Brownian Loops, Cosmic Bubbles and Conformal Fields*

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*Join work with Alberto Gandolfi (Florence) and Matthew Kleban (New York).

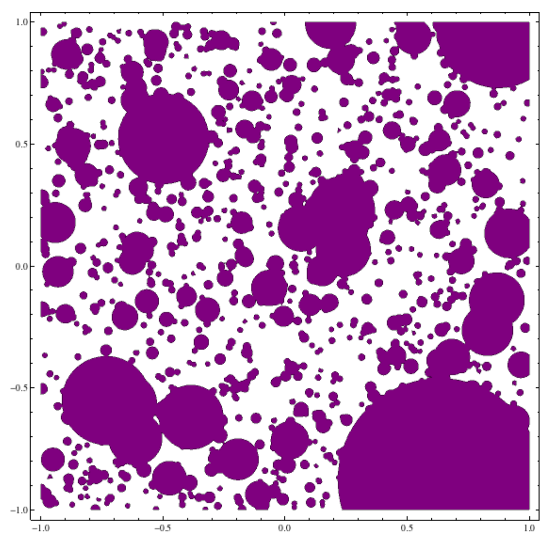
Plan of the Talk

- Introduction: From Fractal Percolation to Cosmology
- Brownian Loop Soup and Conformal Fields
- Some Easy Calculations
- Some Open Questions

Scale-Invariant Boolean Model

Poisson point process with intensity measure $\lambda\, r^{-3}\, dr\, dx\, dy$, $\lambda \in (0, \infty)$:

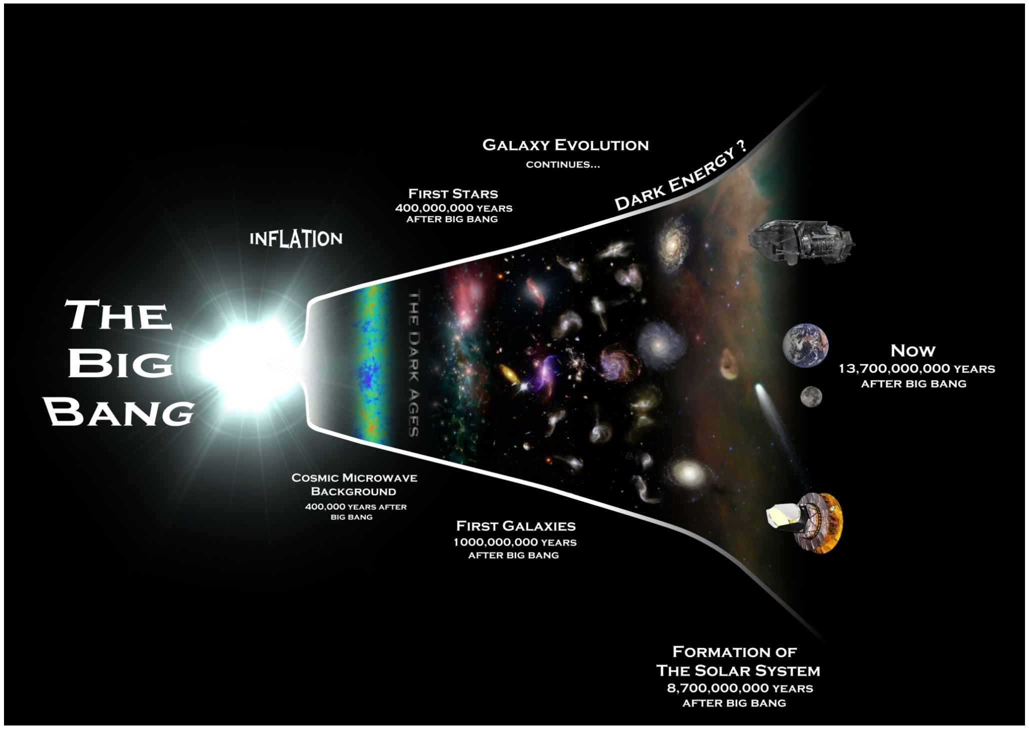
Poissonization of the infinite measure $\lambda \int_D \int_0^\infty r^{-3}\, dr\, dx\, dy$ on space of discs in $D \subseteq \mathbb{R}^2$ with radius r.



Inflation

Inflation is part of the *standard cosmological model*.

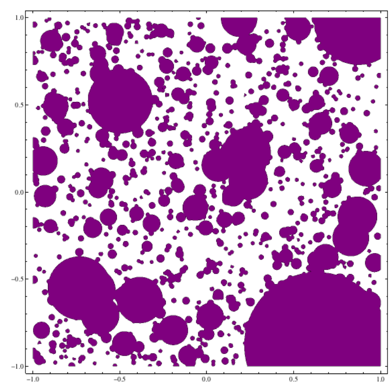
Three main components: Big Bang, *inflation*, slower expansion.



Cosmic Bubbles

Scale-invariant Boolean model as toy model of two-dimensional *eternally inflating* universe.

Intuitive picture: "cosmic bubbles" of slowly expanding space nucleate in inflating universe.



Similar to first order phase transition: inflating (metastable) phase decays to slowly-expanding phase (more stable).

Eternal Inflation and the Multiverse

If inflations is sufficiently fast and nucleation sufficiently slow, cosmic bubbles don't percolate and inflations continues forever.

All models of eternal inflation produce an infinite *multiverse*, typically a fractal.

Our visible universe is inside a single bubble.

Different bubbles may host different laws of physics.

A Conformal Field Theory for Eternal Inflation?

Ben Freivogel and Matthew Kleban tried with construct a *conformal field theory* associated to eternal inflation (*JHEP* **12**, 2009):

Use the scale-invariant Boolean model to generate *spherical* cosmic bubbles in a *two-dimensional* universe; bubbles are randomly assigned one of *two types*.

Consider $N(z) = \#$ type 1 bubbles $- \#$ type 2 bubbles covering $z \in \mathbb{C}$. N is afflicted by logarithmic divergences, so define the "field"

$$e^{i\beta N(z)}, \ \beta \in \mathbb{R} .$$

Replacing Discs with Brownian Loops

Brownian loop soup: Poissonization of the Brownian loop measure μ_D on loops γ in D, i.e.,

$$\lambda\mu_D := \lambda \int_D \int_0^\infty \frac{1}{2\pi t^2}\, \mu^{br}_{z,t}\, \mathbb{1}_{\{\gamma \subset D\}}\, dt\, d\mathbf{A}(z)$$

where $\mathbf{A}$ denotes area and $\mu^{br}_{z,t}$ is the probability measure of a Brownian bridge of time length t started at z.

A loop γ is a continuous function $\gamma : [0, t_\gamma] \to \mathbb{C}$ with $t_\gamma \in (0, \infty)$ and $\gamma(0) = \gamma(t_\gamma)$.

Think of μ_D as a measure on *equivalence classes of loops modulo a time shift*: $\theta_u \gamma_r : t \mapsto \gamma_r(u + t \mod t_{\gamma_r})$.

From Brownian Loops to Conformal Fields

Take two independent Brownian loop soups with intensity $\lambda/2$ in D (bounded). Let, formally,

$N^i(z) =$ # loops in i-th soup that disconnect z from ∞ ($i = 1, 2$)

$N(z) = N^1(z) - N^2(z)$

$V_\beta(z) = e^{i\beta N(z)}$ ("vertex operator")

Conformal Correlation Functions

$\langle \ldots \rangle_{\delta,D}$ = expectation with respect to the Brownian loop soup in D restricted to loops of diameter $\geq \delta$

Theorem. If $n \in \mathbb{N}$, $D \subset \mathbb{C}$ is bounded and simply connected, and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, then

$$\lim_{\delta \to 0} \frac{\left\langle \prod_{j=1}^{n} e^{i\beta_j N(z_j)} \right\rangle_{\delta,D}}{\prod_{j=1}^{n} \delta^{\lambda \frac{\pi}{5}(1-\cos\beta_j)}} = \phi_D(z_1, \ldots, z_n; \boldsymbol{\beta})$$

exists and is finite and real. Moreover, if $\tilde{D}$ is another bounded, s.c. subset of $\mathbb{C}$ and $f : D \to \tilde{D}$ is a conformal map, then

$$\phi_{f(D)}(f(z_1), \ldots, f(z_n); \boldsymbol{\beta}) = \prod_{j=1}^{n} \left| f'(z_j) \right|^{-\lambda \frac{\pi}{5}(1-\cos\beta_j)} \phi_D(z_1, \ldots, z_n; \boldsymbol{\beta}) \, .$$

CFT language: The vertex operators $V_\beta(z) = e^{i\beta N(z)}$ behave like *conformal primaries* with *scaling exponents* $\lambda \frac{\pi}{5}(1 - \cos\beta)$.

Brownian Winding

Let, formally, $\theta(z)$ = total winding around z of all loops from a Brownian loop soup with intensity λ.

Theorem. If $n \in \mathbb{N}$, $D \subset \mathbb{C}$ is bounded and simply connected, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, and $c_k = \frac{1}{2\pi k^2}$ for $k \in \mathbb{Z} \setminus \{0\}$, then

$$\lim_{\delta \to 0} \frac{\left\langle \prod_{j=1}^n e^{i\beta_j \theta(z_j)} \right\rangle_{\delta,D}}{\prod_{j=1}^n \delta^{\lambda \sum_{k \neq 0} c_k (1 - \cos(k\beta_j))}} = \psi_D(z_1, \ldots, z_n; \boldsymbol{\beta})$$

exists and is finite and real. Moreover, if $\tilde{D}$ is another bounded, s.c. subset of $\mathbb{C}$ and $f : D \to \tilde{D}$ is a conformal map, then

$$\begin{aligned} &\psi_{f(D)}(f(z_1), \ldots, f(z_n); \boldsymbol{\beta}) \\ &= \prod_{j=1}^n \left| f'(z_j) \right|^{-\lambda \sum_{k \neq 0} c_k (1 - \cos(k\beta_j))} \psi_D(z_1, \ldots, z_n; \boldsymbol{\beta}) \,. \end{aligned}$$

The One-Point Function in the First Loop Model

$\langle \ldots \rangle_{\delta,R}$ = expectation with respect to the Brownian loop soup in $\mathbb{C}$ restricted to loops γ with $\delta \leq \text{diam}(\gamma) < R$

$A_{\delta,R}(z)$ = set of loops with diameter between δ and R disconnecting z from ∞

$$\begin{aligned}\left\langle e^{i\beta N(z)} \right\rangle_{\delta,R} &= \sum_{n=0}^{\infty} (\cos\beta)^n \, e^{-\lambda\mu_{\mathbb{C}}(A_{\delta,R}(z))} \frac{1}{n!} [\lambda\mu_{\mathbb{C}}(A_{\delta,R}(z))]^n \\ &= e^{-\lambda\mu_{\mathbb{C}}(A_{\delta,R}(z))} \sum_{n=0}^{\infty} \frac{[\lambda\mu_{\mathbb{C}}(A_{\delta,R}(z))]^n}{n!} \\ &= e^{-\lambda\mu_{\mathbb{C}}(A_{\delta,R}(z))(1-\cos\beta)}\end{aligned}$$

Lemma. For any $z \in \mathbb{C}$ and $\delta < R$,

$$\mu_{\mathbb{C}}(A_{\delta,R}(z)) = \frac{\pi}{5} \log \frac{R}{\delta}\,.$$

Using the lemma, we obtain

$$\left\langle e^{i\beta N(z)} \right\rangle_{\delta,R} = \left(\frac{R}{\delta}\right)^{-\lambda\frac{\pi}{5}(1-\cos\beta)} \,.$$

Remark: $\frac{\pi}{5}$ is the expected area of a planar Brownian loop of time length one (Garban and Trujillo Ferreras, CMP **264**, 2006).

Proof of the Lemma

The lemma follows easily from

Theorem. (Werner, 2008) For any simply connected sets $D' \subset D$ and any $z \in D'$,

$$\mu_{\mathbb{C}}(\{\gamma : \gamma \subset D, \gamma \not\subset D', \gamma \text{ disconnects } z \text{ from } \infty\}) = \frac{\pi}{5} \log \Phi'(z)$$

where Φ denote the conformal map from D' to D such that $\Phi(z) = z$ and $\Phi'(z)$ is real and positive.

... and the observation that, if

$A_\delta^R(z)$ = set of loops contained in disc of radius R centered at z but not completely contained in disc of radius δ centered at z and such that they disconnect z from ∞,

then

$$\mu_{\mathbb{C}}(A_{\delta,R}(z)) = \mu_{\mathbb{C}}(A_\delta^R(z)) = \frac{\pi}{5} \log \frac{R}{\delta}\,.$$

The Two-Point Function in Bounded Domains

$A_\delta(z_1, z_2)$ = set of loops with diameter $\geq \delta$ disconnecting z_1 and z_2 from ∞

$$\left\langle e^{i\beta_1 N(z_1)} e^{i\beta_2 N(z_2)} \right\rangle_{\delta,D} = \left\langle e^{i(\beta_1+\beta_2)N_{1,2}} \right\rangle_{\delta,D} \left\langle e^{i\beta_1 N_1} \right\rangle_{\delta,D} \left\langle e^{i\beta_2 N_2} \right\rangle_{\delta,D}$$

$N_{1,2}$ = # loops that disconnect both z_1 and z_2 from ∞
N_1 = # loops that disconnect z_1 but not z_2 from ∞
N_2 = # loops that disconnect z_2 but not z_1 from ∞

$$\left\langle e^{i(\beta_1+\beta_2)N_{1,2}} \right\rangle_{\delta,D} = e^{-\lambda \mu_D(A(z_1,z_2))(1-\cos(\beta_1+\beta_2))}$$

where $A(z_1, z_2)$ is the set of loops that disconnect both z_1 and z_2 from ∞.

Note that $\mu_{f(D)}(A(f(z_1), f(z_2))) = \mu_D(A(z_1, z_2))$.

The One-Point Function in Bounded Domains

$\langle \ldots \rangle_{\delta,D}$ denotes expectation with respect to the Brownian loop soup in D with cutoff $\delta > 0$ on the diameter of the loops.

$A_\delta(z) =$ set of loops with diameter $\geq \delta$ disconnecting z from ∞

$$\mu_D(A_\delta(z)) = \frac{\pi}{5} \log \Phi'(z) + \mu_{B_\delta(z)}(A_\delta(z))$$

where $B_\delta(z) = \{w \in \mathbb{C} : |z - w| < \delta\}$ and Φ is the conformal map from $B_\delta(z)$ to D such that $\Phi(z) = z$ and $\Phi'(z)$ is real and positive.

$$\begin{aligned} \left\langle e^{i\beta N(z)} \right\rangle_{\delta,D} &= e^{-\lambda \mu_D(A_\delta(z))(1-\cos\beta)} \\ &= \left(\Phi'(z)\right)^{-\frac{\pi}{5}\lambda(1-\cos\beta)} \; e^{-\lambda \mu_{B_\delta(z)}(A_\delta(z))(1-\cos\beta)} \end{aligned}$$

$$\frac{\left\langle e^{i\beta N(\tilde{z})}\right\rangle_{\delta,\tilde{D}}}{\left\langle e^{i\beta N(z)}\right\rangle_{\delta,D}} = \exp\left\{\lambda(1-\cos\beta)\left[\mu_D(A_\delta(z)) - \mu_{\tilde{D}}(A_\delta(\tilde{z}))\right]\right\}$$

$$\begin{aligned}
\mu_D(A_\delta(z)) - \mu_{\tilde{D}}(A_\delta(\tilde{z})) &= \frac{\pi}{5}\log\Phi'(z) + \mu_{B_\delta(z)}(A_\delta(z)) \\
&- \left(\frac{\pi}{5}\log\tilde{\Phi}'(\tilde{z}) + \mu_{B_\delta(\tilde{z})}(A_\delta(\tilde{z}))\right) \\
&= \frac{\pi}{5}\log\frac{1}{\Psi'(z)\delta} - \frac{\pi}{5}\log\frac{1}{\tilde{\Psi}'(\tilde{z})\delta} + O(\delta) \\
&= \frac{\pi}{5}\log f'(z) + O(\delta)
\end{aligned}$$

$$\left\langle e^{i\beta N(\tilde{z})}\right\rangle_{\delta,\tilde{D}} = \left(f'(z)\right)^{-\frac{\pi}{5}\lambda(1-\cos\beta)+O(\delta)}\left\langle e^{i\beta N(z)}\right\rangle_{\delta,D}$$

Open Questions

- Existence of limiting field $\varphi_{\boldsymbol{\beta}}$?
  That is, $\phi_D(z_1, \ldots, z_n; \boldsymbol{\beta}) = \left\langle \varphi_{\boldsymbol{\beta}}(z_1) \ldots \varphi_{\boldsymbol{\beta}}(z_n) \right\rangle_D$?

- Scaling limit of model based on random walk loop soup?

- Connections with models of statistical mechanics?

- Connections with cosmology and/or string theory?