Solution of Slow Bond Problem

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Random Conformal Geometry

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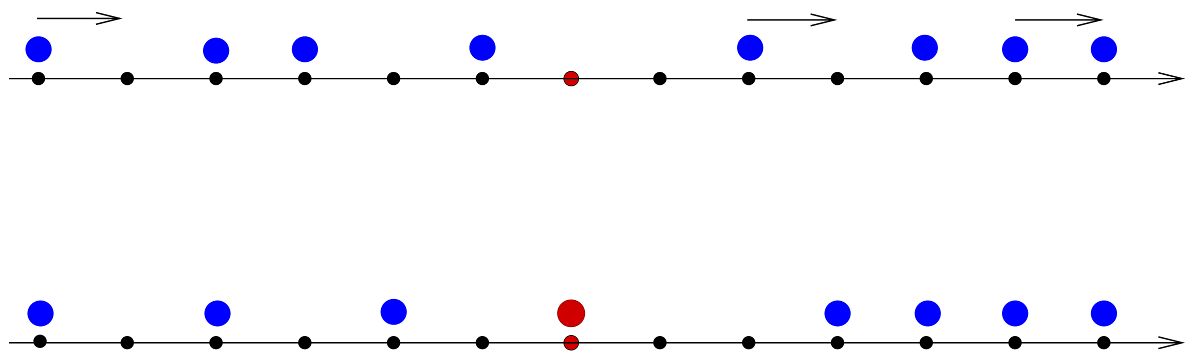


One dimensional TASEP with a slow bond: density of particles $\rho = 1/2$;

Particles jump at rate 1 if they are not at the origin;

Particles jump at rate $1 - r,\ 0 < r < 1$ if they are at the origin.

One of the fundamental questions of equilibrium and non-equilibrium dynamics refers to the following problem: how can localized defect, especially if it is small with respect to certain dynamic parameters, affect the macroscopic behavior of a system?

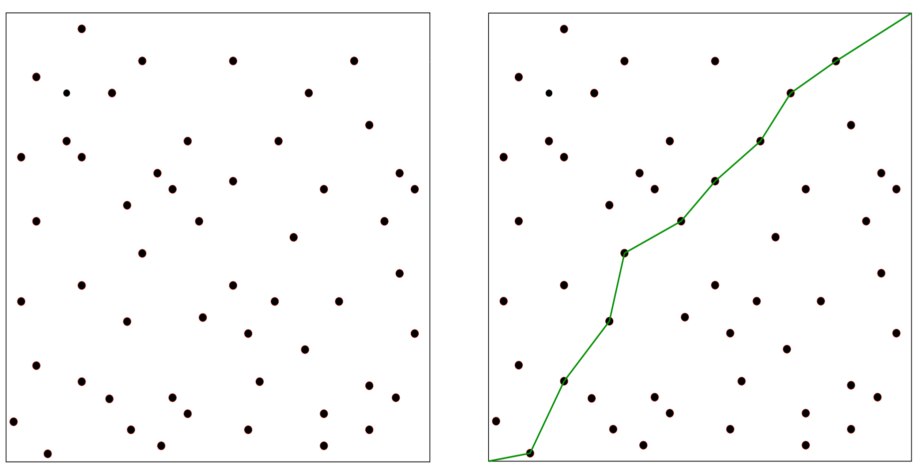
- First/last passage percolation.
- Polymer pinning.
- Driven flows in channels with obstructions.

Such a vanishing presence of the macroscopic effect as a function of the strength of obstruction represents what sometimes in physics literature is called *dynamic phase transition*. The existence of such a transition, its scaling properties, the behavior of the system near the obstruction are among the most important issues.

Ulam's problem or maximal increasing sequence.

Let Π be a Poisson point process of intensity 1 on $\mathbb{R}^2$. For points $u = (0,0)$ and $u' = (n,n)$ let L_n denote the maximum number of points which can be collected along an increasing path from u to u'. We call L_n the *length* of a maximal path from $(0,0)$ to (n,n). It is well known (Vershik and Kerov, Logan and Shep 1977 and Aldous and Diaconis 1995 an alternative proof), that

$$\lim_{n\to\infty} \frac{\mathbb{E} L_n}{n} = 2.$$



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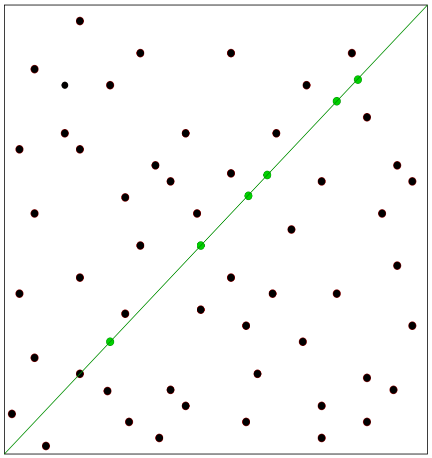
Now, for $\lambda > 0$, let Σ_λ be a one dimensional poisson process of intensity λ on the line $x = y$ independent of Π. Let Π_λ be the point process obtained by superimposing Π and Σ_λ. Let L_n^λ denote the maximum number of points of Π_λ on an increasing path from $(0, 0)$ to (n, n).



Theorem .1. (R. Basu, V.S., A. Sly 2014) *For every $\lambda > 0$,*

$$\lim_{n\to\infty} \frac{\mathbb{E}L_n^\lambda}{n} > 2.$$

Slow bond problem. Consider a discrete last passage percolation on $\mathbb{Z}_+^2$, defined by associating with each vertex $x \in \mathbb{Z}_+^2$ a random variable $\xi_x \sim \exp(1)$, and ξ_x are i.i.d. for all $x \in \mathbb{Z}_+^2$. Let $\pi = \{x_0 = (0,0), x_1, \ldots, x_n = (n,n)\}$ be an oriented path connecting $(0,0)$ to (n,n). Define

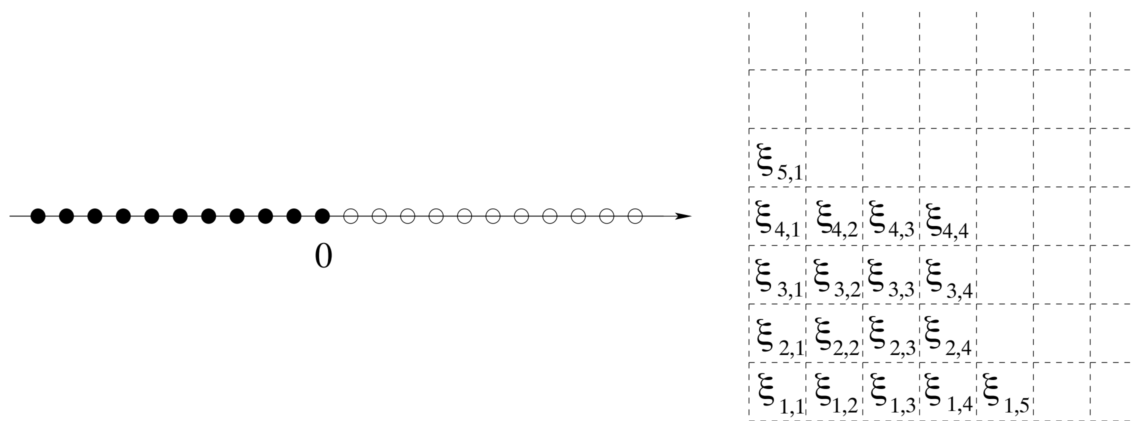
$$L_n^1 = \max_{\pi} \sum_{i=0}^{n} \xi_{x_i};$$

It is well known that

$$\lim_{n \to \infty} \frac{\mathbb{E} L_n^1}{n} = 4$$

This description corresponds to totally asymmetric exclusion process $X(t)$ in continuous time, where the initial configuration is $\mathbb{I}_{(-\infty,0]}(k)$, *i.e.* so called "step initial condition".

Variable $\xi_{x=(i,j)}$ represents the time, which particle η_{-i} has to wait to perform j-th jump, after the time instant that such jump was permitted.



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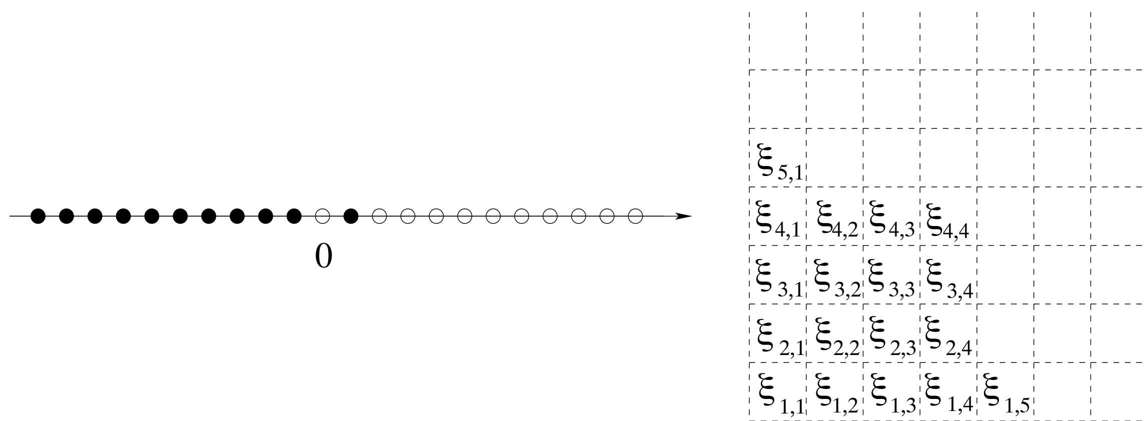


Now let us modify the distribution of passage times, by taking

$$\xi_{(x,y)} \sim \begin{cases} \exp(1) & \text{if } x \neq y, \\ \exp\left(1-\epsilon\right) & \text{if } x = y. \end{cases}$$

and ask the same question: does the law of large numbers change for any $\epsilon > 0$.

In TASEP representation this change corresponds to local modification of the dynamics: particles are jumping across the edges of $\mathcal{E}(\mathbb{Z}) \setminus \langle 0, 1 \rangle$ with intensity 1, and the edge $\langle 0, 1 \rangle$ is crossed at intensity $1 - \epsilon$.

This version of the process with "slow bond" was proposed by Janowsky and Lebowitz 1991 in an attempt to understand non-equilibrium stationary states.

The jump-rate decrease at the origin will increase the particle density to the immediate left of such "blockage" bond, and decrease the density to its immediate right.

The difficulty to analyze this process comes from the fact that effect of any local perturbation in non-equilibrium systems carrying fluxes of conserved quantities is felt at large scales, and what was not obvious, if this perturbation, in addition to local effects, may have a global effect, in particular change the current in the system. *i.e.* weather $\epsilon_c > 0$ or LLN for L_n^ϵ changes for any value $\epsilon > 0$.

Related work

Covert and Rezakhanlou 1997

Liggett 1999

Seppalainen 2000

Baik and Rains 2000

den Nijs et al. 2003

Beffara, Sidoravicius 2011

The question generated certain controversy in theoretical physics and mathematical community, which was supported from opposite sides by numerical analysis and some theoretical arguments, and became known in the literature as “slow bond problem”. Our second result settles this problem:

Theorem .2. (R. Basu, V.S., A. Sly 2014) *In discrete last passage percolation model for every $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{\mathbb{E}L_n^{\epsilon}}{n} > 4. \tag{1}$$

Tracy-Widom Limit, Moderate Deviations and $n^{2/3}$ Fluctuations

The two models that we consider (i.e., the longest increasing subsequence and the exponential last passage percolation) are exactly solvable in absence of defect and it is possible to obtain scaling limits and precise moderate deviation tail bounds for L_n. We shall treat these results from the exactly solvable models as 'black box' in our arguments, and as we shall see using these estimates the problems at hand can be treated as percolation type questions. Here we collect these 'black box' results for the longest increasing subsequence model as this is the model we shall primarily work with in this paper.

Scaling limit Baik, Deift and Johansonn in 1999 proved the following fundamental result about fluctuations of L_n. Let Π be a homogeneous Poisson point process on $\mathbb{R}^2$ with rate 1. Let u_λ be a point on the first quadrant of $\mathbb{R}^2$ such that the area of the rectangle with bottom left corner $(0,0)$ and the top right corner u_λ (λ here is unrelated to the reinforcement parameter). Let X_{u_λ} denote the maximum number of points on Π on an increasing path from $(0,0)$ to u_λ. By the scaling of Possion point process it is clear that the distribution of $X_\lambda = X_{u_\lambda}$ depends on u_λ only through λ.

Theorem .3. *(BDJ99) Let F_{TW} be the Tracy-Widom distribution. As $\lambda \to \infty$,*

$$\frac{X_\lambda - 2\sqrt{\lambda}}{\lambda^{1/6}} \xrightarrow{d} F_{TW} \tag{2}$$

where $\xrightarrow{d}$ denotes convergence in distribution.

Moderate deviation estimates We quote the following moderate deviation estimates for upper and lower tails of longest increasing subsequence by Löwe, Merkl and Löve, Merkl and Rolles, respectively.

Theorem .4. *There exists absolute constants* C_1, t_0 *and* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ *and* $t > t_0$, *the following holds.*

$$\mathbb{P}[X_\lambda \geq 2\sqrt{\lambda} + t\lambda^{1/6}] \leq e^{-C_1 t^{3/2}}. \tag{3}$$

Theorem .5. *There exists absolute constants* C_1, t_0 *and* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ *and* $t > t_0$, *the following holds.*

$$\mathbb{P}[X_{u,u'} \leq 2\sqrt{\lambda} - t\lambda^{1/6}] \leq e^{-C_1 t^{3/2}}. \tag{4}$$

Observe that λ_0, t_0 and C_1 can be taken to be same in Theorem .4 and Theorem .5 above.

It is also clear by the translation invariance of Poisson process that the same bounds can be obtained for the the number of points on a maximal increasing path on any pair of points that determine a rectangle with area λ.

Transversal Fluctuation

Consider all increasing paths γ from $(0,0)$ to (n,n) in Π containing the maximum number of points from. The maximum transversal fluctuation F_n is defined as $max_{x\in[0,n],\gamma}|\gamma(x)-x|$. The scaling exponent for the transversal fluctuation ξ is defined by

$$\xi = \inf\{\theta > 0 : \liminf_n \mathbb{P}[F_n \geq n^\theta] = 0\}.$$

Johansson (2000) proved the following theorem.

Theorem .6. *In the above set-up we have* $\xi = \frac{2}{3}$.

So this theorem tell us the the maximal fluctuation of the maximal paths from the diagonal is typically of the order $n^{2/3}$. This motivates a lot of our construction. However for our proof, we need a slightly sharper estimate which we establish using Theorem .5 and Theorem .4.

Outline of the proof for the case of the continuum last passage percolation.

Due to superadditivity of the passage times

$$\mathbb{E}L^{\lambda}_{n+m} \geq \mathbb{E}L^{\lambda}_{n} + \mathbb{E}L^{\lambda}_{m}$$

for any $\lambda > 0$, it suffices to prove that for some N

$$\mathbb{E}[L^{\lambda}_{N}] > 2N. \tag{5}$$

Using Tracy-Widom Limit Theorem.3 and moderate deviation inequalities from Theorem.4 and Theorem.5 we have

$$\mathbb{E}[L_N^\lambda] = 2N - O(N^{1/3}). \tag{6}$$

Thus, in order to obtain (5), it is enough to prove that

in the environment with the diagonal reinforced by one-dimensional Poisson point process of intensity $\lambda > 0$, the length of longest increasing path increases by at least $cN^{1/3}$ for arbitrarily large positive constant $c > 0$. (7)

Most of the work is dedicated to show (7). It is done in several steps.

Small improvement

First, we observe that the maximal path in unperturbed environment, *i.e.* with $\lambda = 0$, is expected to spend $O(N^{1/3})$ of time near the diagonal (within finite distance from it). If this happens, then essentially without an additional cost it results in average increase of the length of the maximal path by $\lambda O(N^{1/3})$, once the reinforcement $\lambda > 0$ is added to the diagonal.

On the other hand, if the maximal path in unperturbed environment deviates from the diagonal for substantially long time, then we search for an alternative path in unperturbed environment which returns to the diagonal more frequently and does it in such a manner, that the loss of the length due to such maneuver in unperturbed configuration at the end becomes compensated, and actually improved once the reinforcement $\lambda > 0$ is added to the diagonal.

Bootstrap

Next, we need to "bootstrap" small improvement $\lambda N^{1/3}$ to $cN^{1/3}$ for arbitrarily large constant $c > 0$.

Consider translates of the diagonal $\ell_m = \{y = x + m\}$, where $m \in [-Kn^{2/3}, Kn^{2/3}]$, for some large $K \in \mathbb{N}$.

For each $m \in [-Kn^{2/3}, Kn^{2/3}]$ and $\lambda > 0$ consider new reinforced environment, obtained from the original one, by adding Poisson point process of intensity λ on ℓ_m.

Let $L_n^{\lambda,m}$ be the length of the longest increasing path from $(0,0)$ to (n,n) in the environment with reinforced ℓ_m.

Claim. For any $\lambda > 0$ and arbitrarily large $c_1 > 0$ there exists $m \in [-Kn^{2/3}, Kn^{2/3}]$, such that

$$\mathbb{E}[L_n^{\lambda,m}] > 2n + c_1 n^{1/3}, \tag{8}$$

for any n large enough. Once this is obtained, a little extra work is needed to show that

$$\mathbb{E}[L_n^{\lambda}] \equiv \mathbb{E}[L_n^{\lambda,0}] > 2n + c_2 n^{1/3}, \tag{9}$$

where $c_2 > 0$ can be chosen arbitrarily large, which implies (7).

To obtain (8) we analyze the unperturbed environment at different scales simultaneously.

• For fixed length scale r, and spatial location $x = kr, \ k \in [n/r]$, we define a rectangular box $B_{r,x} := [kr, (k+1)r) \times [0, n]$ of width r and hight n.

• Unperturbed environment in the unique way determines the *top most* maximal path from $(0, 0)$ to (n, n), a random curve which we denote by π^*.

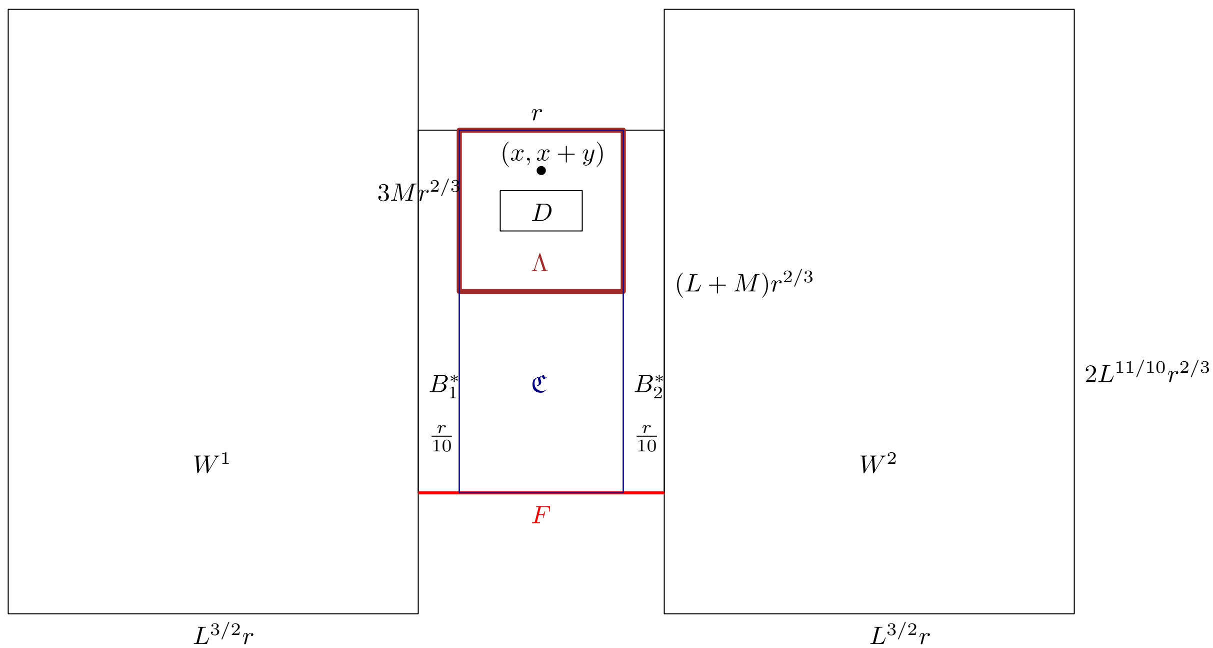
Our argument relies on the fact that there is a reasonable chance that π^* "behaves nicely" while crossing $B_{r,x}$. By saying that we mean that with probability $p_\lambda > 0$, bounded away from 0 and independent of the scale r, the following event occurs:

there exists non-empty random set of indices $I_{r,x} \subset [-Kn^{2/3}, Kn^{2/3}]$, depending on the chosen scale r and location x, or, more precisely, depending on the shape and spatial localization of π^* within $B_{r,x}$, such that if the line ℓ_i, for some fixed $i \in I_{r,x}$, was reinforced by an independent one dimensional Poisson point process of intensity $\lambda > 0$, then there exists a modification of π^* within $B_{r,x}$, called a local modification at scale r and denoted by $\pi^*_{r,x}$, which has the following properties:

- *$\pi^*_{x,r}$ coincides with π^* from $(0,0)$ to the last Poisson point in unperturbed configuration before entering $B_{r,x}$, denote this point by $x^*_{r,x}$, and coincides with π^* from the first Poisson point in unperturbed configuration after exiting $B_{r,x}$, denote this point by $y^*_{r,x}$, till reaching (n,n).*
- *the restriction of the new path $\pi^*_{x,r}$ within the box $B_{x,r}$, i.e. starting at $x^*_{r,x}$ and ending at $y^*_{r,x}$, has transversal fluctuations of order $O(r^{2/3})$;*
- *Increase of the length obtained from such local modification at the scale r within $B_{r,x}$, integrated over all lines ℓ_i, $i \in [-Kn^{2/3}, Kn^{2/3}]$, and averaged over reinforcements by $\lambda > 0$ is at least $c(\lambda)r$.*

Thus the total average improvement from doing local perturbations at fixed scale r in all boxes of scale r between $(0,0)$ and (n,n) would result in increase $\widetilde{c}(\lambda)n$.

By choosing scales properly we ensure that at given location and given m the improvement is obtained only at one scale. This allows us to sum up the integrated improvement over different scales, and by considering a large number of scales we obtain a total improvement



Anatomy of a butterfly $\mathbb{B}(x, y, r)$