Counting Self-Avoiding Walks

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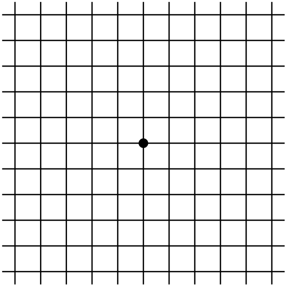
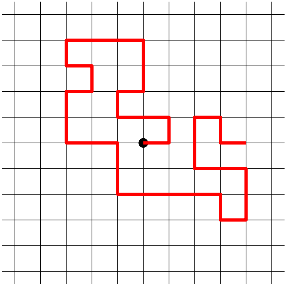
Seoul, 12 August 2014

Self-avoiding walks

Square lattice: $\mathbb{Z}^2$

Self-avoiding walk (SAW): a non-self-intersecting walk





a 36-step SAW from origin

Why? Polymerization

Paul Flory (1910–1985)

- statistical mechanics of polymers
- how many are there?

Basic questions

σ_n := number of n-step SAWs from origin of $\mathbb{Z}^2$

What can be said about the sequence σ_n?

- σ_n grows approximately exponentially: $\sigma_n = \mu^{n(1+o(1))}$
- what is the value of the connective constant $\mu = \mu(\mathbb{Z}^2)$?
- finer order asymptotics?

- known that μ exists and $2.6256 < \mu \leq 2.6792$
- believed that $\sigma_n \sim An^{11/32}\mu^n$
- known that $\sigma_{n+2}/\sigma_n \to \mu^2$
- believed that $\sigma_{n+1}/\sigma_n \to \mu$

Hammersley, Kesten, Hara, Slade +

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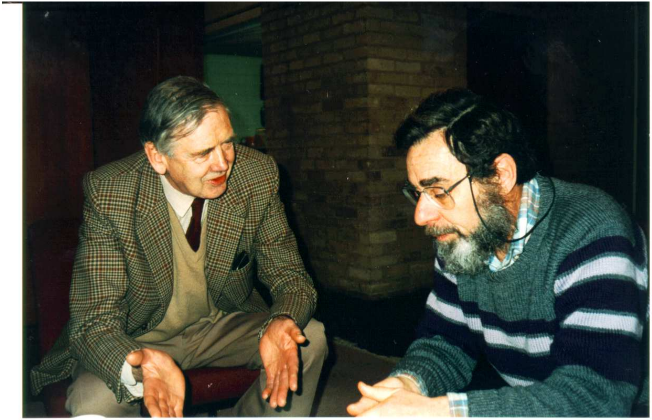
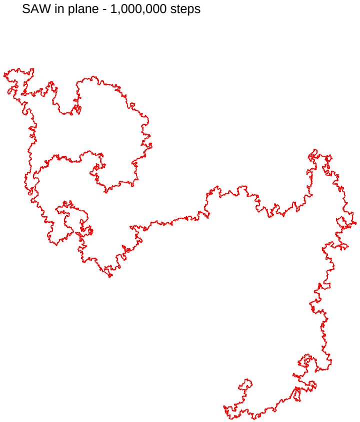
Hammersley and Kesten 1993



What does a typical n-step SAW look like?



Tom Kennedy

Problem: Prove that random SAW $\Rightarrow$ $SLE_{8/3}$

Where is the starting point?

SAWs on a general graph G

G: infinite, quasi-transitive, connected (possibly multi/di-) graph

$\sigma_n(v)$: number of n-step SAWs from v

$\sigma_n := \sup_v \sigma_n(v)$

subadditivity: $\sigma_{m+n} \leq \sigma_m \sigma_n$

Theorem (Hammersley 1957)

For a quasi-transitive graph G, there exists a connective constant $\mu = \mu(G)$ such that

$$\sigma_n^{1/n} \to \mu$$

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Properties of connective constants?

- calculate μ
- approximate μ
- is $\mu(G)$ strictly monotone in G?

Which graphs?: infinite, connected, (vertex-)transitive, d-regular (multi-)graphs

Bounds for μ

Theorem (G + Li 2012)

Let G be an infinite, connected, d-regular, vertex-transitive, simple graph. Then $\sqrt{d-1} \leq \mu(G) \leq d-1$.

Upper bound: trivial, sharp iff tree

Lower bound: less trivial, less sharp?

Question: What is the sharp lower bound for $\mu(G)$?

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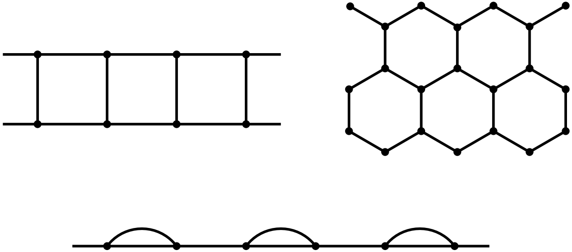
Cubic graphs, $d = 3$



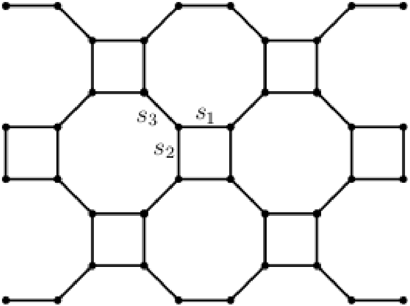
$$\mu(\text{ladder}) = \tfrac{1}{2}(1+\sqrt{5}), \quad \mu(\text{hex}) = \sqrt{2+\sqrt{2}}, \quad \mu(\text{bridge}) = \sqrt{2}.$$

Question: For simple G: $\mu(G) \geq \frac{1}{2}(1+\sqrt{5})$?

d-regular graphs?

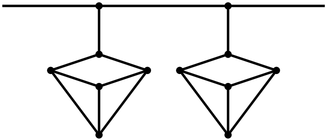
A question

Question: How about the square–octagon lattice, $(4, 8^2)$?



Is it the case that $\mu \geq \frac{1}{2}(1 + \sqrt{5})$?

Why not quasi-transitive graphs?



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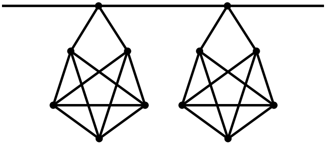

$d = 4$

Strict inequalities I

G: infinite, connected, transitive, simple

$\overline{G}$: G plus extra non-trivial edge-set

Theorem (G + Li 2013)

Let Γ be a group acting transitively on G, and let $\mathcal{A} \subseteq Aut(\overline{G})$ be a normal subgroup of Γ acting quasi-transitively on G. Then

$$\mu(G) < \mu(\overline{G}).$$

Question: can normality be relaxed at all?

Kesten pattern theorem

Strict inequalities II

G: infinite, connected, transitive, simple

$\vec{G}$: (directed) quotient graph $G/\mathcal{A}$

L = length of shortest SAW in G with distinct endpoints in same orbit of $\mathcal{A}$

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if either:

1. $L \neq 2$,
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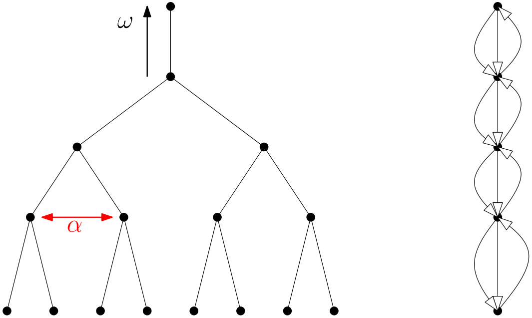

G: binary tree with end ω

Γ: automorphisms that preserve ω

$\mathcal{A}$: normal subgroup of Γ generated by α

but $\mu(\vec{G}) = \mu(G) = 2$

Applications to Cayley graphs

$\mathcal{G} = \langle S \mid R \rangle$: infinite group

S: finite set of generators (satisfying $S = S^{-1}$)

R: set of relators

Cayley graph G: vertex-set $\mathcal{G}$, edges $\langle g, gs \rangle$ for $g \in \mathcal{G}$, $s \in S$

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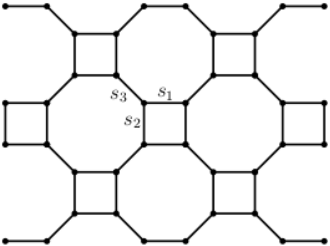

Square–octagon lattice SO as a Cayley graph:

$$S = \{s_1, s_2, s_3\}, \quad R = \left\{s_1^2, s_2^2, s_3^2, s_1 s_2 s_1 s_2, s_1 s_3 s_2 s_3 s_1 s_3 s_2 s_3\right\}$$

Ladder graph: obtained by adding the relator $s_2 s_3 s_2 s_3$

Therefore, $\mu(\mathrm{SO}) > \frac{1}{2}(1 + \sqrt{5})$

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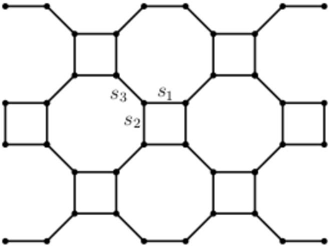

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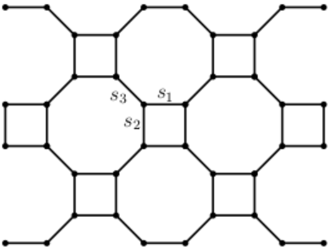

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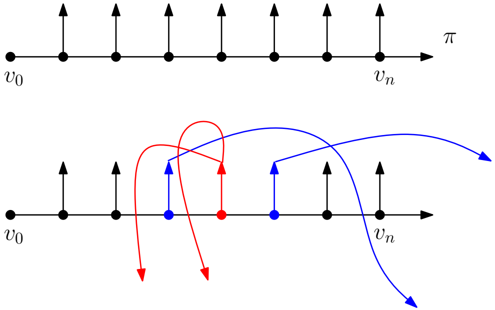
Proof of $\mu \geq \sqrt{2}$ for cubic graphs



Each vertical edge is blue if it extends to an infinite SAW from v_0

$$\#\{\text{blue edges}\} - \#\{\text{red edges}\} \geq 0$$
$$\#\{\text{blue edges}\} + \#\{\text{red edges}\} = n$$

Therefore, $\#\{\text{blue edges}\} \geq \frac{1}{2}n$

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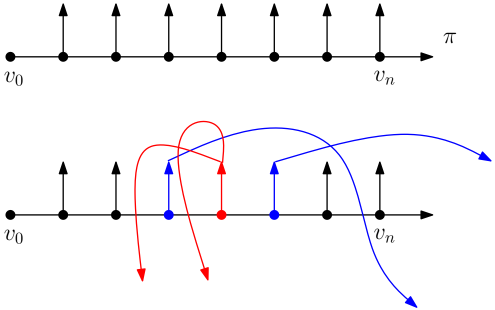

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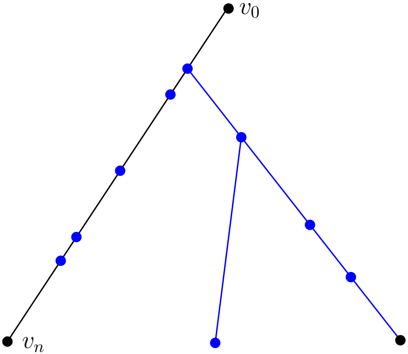
SAW tree



The blue extendable' tree is a binary tree of height $\frac{1}{2}n$. Therefore,

$$\sigma_n \geq \#\{n\text{-step extendable walks}\} \geq 2^{n/2} = (\sqrt{2})^n$$

Extendable SAWs

a SAW is forward extendable if extendable forwards to ∞

similarly, backward extendable and doubly extendable

hence σ_n^{F}, σ_n^{B}, σ_n^{FB}, etc

Theorem (G, Holroyd, Peres 2013)

Let G be an infinite, strongly connected, quasi-transitive digraph. The connective constants μ^F, μ^B, μ^{FB} exist and satisfy

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Idea I of proof

Tree T, rooted at ρ.

$W_n = \{\text{vertices at depth } n\}$

growth(T)= $\lim_{n\to\infty} |W_n|^{1/n}$

branching rate(T)= $\sup\left\{\lambda : \inf_{\Pi} \sum_{e\in\Pi} \lambda^{-|e|} > 0\right\}$, where infimum is over cutsets Π separating ρ from ∞

Theorem (Furstenberg 1967)

If T is subperiodic, growth(T) = branching rate(T).

Apply this to the various SAW trees, so that growth rates are connective constants. Hence $\mu = \mu^{\mathsf{F}}$ and $\mu^{\mathsf{B}} = \mu^{\mathsf{FB}}$.

Idea II of proof

Two cases:

- G unimodular: use mass-transport principle to obtain $\mu^{\mathsf{B}} = \mu$.
- G non-unimodular: use the fact that the modular function is unbounded to construct a certain type of 'geodesic'. Then use combinatorics of paths to obtain $\mu^{\mathsf{B}} = \mu$.

Question: is there a unified proof?

A problem

Question: For what graphs is it the case that $0 < c \leq \frac{\sigma_n^{\mathsf{F}}}{\sigma_n} \leq 1$, etc?

YES: for $\mathbb{Z}^d$ and $d \geq 5$
PERHAPS: for $\mathbb{Z}^2$

Hara/Slade, lace expansion
approximation by $\mathsf{SLE}_{8/3}$

Locality of connective constants

Question: If G and H are alike on a ball around the origin, are $\mu(G)$ and $\mu(H)$ close?

1. critical percolation probabilities (Benjamini, Nachmias, Peres for tree-like graphs, Martineau, Tassion for Cayley graphs of abelian groups)
2. Ising critical temperature?
3. random-cluster critical point?

Warning: at least as hard as proving

"slab critical points $\to$ full-space critical point"

Grimmett–Marstrand, Aizenman, Bodineau

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Graphs as a metric space

Rooted graphs G, H

$$B_k(G) = \text{ball within distance } k \text{ from root}$$

Distance $d(G, H)$:

$$K = \max\{k : B_k(G) \simeq B_k(H)\}$$
$$d(G, H) = 2^{-K}$$

Babai 1991

Question: For what types of graph is it the case that

$$|\mu(G) - \mu(H)| \text{ is small when } d(G, H) \text{ is small?}$$

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OK for many Cayley graphs of finitely presented groups, e.g., of infinite abelian groups, free nilpotent, free solvable, + others satisfying a certain condition on the presentation.

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Ideas of proof . . .

A percolation problem

For any connected graph G with bounded degrees,

$$\overline{\mu}_p(v) := \limsup_{n\to\infty}\{\sigma_n(v)^{1/n}\}$$

exists and is independent of v.

Let $G :=$ the infinite cluster of supercritical bond percolation on $\mathbb{Z}^d$

Question: does $\sigma_n(v)^{1/n}$ converge a.s.?

Theorem (Lacoin 2012: annealed < quenched)

We have $\overline{\mu}_p < p\mu_1$ for $p_c < p < \overline{p_c}(d)$, where $\overline{p_c}(2) = 1$ and $\overline{p_c}(d) > p_c$ for large d.

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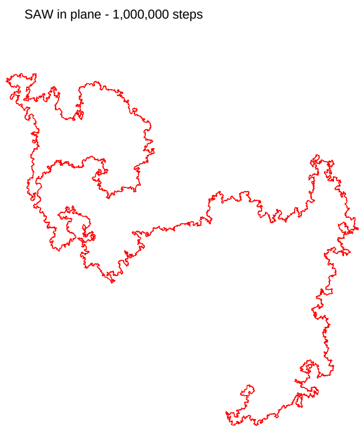
Finally



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