Random Geometry in the Spectral Measure of the Circular Beta Ensemble

Tom Alberts

University of Utah

Joint work with
Bálint Virág (Toronto), Raoul Normand (Academia Sinica)

Circular Beta Ensemble

- $\beta \geq 0$
- Point process of n points on the unit circle $\partial\mathbb{D}$
- Points $e^{i\theta_1}, \ldots, e^{i\theta_n}$ distributed according to

$$\frac{1}{Z_n(\beta)} \prod_{1 \leq j \neq k \leq n} \left| e^{i\theta_j} - e^{i\theta_k} \right|^\beta \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi}$$

- **Partition function** $Z_n(\beta)$ is explicitly known [Wil62, Goo70]

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- Natural question:

Are there **random**, **unitary** *matrices*
whose eigenvalue distribution is Circular-β?

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- For $\beta = 2$ this is well known:
  - $\beta = 2$: Haar distributed random matrices from $SU(n)$

Circular Beta Ensemble

- For general β the answer is also **yes** [KN04, KN07]
- Uses the **CMV representation** [CMV03]
- CMV is an analogue of the **tri-diagonal representation** for self-adjoint operators
- CMV is 5-diagonal
- Is the sparsest possible representation of a unitary matrix

Recipe for CMV

- Input is a sequence $\{\alpha_j, j \geq 0\}$, taking values in $\bar{\mathbb{D}}$
- Called the **Verblunsky coefficients**
- Given α_j, define the 2×2 unitary matrix Θ_j by

$$\Theta_j = \begin{pmatrix} \bar{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix}, \quad \rho_j = \sqrt{1 - |\alpha_j|^2}$$

- Let $\mathcal{L}$ and $\mathcal{M}$ be the infinite matrices

$$\mathcal{L} = \begin{pmatrix} \Theta_0 & & & \\ & \Theta_2 & & \\ & & \Theta_4 & \\ & & & \ddots \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 1 & & & \\ & \Theta_1 & & \\ & & \Theta_3 & \\ & & & \ddots \end{pmatrix},$$

- Then $\mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots) = \mathcal{L}\mathcal{M}$ is the **CMV matrix** determined by the Verblunskies $\{\alpha_j, j \geq 0\}$

Recipe for CMV

- Note that $\mathcal{C}$ as defined is an infinite matrix
- Treated as an operator on

$$\ell^2(\{0,1,2,\ldots\}) = \left\{ (a_k)_{k=0}^{\infty} : \sum_{k=0}^{\infty} |a_k|^2 < \infty \right\}$$

- To get an operator on $\mathbb{C}^n$, simply set $|\alpha_{n-1}| = 1$ and take the upper $n \times n$ block of $\mathcal{C}$

$$\mathcal{C} = \begin{pmatrix} \bar{\alpha}_0 & \bar{\alpha}_1\rho_0 & 0 & 0 & 0 & \ldots \\ \rho_0 & -\bar{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \ldots \\ 0 & \bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_3\rho_2 & \ldots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \ldots \\ 0 & 0 & 0 & \bar{\alpha}_4\rho_3 & -\bar{\alpha}_4\alpha_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Circular Beta Ensemble

- For eigenvalues with the Circular-β distribution, choose the α_j **independent** with marginal distributions

$$\alpha_j \sim e^{2\pi i \,\mathrm{Uniform}(0,1)} \sqrt{\mathrm{Beta}\left(1, \frac{\beta(j+1)}{2}\right)}$$

which has pdf

$$\frac{\beta(j+1)}{2\pi}(1-|z|^2)^{\frac{\beta(j+1)}{2}-1}\, d^2 z$$

with respect to Lebesgue measure on $\mathbb{D}$

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- Note α_j are **rotationally invariant**,

$$\mathbf{E}[\alpha_j] = 0, \quad \mathbf{E}[|\alpha_j|^2] = \frac{2}{\beta(j+1)+2}$$

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- **Theorem: [KN04, KR10]** Let $U \sim \mathrm{Uniform}(0,1)$ be independent of the Verblunskies. Then for each fixed n the matrix

$$\mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, e^{2\pi i U}, 0, 0, 0, \ldots)$$

has eigenvalues distributed according to Circular-β

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- Note the “nesting” property of these matrices

- Suggests that it is worthwhile to study the infinite matrix $\mathcal{C}(\alpha_0, \alpha_1, \alpha_2, \ldots)$

Circular Beta Ensemble

$$\mathcal{C}(\alpha_0, \alpha_1, \alpha_2, \ldots)$$

- Operator is well-defined. What happens to the eigenvalues?
- Need a notion of a limit of a point process
- Could study the limit of the **empirical measure**

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{e^{i\theta_k}}$$

but it converges almost surely to uniform measure on $\partial\mathbb{D}$, for all β

- Instead study the **spectral measure**

Spectral Measure for CMV Matrices

- Assume all Verblunskies have $|\alpha_n| < 1$ and let $\mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots)$
- Let $\delta_0 = (1, 0, 0, \ldots) \in \ell^2$. Can be shown that linear combinations of $\{\mathcal{C}^m \delta_0\}_{m \in \mathbb{Z}}$ are dense in ℓ^2

Spectral Theorem for CMV

- There exists a probability measure μ on $\partial\mathbb{D}$ such that the mapping $V : \ell^2 \to L^2(\partial\mathbb{D}, d\mu)$ defined by

$$\begin{array}{rcl} V : \ell^2 & \longrightarrow & L^2(\partial\mathbb{D}, d\mu) \\ \mathcal{C}^m \delta_0 & \mapsto & z^m \end{array}$$

  is **unitary**, and $V(\mathcal{C}\mathbf{x}) = zV(\mathbf{x})$

- *In short:* Make the space more complicated ($L^2(\partial\mathbb{D}, d\mu)$ instead of ℓ^2), but the action of the operator simpler (multiplication by the function z)

Examples of Spectral Measure

- $\alpha_n \equiv 0$

$$\mu = \text{Lebesgue}$$

- $\alpha_0 = \zeta \in \mathbb{D}, \alpha_n = 0$ for $n \geq 1$

$$d\mu(\theta) = \frac{1 - |\zeta|^2}{|1 - \zeta e^{i\theta}|^2} \frac{d\theta}{2\pi}$$

- $\alpha_n = -1/(n+2)$

$$d\mu(\theta) = 1 - \cos\theta \, \frac{d\theta}{2\pi}$$

Orthogonal Polynomials on the Unit Circle

- Spectral theorem gives a way of going from CMV matrix to probability measures
- **Orthogonal polynomials** gives a way of going in reverse

$$\Phi_n(z) := P_n[z^n], P_n := \text{ projection onto } \{1, z, \ldots, z^{n-1}\}^\perp, \Phi_0 \equiv 1$$

Projection is in the $L^2(\partial\mathbb{D}, d\mu)$ inner product

- **Szego Recursion:**

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z)$$

where $\alpha_n \in \mathbb{D}$, and

$$\left(\sum_{k=0}^{n} a_k z^k \right)^* = \sum_{k=0}^{n} \bar{a}_{n-k} z^k$$

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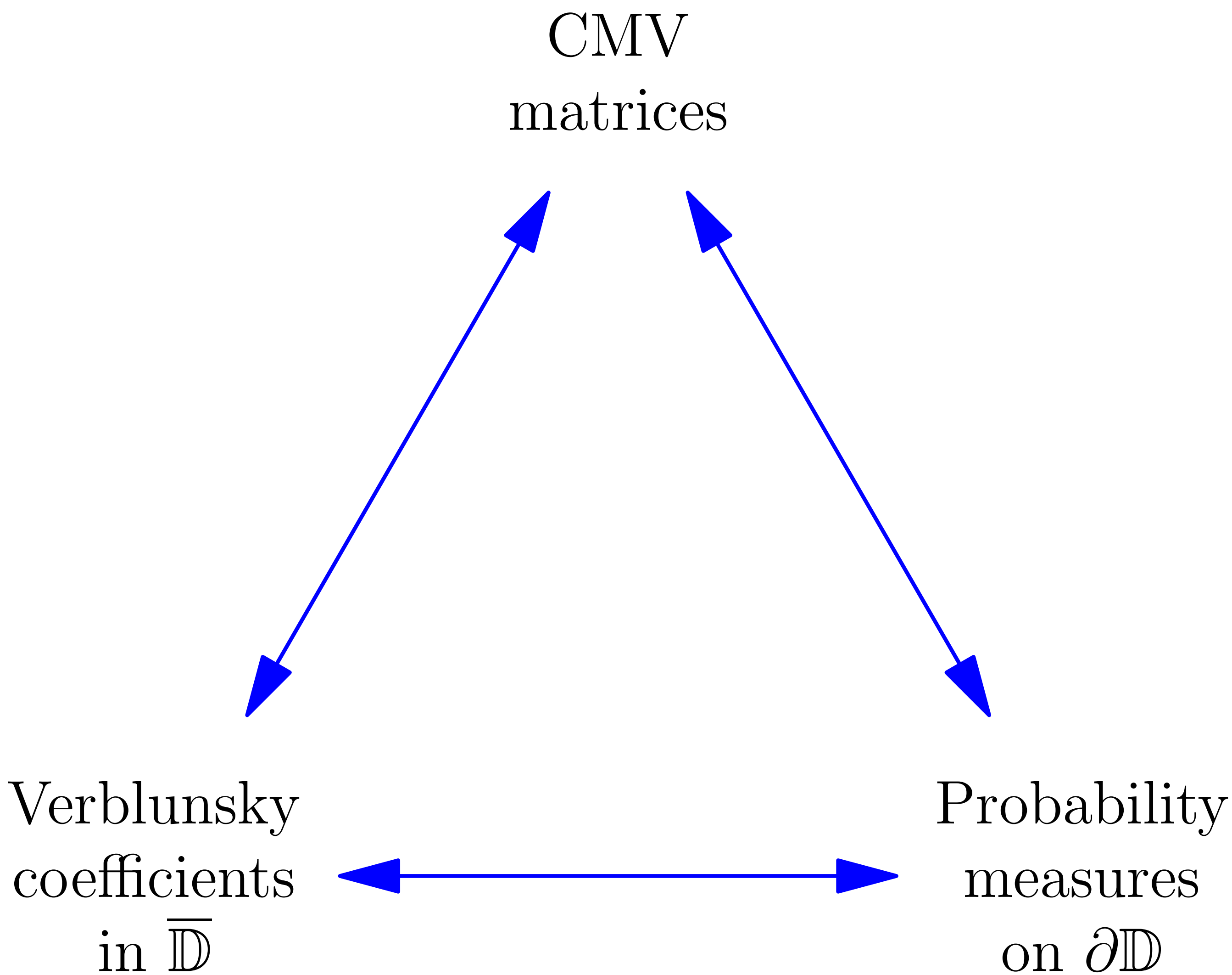
where $\alpha_n \in \mathbb{D}$, and $\mathcal{C}(\alpha_0, \alpha_1, \alpha_2, \ldots)$ has spectral measure μ

Triumvirate of Objects



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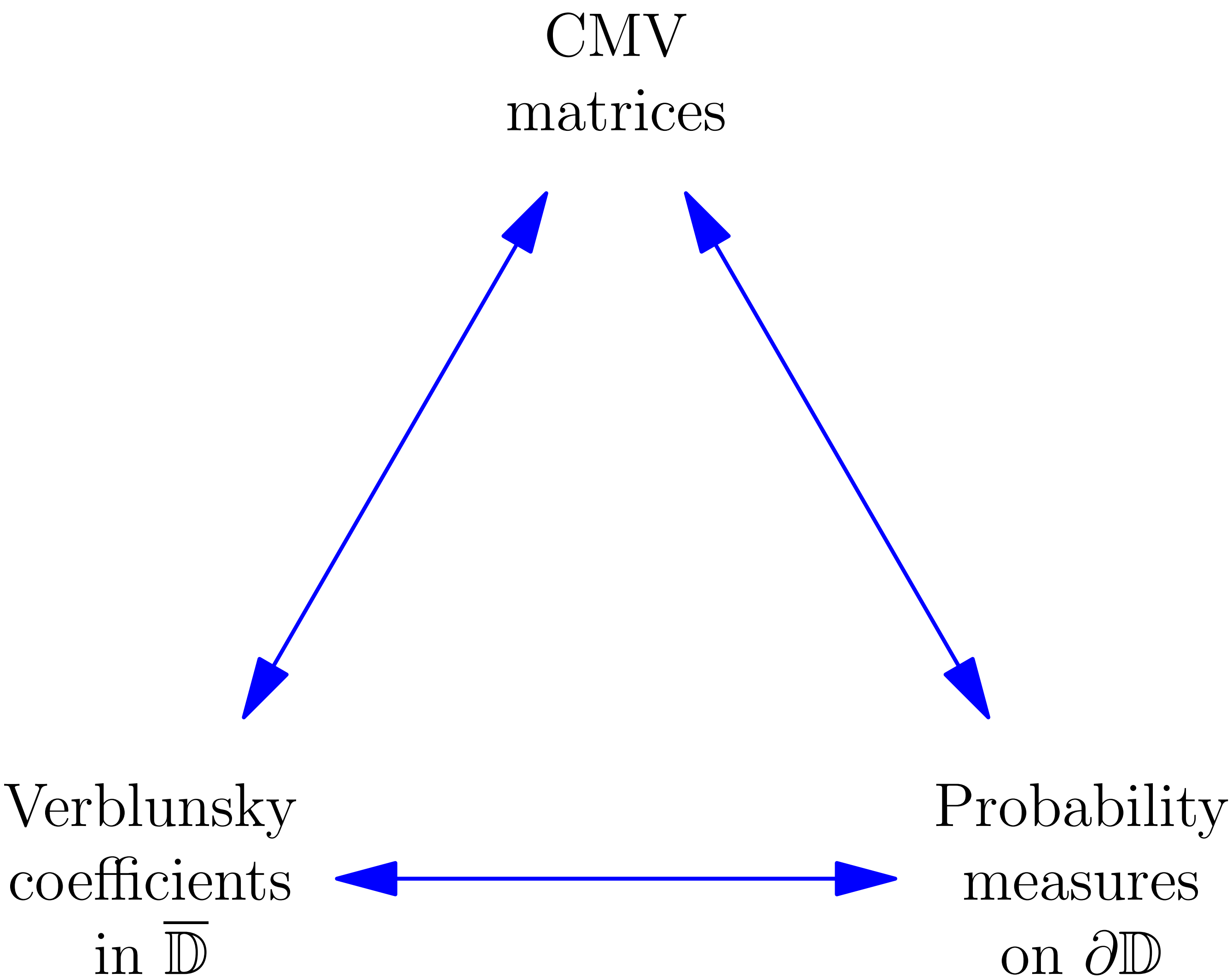
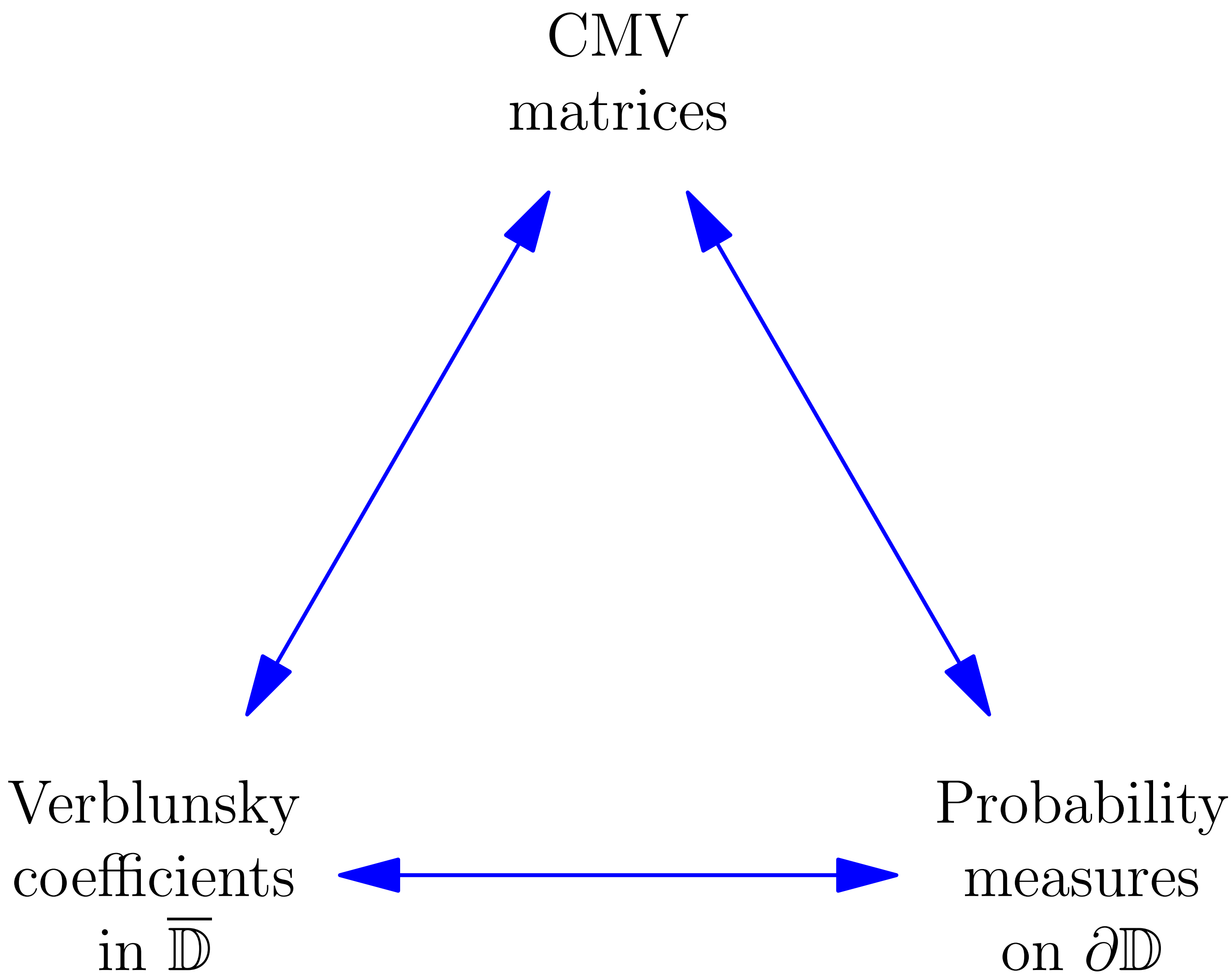


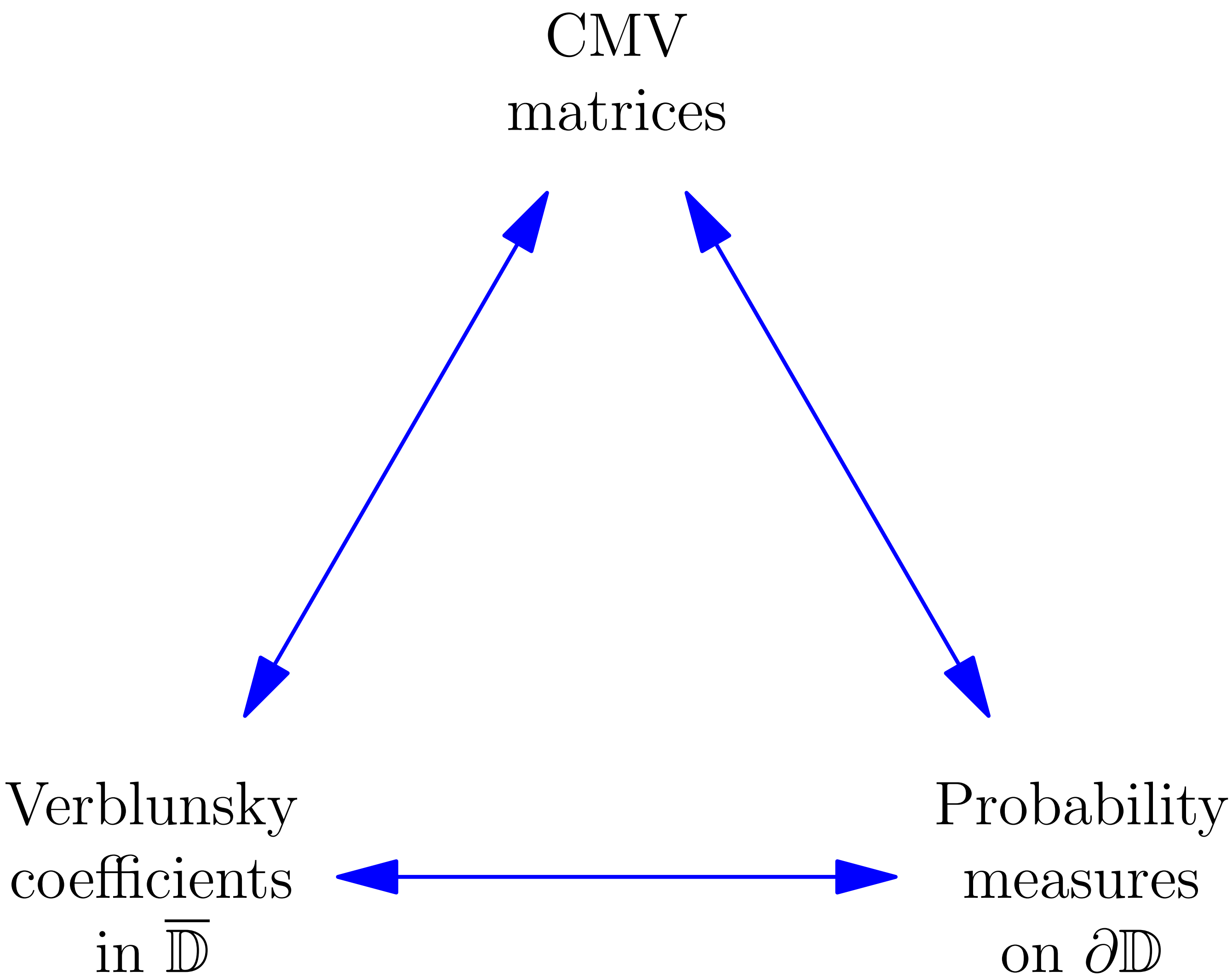
- There is a bijection between any two of them

Triumvirate of Objects



- For circular-β ensemble, the Verblunsky coefficients are the simplest objects

Triumvirate of Objects



- Can we translate properties of Verblunskies into properties of operators/spectral measures?

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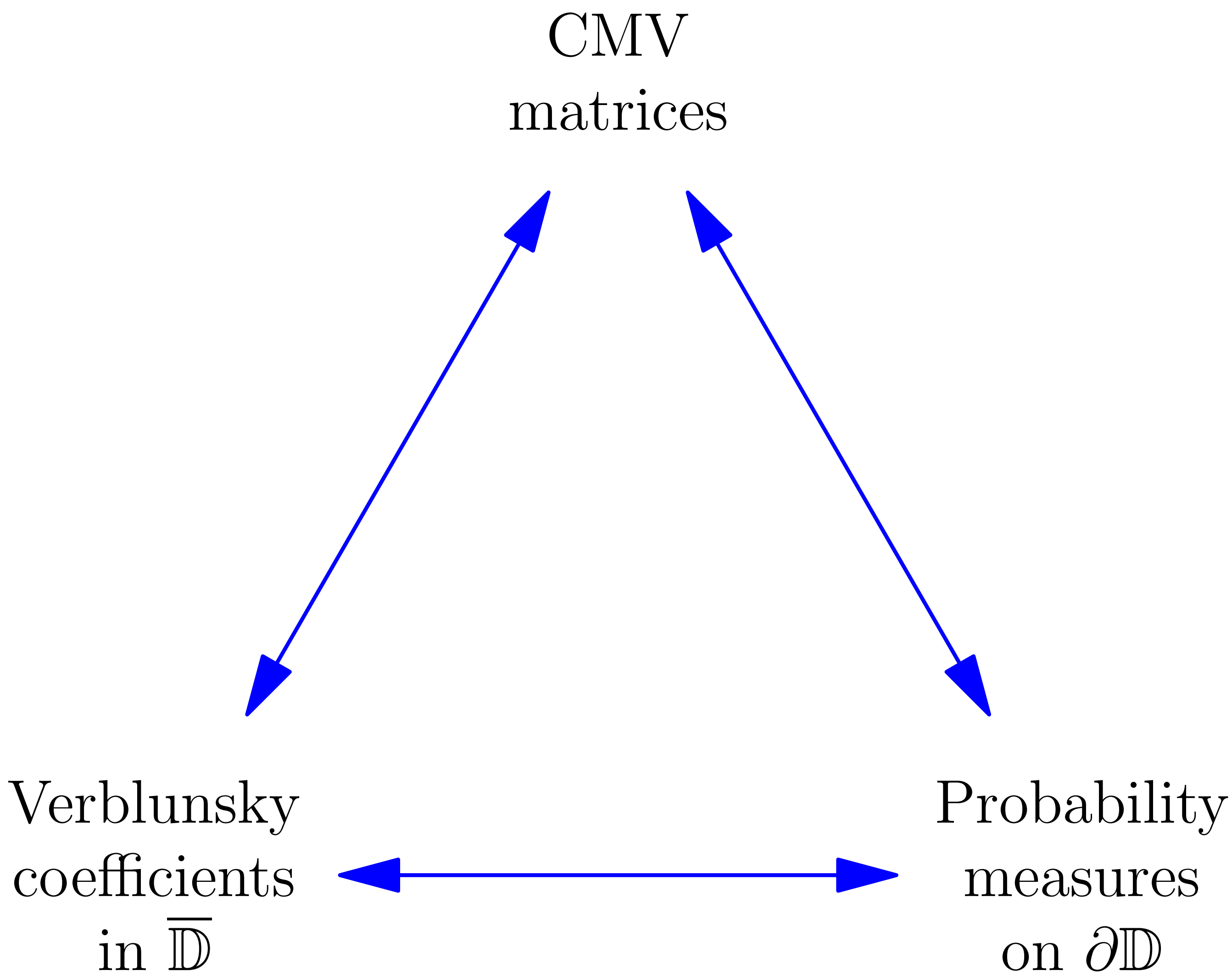

- Subject of Barry Simon's two volumes *Orthogonal Polynomials on the Unit Circle* [Sim05a, Sim05b]

Spectral Measure for Circular-β

- For the infinite-dimensional circular-β operator, the spectral measure exhibits a rich random geometry

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- For the infinite-dimensional circular-β operator, the spectral measure exhibits a rich random geometry

Proposition [Sim05b]:

- For $\beta \geq 2$, the spectral measure is purely singular continuous with respect to Lebesgue and has exact Hausdorff dimension $1 - 2/\beta$

(in the sense that there is a set A of dimension $1 - 2/\beta$ with $\mu(\partial\mathbb{D}\setminus A) = 0$ and μ assigns zeros mass to any subset of $\partial\mathbb{D}$ with Hausdorff dimension strictly less than $1 - 2/\beta$)

- For $\beta < 2$ the spectral measure is pure point but supported on a dense subset of $\partial\mathbb{D}$

- The phase transition at $\beta = 2$ is reminiscent of that for the **Liouville quantum gravity measure** on $\partial\mathbb{D}$

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- The phase transition at $\beta = 2$ is reminiscent of that for the **Liouville quantum gravity measure** on $\partial\mathbb{D}$ ($\beta = 8/\kappa$)

Spectral Measure for Circular-β

- For the infinite-dimensional circular-β operator, the spectral measure exhibits a rich random geometry

- **Theorem in Progress: [Alberts-Normand-Virag]:** In the $\beta \geq 2$ phase we can compute the multifractal spectrum of the spectral measure for Circular-β.

  Fix a realization of the spectral measure μ. The multifractal spectrum is the function

$$\zeta \mapsto \dim_H \left\{ \theta : \limsup_{r \to 0} \frac{\log \mu(B(\theta, r))}{\log r} \geq \zeta \right\}$$

  The spectrum is an almost sure quantity.

- Proof is an adaptation of that used for **Eggleston measures** or **multiplicative cascades**

Spectral Measure for Circular-β

- Four key tools used in Simon's proof:
  - **Szego recursion**
  - **transfer matrices**
  - **Bernstein-Szego approximation**
  - **Jitomirskaya-Last inequalities**

Bernstein-Szego approximation

- Recall $\Phi_n(z)$ is the monic orthogonal polynomial of degree n, in $L^2(\partial\mathbb{D}, d\mu)$
- Define φ_n to be the **normalized** orthogonal polynomial

$$\varphi_n := \frac{\Phi_n}{||\Phi_n||}, \quad ||\Phi_n||^2 = \int_{\partial\mathbb{D}} |\Phi_n(e^{i\theta})|^2 \frac{d\theta}{2\pi}$$

- **Bernstein-Szego Approximation:** The measures

$$|\varphi_n(e^{i\theta})|^{-2} \frac{d\theta}{2\pi}$$

are probability measures, and they converge to μ as $n \to \infty$

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- **Lemma: [ANR]** For independent and rotationally invariant Verblunskies, $n \mapsto |\varphi_n(e^{i\theta})|^{-2}$ is a martingale

(gives a probabilistic proof of Bernstein-Szego!)

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  are probability measures, and they converge to μ as $n \to \infty$
- Key computation in Simon's proof: with probability one

$$\lim_{n\to\infty} \frac{\log|\varphi_n(1)|^{-2}}{\log n} = -\frac{2}{\beta}$$

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with respect to the measure $dQ(\{\alpha\}_{n=0}^{\infty})\,\delta_0(\theta)$

Bernstein-Szego approximation

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(dQ the measure under which Verblunskies $\{\alpha\}_{n=0}^{\infty}$ have Circular-β distribution)

Bernstein-Szego approximation

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- Key computation in our analysis: compute the same limit under the measure $dQ(\{\alpha_n\}_{n=0}^{\infty})\,d\mu_\alpha(\theta)$, where μ_α is the measure determined by the Verblunskies

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- Key computation in our analysis: compute the same limit under the measure $dQ(\{\alpha_n\}_{n=0}^{\infty})\, d\mu_\alpha(\theta)$, where μ_α is the measure determined by the Verblunskies

- Rewrite using marginal of θ (Lebesgue) and conditional of $\{\alpha_n\}_{n=0}^{\infty}$ as

$$dQ(\{\alpha_n\})\, d\mu_\alpha(\theta) = dQ_\theta(\{\alpha_n\})\, \frac{d\theta}{2\pi}$$

Bernstein-Szego approximation

$$dQ(\{\alpha_n\})\, d\mu_\alpha(\theta) = dQ_\theta(\{\alpha_n\})\, \frac{d\theta}{2\pi}$$

- **Theorem [Alberts-Normand-Virag]:** We understand the measures Q_θ completely. Under Q_0, the Verblunskies form a **Markov Chain** with an explicit transition kernel

$$Q_0(\alpha_{n+1} \in dz | \alpha_n, V_n) = \frac{1-|z|^2}{|V_{n+1}-z|^2} Q(\alpha_{n+1} \in dz)$$

with $V_{n+1} \in \partial\mathbb{D}$ determined by $\alpha_0, \ldots, \alpha_n$ from

$$V_{n+1} = \frac{V_n - \alpha_n}{1 - \bar{\alpha}_n V_n}, V_0 = 1$$

Moreover, there exists a deterministic algorithm using conformal maps that turns Verblunskies distributed according to Q into Verblunskies distributed according to Q_θ

Bernstein-Szego approximation

$$dQ(\{\alpha_n\})\, d\mu_\alpha(\theta) = dQ_\theta(\{\alpha_n\})\, \frac{d\theta}{2\pi}$$

- With probability one

$$\lim_{n\to\infty} \frac{\log|\varphi_n(e^{i\theta})|^{-2}}{\log n} = +\frac{2}{\beta}$$

with respect to the measure $dQ(\{\alpha\}_{n=0}^{\infty})\, d\mu_\alpha(\theta)$

- **Other Result:** The V_n process picks up a speed

Future Work

- Establish a **(strong) KPZ relation** for the spectral measure of Circular-β
- The spectral measures for Circular-β induce random metrics on ∂D
- It should be possible to show that Hausdorff dimensions transform the same way as for multiplicative cascades (Benjamini-Schramm)

Future Work

- Understand connections with **multiple SLE**
- Consider the Loewner equation

$$\partial_t g_t(z) = -g_t(z) \sum_{k=1}^{n} \frac{g_t(z) + e^{i\theta_k(t)}}{g_t(z) - e^{i\theta_k(t)}}$$

where $t \mapsto (\theta_1(t), \ldots, \theta_n(t))$ evolves by **Dyson's Brownian motion**

$$d\theta_j(t) = \sum_{k \neq j} \cot((\theta_k(t) - \theta_j(t))/2)\, dt + \sqrt{\kappa} dB_j(t)$$

- Circular-β is stationary distribution, with $\beta = 8/\kappa$

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- Circular-β is stationary distribution, with $\beta = 8/\kappa$
- Note all curves grow at the same rate
- **Cardy:** *It is unclear that this produces conformal invariance*

Slides Produced With

Asymptote: The Vector Graphics Language

`http://asymptote.sf.net`

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