Topology of Quadrature Domains

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Joint work with Nikolai Makarov

– Topology of quadrature domains (arXiv:1307.0487)
– Sharpness of connectivity bounds for quadrature domains (arXiv:14??.????)

Quadrature domain

An open connected set $\Omega \subset \widehat{\mathbb{C}}$ is a **quadrature domain (QD)** if the identity

$$\int_{\Omega} f(z)\, dA(z) = \sum_{j} c_j f^{(k_j)}(z_j)$$

holds for any f analytic on Ω.

c_j's are complex numbers, k_j's are the orders of derivatives.

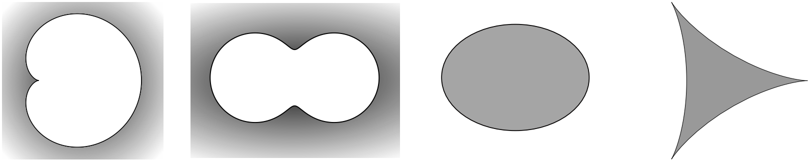
Using residue calculation, above can be rewritten.

$$\int_{\Omega} f(z)\, dA(z) = \frac{1}{2i} \oint_{\partial\Omega} f(w)\, r(w)\, dw,$$

for some rational function r.

We call the above identity **quadrature identity** and the function r **quadrature function**.

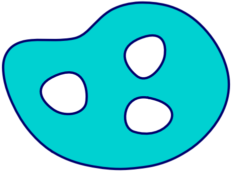
Examples of QD



(The complements of QDs are shaded.)

- disk centered at a with radius R: $r(z) = R^2/(z-a)$;
- cardioid: $r(z) = \dfrac{3}{2z} + \dfrac{1}{2z^2}$;
- Neumann oval: $r(z) = \dfrac{b}{z-a} + \dfrac{b}{z+a}$;
- exterior of an ellipse: $r(z) = az + b$;
- deltoid: $r(z) = z^2$.

(Given Ω, the quadrature function r is unique. Not the other way.)



Let us define the **connectivity of** Ω by

$$\operatorname{conn}\Omega = (\text{number of components in } \Omega^c).$$

In general, for a given r, $\operatorname{conn}\Omega$ is hard to find.

Theorem [Lee-Makarov] When Ω is an unbounded QD:

$$\operatorname{conn}\Omega \leq \min\{d+n-1, 2d-2\}.$$

When Ω is a bounded QD:

$$\operatorname{conn}\Omega \leq \min\{d+n-2, 2d-4\}.$$

Here, we defined d : the degree of the rational map r;

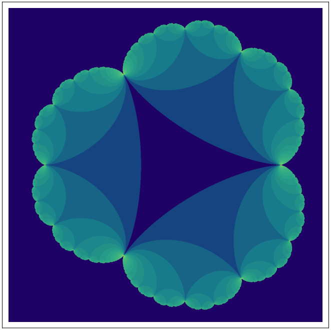
n : the number of distinct poles in r.

Proof: The main steps

The proof of this theorem uses **Schwarz function of quadrature domain**.

We deform the Schwarz function into a rational map, $r(z)$, such that each **component of** Ω^c (whose boundary is fixed under the Schwarz reflection) becomes the **attracting fixed point** of the map $z \mapsto \overline{r(z)}$.

Then we count the attracting fixed point using the Fatou argument in complex dynamics.



Theorem [Lee-Makarov] The connectivity bounds are sharp. I.e. there exists a quadrature domain that saturates the bound.

We will show this by explicitly constructing a "maximal" quadrature domain.

Given an external potential Q with appropriate boundary condition, consider the **equilibrium measure** that minimizes the Coulomb (logarithmic potential) energy:

$$I[\mu] := \int_{\mathbb{C}} Q(z)\, d\mu(z) + \iint \log \frac{1}{|z-w|}\, d\mu(z) d\mu(w).$$

Let **droplet = support of equilibrium measure**, and denote it by K.

Let us say Q is an "Algebraic Hele-Shaw potential" if

$$Q(z) = |z|^2 - 2\,\mathrm{Re} \int^z R(w)\, dw \qquad z \in \operatorname{supp} K,$$

for some rational function R. (All the poles of R are then outside K.)

Examples:
– For $R(z) = a\,z$, $|a| < 1$ the droplet is an ellipse.
– For $R(z) = z^2$ the droplet can be a deltoid.

If Q is an "algebraic Hele-Shaw potential", **the complement of droplet** satisfies quadrature identity:

$$\int_{K^c} f(z)\,dA(z) = \frac{1}{2i}\oint_{\partial K} f(z)\,R(z)\,dz.$$

(If K^c is connected, K^c is a QD with quadrature function R.)

For K^c with **multiple** components, let us denote each connected component of K^c by Ω_j, i.e.

$$K^c = \bigsqcup_j \Omega_j.$$

Then there is a corresponding decomposition of R:

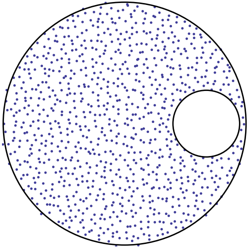
$$R(z) = \sum_j r_j(z)$$

such that $r_j(z)$ has singularities only on Ω_j and $r_j(\infty) = 0$ for any bounded Ω_j.

Lemma. Ω_j is a QD with quadrature function r_j.

Lemma (converse) Given QDs Ω_j's with quadrature functions $r_j(z)$'s, the complement of $\bigsqcup_j \Omega_j$ makes a droplet with the external potential:

$$Q(z) = |z|^2 - 2\,\mathrm{Re}\int^z R(w), \quad R(z) = \sum_j r_j(z).$$



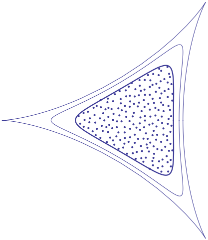
$$R(z) = \frac{1}{6(z-1)} \quad \left(Q(z) = |z|^2 + \frac{1}{3}\log\frac{1}{|z-1|}\right)$$

Consider **deformation of unbounded QD** such that the quadrature function remains fixed.

By the previous observation, it corresponds to **deformation of droplet** under a fixed R or under a fixed external potential Q.

Changing the total mass (area of K) = **Hele-Shaw flow from a source at** ∞.

$$R(z) = 3z^2 \quad \left(Q(z) = |z|^2 + 2\mathrm{Re}\, z^3\right)$$



(Deformation of bounded QD = **Hele-Shaw from a finite source**.)

Two properties of the Hele-Shaw flow

For algebraic Hele-Shaw potential, the boundary ∂K is an algebraic curve (Gustafsson '83), i.e.

$$\partial K = \{(x, y) : \mathrm{Poly}(x, y) = 0\}.$$

For such boundary, K shrinks monotonically as $t \searrow$ where t is the area of the droplet. That is,

$$K(t') \subset \mathrm{Int}\, K(t) \quad \text{for } t' < t.$$

(In fact, K should be the polynomial convex hull of K to be exact.)

Continuity: $K(t) = \mathrm{clos} \bigcup_{t' < t} K(t')$.

Strategy of the proof

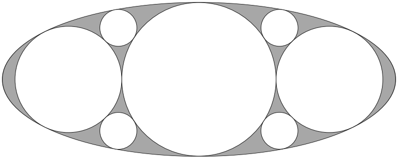
1. Existence of a special QDs called **Suffridge domains**. These will be used as "building blocks".

2. Construction of "maximal" QD by combining (i.e. disjoint union of) "building blocks".

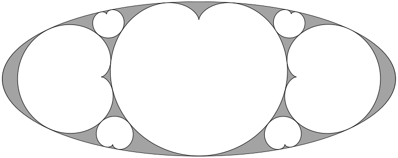
3. Hele-Shaw flow to glue these multiple QDs into a single QD.

Explicit construction

(Apollonian) packing of m disks



Packing m cardioids



The "quadrature function" of the union of QDs is given by the summation of each quadrature function.

Small (backward) Hele-Shaw flow $\Rightarrow$ A connected (unbounded) QD

$$\operatorname{conn}\Omega = 2m, \quad d = m+1, \quad n = m+1,$$

$$\operatorname{conn}\Omega = 3m. \quad d = 2m+1. \quad n = m+1.$$

Theorem says that the maximal connectivities are:

$$\operatorname{conn}\Omega \le d + n - 2 = \begin{cases} (m+1)+(m+1)-2 = 2m & \text{(Left);} \\ (2m+1)+(m+1)-2 = 3m & \text{(Right).} \end{cases}$$

For higher order poles in quadrature function

Theorem (Aharonov-Shapiro '76)

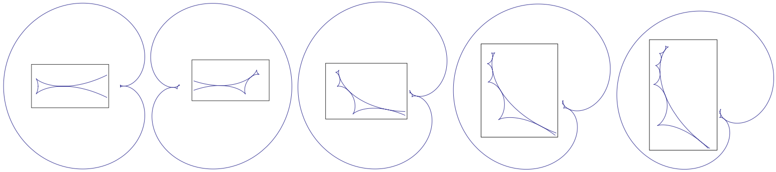
Given a univalent polynomial P of degree d (with $P(0) = 0$), the conformal image $P(\mathbb{D})$ is a QD whose quadrature function has a pole singularity (of order d) only at the origin.

Example: **Cardioid** is the simplest non-trivial case, with double pole at the origin.

$$\text{(Cardioid)} = P(\mathbb{D}), \quad P(z) = z + \frac{1}{2}z^2, \quad r(z) = \frac{3}{2z} + \frac{1}{2z^2}.$$

Suffridge curves (building blocks)

Suffridge domains: Conformal image $P(\mathbb{D})$ the univalent polynomial, P, of degree d such that all the $d-1$ critical points are on the unit circle, and there are exactly $d-1$ components in the complement of $\operatorname{clos} P(\mathbb{D})$ (or, $d-2$ double points).



$d=3$ $d=4$ $d=5$ $d=6$ $d=7$

Theorem (Suffridge '72)

For any $d \geq 1$ there exists a Suffridge curve.

We will give more geometric proof than the original proof by Suffridge.

Packing by Suffridge curves

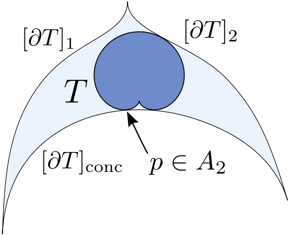
Jordan curve is **Cardioid-like** if it is smooth with positive curvature (convex) except a single cusp.

Jordan curve is **deltoid-like** if it has three outward cusps and at least one concave side (i.e. negative curvature).

All the Suffridge curves have **Cardioid-like** outer boundary and **Deltoid-like** inner boundaries.

The following guarantees the packing construction.

Lemma. Given a cardioid-like curve and a deltoid-like curve, one can inscribe the former in the latter with four intersection such that there are four deltoid-like curves.



Let $\mathcal{S}_d = \{P(z) = z + a_2 z^2 + \cdots + a_{d-1} z^{d-1} + \frac{1}{d} z^d \colon P \text{ is univalent in } \mathbb{D}\}$.

Lemma. For $P \in \mathcal{S}_d$, $P(\partial\mathbb{D})$ has a constant conformal curvature:

$$\frac{d}{d\theta} \arg \frac{dP(e^{i\theta})}{d\theta} = \frac{d+1}{2}.$$

Proof) The product of $d-1$ critical points is ± 1.

By the univalency, there should not be any critical point in $\mathbb{D}$.

$\Longrightarrow$ All the critical points of $P(z)$ are on the unit circle.

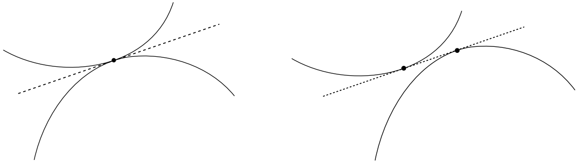
$\Longrightarrow$ $P'(z) = z^{d-1}\overline{P'(1/\overline{z})}$ (Self-inversive polynomial).

$\Longrightarrow$ $P'(z) \propto z^{\frac{d-1}{2}}$ for $z \in \partial\mathbb{D}$. QED.

Existence of Suffridge curve

Theorem [Lee-Makarov]. Extremal points of $\mathcal{S}_d$ are Suffridge polynomials. I.e. the conformal image of the circle is a Suffridge curve (having $d-2$ double points and $d-1$ cusps).

Proof is by contradiction. We assume that the extremal point, say P, of $\mathcal{S}_d$ is *not* a Suffridge polynomial, i.e. $P(\mathbb{T})$ has $N < d-2$ double points.
i) $P \in \mathcal{S}_d$ has $d-2$ real dimension.
ii) At each double point, the tangential deformation is allowed. But the perpendicular deformation is *not allowed*. So there are N constraints.



iii) There remains $d-2-N$ real dimensional space where one can wiggle P within $\mathcal{S}_d$. CONTRADICTION.

Fundamental theorem of algebra for harmonic polynomials:

Given $(\deg p, \deg q) = (n, m)$, find the maximal number of roots satisfying:

$$p(z) - \overline{q(z)} = 0.$$

– For $m = 0$, n is the maximal number of roots.
– For $m = n - 1$, n^2 is the maximal number of roots (Wilmshurst '94).
– For $m = 1$, $3n - 2$ is the maximal number of roots (Khavinson-Świątek, '03; **Geyer, '08**).
– For an arbitrary (n, m), Wilmshurst conjectured that $m(m-1) + 3n - 2$ is the maximal number of roots. However, counter-examples were found in [Lee-Lerario-Lundberg, '14].

Relation

For $m = 1$: the roots of $P(z) = \overline{z}$ is the critical points of

$$Q(z) = |z|^2 - 2\mathrm{Re} \int P(w)\, dw.$$

Critical points are either **local minima** or saddle points.

Local minimum is where one can have a **birth of droplet**.

$\#$**(local minima) =** $\#$**(droplet components) = #(connectivity of** K^c**).**

Open problems

For a degree d polynomial $P(z)$, how many roots are there for the following equation?

$$P(z) = \bar{z}^2.$$

Conjecture: $3d$.

Thank you!