z}^2$ with

$$[\Delta a](z) = \delta_0(z);$$

$$a(z) = \frac{2}{\pi} \log |z| + O(1), \quad |z| \to \infty;$$

$$[\Delta f](z) = \frac{1}{4} \sum_{w \sim z} [f(w) - f(z)] .$$

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Extend (a and ϕ) to vertices of dual graph $\mathbb{Z}^2_*$ by setting 0 there $\Rightarrow$ definition on vertices of diamond graph $\mathbb{Z}^2_\diamond = \mathbb{Z}^2 \cup \mathbb{Z}^2_*$.

We get the multipoint correlations from the two-point functions via Wick's formula: if X_j are centered Gaussians, then

$$\left\langle X_1 \cdots X_n \right\rangle = \sum \prod_k \left\langle X_{i_k} X_{j_k} \right\rangle,$$

where the sum is over all partitions of $\{1, \ldots, n\}$ into disjoint pairs (i_k, j_k).

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We work with a vector space of measures.

- We call (complex) Gibbs measures on $\mathbb{Z}^2$ assignments of a complex Borel measure $\mu|_{\mathcal{G}}$ on $\mathbb{R}^{|\mathcal{G}|}$ to each finite $\mathcal{G} \subset \mathbb{Z}^2$ with the natural compatibility condition. The law of ϕ on $\mathbb{Z}^2$ gives such a measure, μ_{GFF}. (Write also $\mu_{GFF}|_{\mathcal{G}}$.)

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- If $\nu|_{\mathcal{G}} << \mu|_{\mathcal{G}}$ for all finite subgraphs $\mathcal{G}$, then ν is called a change of measure of μ.
- If $\mathcal{M}$ is a linear space of complex Gibbs measures, we call a linear operator $T : \mathcal{M} \to \mathcal{M}$ such that $T\mu$ is a change of measure of μ a change of measure operator.

Thm. There is a space $\mathcal{M}$ of changes of measure of μ_{GFF} and for $n \in \mathbb{Z}$ explicit change of measure operators $L_n : \mathcal{M} \to \mathcal{M}$ that yield a representation of the Virasoro algebra of central charge $c = 1$, i.e.,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\mathrm{Id}_{\mathcal{M}}.$$

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We will give the main ideas for the construction of the L_n. Easier to work on the level of "insertions", i.e., directly with correlation functions. Then we will "lift" to $\mathcal{M}$.

Basic idea: define discrete version of "modes", a_n:s, of the current $J(z) := \partial\phi$ acting by contour integrals:

$$\text{“}\Big\langle \mathfrak{a}_n(0)\phi(z_1)\cdots\phi(z_k)\Big\rangle = \oint_{\mathcal{C}} \Big\langle J(z)\phi(z_1)\cdots\phi(z_k)\Big\rangle z^n dz\text{”}.$$

(By inserting polynomials, defines a measure using L^p space duality.

Look at commutations of a_n:s, then build L_n:s by summing suitable "products" of a_n:s. Similar constructions (in the continuum) appear in the physics literature – Sugawara construction.)

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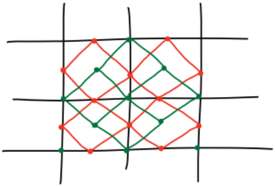
To discretize: Need four ingredients: discrete - derivatives, current, monomials, contour integrals (+ formulas).

The diamond graph, $\mathbb{Z}^2_\diamond = \mathbb{Z}^2 \cup \mathbb{Z}^2_*$ (green). The medial graph, $\mathbb{Z}^2_m$ (red) is the dual.



Ingredient 1: If f defined on either of these,

$$2[\partial f](z) := f\left(z + \frac{1}{2}\right) - f\left(z - \frac{1}{2}\right) - i\left[f\left(z + \frac{i}{2}\right) - f\left(z - \frac{i}{2}\right)\right].$$

If $[\bar{\partial} f](z) = 0$, then f is called discrete holomorphic at z.

We call $J(z) := [\partial\phi](z)$ the "current". (z is on the medial graph.)

Ingredient 2: discrete monomials. Duffin, Mercat, and others. For $n \geq 0$ we define $z^{[n]}$ living on the medial and dual medial graphs: start with

$$z^{[0]} \equiv 1, \; z^{[1]} = z.$$

Then define higher powers by successive discrete integration so that

$$\partial z^{[n]} = n z^{[n-1]}$$

with $z^{[n]}$ discrete holomorphic for each $n \geq 0$. Straight-forward, some normalizations needed. Integral here means

$$\int_{e_{xy}} f(z) dz = (y - x) f(\frac{1}{2}(x + y)).$$

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A discrete version of $1/z$, the discrete Cauchy kernel in the plane:

$$K(z) := [\partial a](z),$$

with

$$[\bar{\partial} K](z) = \delta_0(z), \quad K(z) \to 0, \quad (|z| \to \infty).$$

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Define

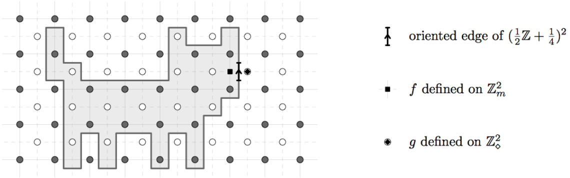
$$z^{[-1]} := 2\pi K(z), \quad z \in \mathbb{Z}_m^2;$$

$+$ use a linear combination and translation to define on $\mathbb{Z}_\diamond^2$.

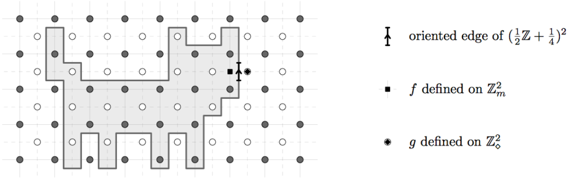
Then repeatedly differentiate and multiply by constants:

$$z^{[-2]} := -2\pi \, [\partial K] \, (z), \ldots$$

Then $z^{[n]}, n \leqslant -1,$ are discrete holomorphic sufficiently far away from 0 and $[\partial z^{[n]}](z) = n z^{[n-1]}$.



Ingredient 3: With a proper notion of discrete contour integral (contours live between the medial and diamond grids) one has a discrete residue formula for the product of two functions on each of these grids,



Ingredient 3: With a proper notion of discrete contour integral (contours live between the medial and diamond grids) one has a discrete residue formula for the product of two functions on each of these grids, e.g., for $[\gamma]$ sufficiently large:

$$\frac{1}{2\pi i}\oint_{[\gamma]} z^{[m]} z^{[n]} dz = \delta_{m+n,-1}.$$

Ingredient 4: The current again. Recall that $J(z) := \partial\phi(z)$ is a centered Gaussian field defined on the medial graph. The relevant correlations are, e.g.,

$$\left\langle J(z)\phi(w)\right\rangle = K(z) - K(z-w); \quad (w \in \mathbb{Z}^2)$$

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...putting these things together, the following makes sense

$$\int_{[\gamma]} \left\langle J(z)\phi(z_1)\cdots\phi(z_k)\right\rangle z^{[n]}dz,$$

if $[\gamma]$ encircles 0 and the $\{z_j\}$. Doesn't depend on $[\gamma]$. (At least if $[\gamma]$ sufficiently large, can be quantified.)

Can consider modes of the current $J(z)$ "acting by contour integrals on field insertions" of any f in $L^p(\mu_{GFF})$ for some $p > 1$:

$$\left\langle \mathfrak{a}_n f(\phi|_{\mathcal{G}}) \right\rangle_\mu := \frac{1}{\sqrt{\pi}} \oint_{[\gamma]} \left\langle J(z) f(\phi|_{\mathcal{G}}) g(\phi|_{\mathcal{G}}) \right\rangle z^{[n]} \, dz,$$

where $[\gamma]$ is a sufficiently large discrete contour separating $\mathcal{G}$ from ∞. Independent on choice of contour. g is the RN derivative coming from μ on $\mathcal{G}$. Varying f, this determines $\mu \mapsto a_n \mu$.

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$\left\langle \mathfrak{a}_{n_j} \cdots \mathfrak{a}_{n_1} f(\phi|_{\mathcal{G}}) \right\rangle_\mu$ defined iteratively using radially ordered contours.

Using Wick's formula and the discrete residue formula we can compute (via insertions of polynomials) the commutation relations of the a_n:

Thm. The discrete current mode operators (a_n), $n \in \mathbb{Z}$, satisfy the commutation relations

$$[a_m, a_n] = m\delta_{m+n,0}\mathrm{Id}.$$

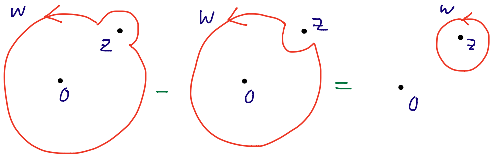
Example computation... For example, if $[\gamma_2]$ separates $[\gamma_1]$ from ∞:

$$\left\langle \mathfrak{a}_m \mathfrak{a}_n \phi(z_1) \cdots \phi(z_k) \right\rangle = \frac{1}{\pi} \oint_{[\gamma_2]} \oint_{[\gamma_1]} \left\langle J(z) J(w) \phi(z_1) \cdots \phi(z_k) \right\rangle z^{[n]} w^{[m]} \, dz \, dw.$$

Expanding using Wick's formula gives commuting terms $+$.

$$\frac{1}{\pi} \oint_{[\gamma_2]} \oint_{[\gamma_1]} \left\langle J(z) J(w) \right\rangle z^{[n]} w^{[m]} \, dz \, dw \times \left\langle \phi(z_1) \cdots \phi(z_k) \right\rangle.$$

Subtracting $\left\langle \mathfrak{a}_n \mathfrak{a}_m \phi(z_1) \cdots \phi(z_k) \right\rangle$ gives a "satellite integral"



and using a discrete Cauchy formula one ends up with an integral of the type:

$$\oint_{[\gamma_1]} \oint_{[\gamma_z]} \left\langle J(z) J(w) \right\rangle w^{[m]} z^{[n]} dw dz$$

$$= \frac{-im}{2} \oint_{[\gamma_1]} z^{[m-1]} z^{[n]} dz = \pi m \delta_0(m+n).$$

Using the a_n we define, via insertions,:

$$\mathfrak{L}_n := \frac{1}{2}\sum_{j\geq 0} \mathfrak{a}_{n-j}\mathfrak{a}_j + \frac{1}{2}\sum_{j\leqslant -1} \mathfrak{a}_j\mathfrak{a}_{n-j}, \quad n \in \mathbb{Z}.$$

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So,

$$\left\langle \mathfrak{L}_n \phi(z_1) \cdots \phi(z_k) \right\rangle$$

makes sense to consider and also defines an operator acting on the space of complex measures.

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Tedious exercise in keeping track of indices. E.g., if $m \geq 1$ then one gets $(m-n)L_{m+n} +$

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So, our operators yield a representation of the Virasoro algebra with central charge $c = 1$.

Summary: We constructed L_n-operators acting on a space of changes of measures of the discrete Gaussian free field. The L_n:s give a representation of the Virasoro algebra, and we can identify the "correct" central charge 1 directly on the discrete level.

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Main ingredients:

- discrete current $\partial\phi$ of the dGFF;
- discrete complex analysis: nice form for correlations, Wick's formula, contour integrals + Cauchy formulas, discrete monomials;
- Discrete current modes acting on insertions, which lift to operators;
- Sugawara construction from the current modes.

Thank you for your attention!

Additional remarks:

- A similar construction for the dGFF in $\mathbb{H}$ with 0 bundary condition also works.
- Coulomb gas: Take $b \in \mathbb{R}$, and define $L_n^b = L_n + b(n+1)a_n$. The commutations of L_n^b are those of the Virasoro algebra with central charge $c = 1 - 12b^2$. However, only $b = 0$ gives parity preserving L_n^b:s; we loose the $+/-$ symmetry of ϕ.
- We (strongly) believe an analogous result is true for the Ising model, but with $c = 1/2$. Work in progress.

A similar construction for the upper half-plane also works:

Let $\tilde{\phi}(z)$ be a discrete Gaussian field on $\mathbb{H}$ with Dirichlet boundary condition. Then $\left\langle \tilde{\phi}(z)\tilde{\phi}(w) \right\rangle = \tilde{G}(z, w) := G(z, w) - G(z, \bar{w})$.

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Define

$$\left\langle \tilde{a}_n \prod_{j=1}^{k} \tilde{\phi}(z_j) \right\rangle = \frac{1}{i} \int_S \left\langle \partial\tilde{\phi}(z) \prod_{j=1}^{k} \tilde{\phi}(z_j) \right\rangle z^{[n]} dz + \frac{1}{i} \int_S \left\langle \bar{\partial}\tilde{\phi}(z) \prod_{j=1}^{k} \tilde{\phi}(z_j) \right\rangle \bar{z}^{[n]} d\bar{z},$$

where S is a discrete "half-contour". The half-contour integrals combine exactly to full contour integrals. (After Wick expansion and using reflection symmetries.)

Remark: Formally, in the continuum CFT, the L_n are often given as modes of the so-called stress-energy tensor, $T(z)$:

$$\left\langle L_n \phi(z_1) \cdots \phi(z_k) \right\rangle = \frac{1}{2\pi i} \oint_{\mathcal{C}} \left\langle T(z) \phi(z_1) \cdots \phi(z_k) \right\rangle z^{n+1} dz.$$

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$T(z)$ is an object that in CFT represents the variation of the fields/correlations under $z \mapsto z + \epsilon\alpha(z)$, e.g.:

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \left\langle T(z) \phi(z_1) \cdots \phi(z_k) \right\rangle \alpha(z) dz = \sum_j (h\alpha'(z_j) + \alpha(z_j)\partial_{z_j}) \left\langle \phi(z_1) \cdots \phi(z_k) \right\rangle.$$

Remark: why this definition of L_n? Formal computation:

$$\partial\phi(z) \sim \sum_n \frac{a_n}{z^{n+1}}, \quad [\partial\phi(z)]^2 \sim \sum_n \frac{\sum_j a_j a_{n-j}}{z^{n+2}}.$$

If $T = (1/2)[\partial\phi]^2$ and

$$T(z) \sim \sum_n \frac{L_n}{z^{n+2}},$$

we can try to identify coefficients.