Complex Analysis in Backward SLE

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Everything is complex
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Based on a joint work with Steffen Rohde.

Preliminary

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Sketch proof

Schramm's SLE process is successful in describing random fractal curves, which are the scaling limit of some critical two-dimensional lattice models, which include critical percolation ([Smi01]), loop-erased random walk and uniform spanning tree ([LSW04]), critical Ising model and critical FK-Ising model ([CDCH+13]), and etc. The definition of SLE combines the Loewner's differential equation with a random driving function: Brownian motion.

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Backward SLE uses backward Loewner equation, which differs from the forward equation by a minus sign. The goal of the joint work was to study the backward SLE process as a whole instead of only the hulls at fixed capacity times. Prior to our work, S. Sheffield proved the existence of a coupling of a backward chordal SLE_κ with a free boundary GFF such that real intervals $[x,0]$ and $[0,y]$ with the same quantum weight are welded by the backward SLE process.

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To explain the idea, let me briefly recall some notation.

- $\mathbb{H} := \{z \in \mathbb{C} : \Im z > 0\}$ is the upper half plane.
- An $\mathbb{H}$-hull is a bounded set $K \subset \mathbb{H}$ such that $\mathbb{H} \setminus K$ is a simply connected domain.
- For an $\mathbb{H}$-hull K, g_K is the unique conformal map from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ such that $g_K(z) = z + \frac{c(K)}{z} + O(1/z^2)$ as $z \to \infty$. Let $f_K = g_K^{-1}$.
- $\mathrm{hcap}(K) := c(K)$ is called the $\mathbb{H}$-capacity of K. We have $\mathrm{hcap}(\emptyset) = 0$ and $\mathrm{hcap}(K_1) < \mathrm{hcap}(K_2)$ if $K_1 \subsetneqq K_2$.

The double of K: K^{doub} is the union of $\overline{K}$ and the reflection of K about $\mathbb{R}$. By Schwarz reflection principle, g_K extends to a conformal map from $\mathbb{C} \setminus K^{\text{doub}}$ onto $\mathbb{C} \setminus S_K$ for some compact $S_K \subset \mathbb{R}$ called the support of K.

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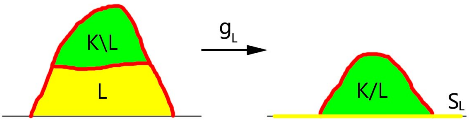
If $K_1\subset K_2$ are two $\mathbb{H}$-hulls, we define $K_2/K_1=g_{K_1}(K_2\setminus K_1)$, which is also an $\mathbb{H}$-hull. We call K_2/K_1 a quotient hull of K_2, and write $K_2/K_1\prec K_2$. If $K_3\prec K_2$, then $\text{hcap}(K_3)\le\text{hcap}(K_2)$, $S_{K_3}\subset S_{K_2}$, and there is a unique $K_1\subset K_2$ s.t. $K_3=K_2/K_1$. We write $K_1=K_2:K_3$.



An $\mathbb{H}$-Loewner chain is a strictly increasing family of $\mathbb{H}$-hulls $(K_t)_{0\le t<T}$, which starts from $K_0=\emptyset$, and satisfies that

$$\bigcap_{0<\varepsilon<T-t}\overline{K_{t+\varepsilon}/K_t}=\{\lambda_t\},\quad 0\le t<T,$$

for some real continuous function λ_t, $0\le t<T$.

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If u is a continuously (strictly) increasing function with $u(0)=0$, then $K_{u^{-1}(t)}$, $0\le t<u(T)$, is also an $\mathbb{H}$-Loewner chain, and is called a time-change of (K_t). An $\mathbb{H}$-Loewner chain is called normalized if $\mathrm{hcap}(K_t)=2t$ for each t. Every $\mathbb{H}$-Loewner chain can be normalized by applying a time-change.

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Example. We say that γ_t, $0\le t<T$, an $\mathbb{H}$-simple curve, if $\gamma_0\in\mathbb{R}$ and $\gamma_t\in\mathbb{H}$ for $t>0$. An $\mathbb{H}$-simple curve γ generates an $\mathbb{H}$-Loewner chain: $K_t=\gamma(0,t]$, $0\le t<T$.

Let $\lambda \in C([0,T),\mathbb{R})$. The (forward) chordal Loewner equation driven by λ is

$$\partial_t g_t(z) = \frac{2}{g_t(z)-\lambda_t}, \quad g_0(z) = z.$$

For $0 \le t < T$, let K_t denote the set of $z \in \mathbb{H}$ such that the solution $s \mapsto g_s(z)$ blows up before or at time t. Then each K_t is an $\mathbb{H}$-hull with $\mathrm{hcap}(K_t) = 2t$ and $g_{K_t} = g_t$. We call g_t and K_t the chordal Loewner maps and hulls driven by λ. Chordal SLE$_\kappa$ is defined by taking $\lambda_t = \sqrt{\kappa} B_t$, where $\kappa > 0$ and B_t is a standard Brownian motion.

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Proposition [LSW01]

(K_t) are chordal Loewner hulls driven by some continuous function iff it is a normalized $\mathbb{H}$-Loewner chain. Moreover, when the above holds, the driving function λ is given by $\bigcap_{0<\varepsilon<T-t} \overline{K_{t+\varepsilon}/K_t} = \{\lambda_t\}$, $0 \le t < T$.

The backward chordal Loewner equation driven by λ is

$$\partial_t f_t(z) = \frac{-2}{f_t(z) - \lambda_t}, \quad f_0(z) = z.$$

For every t, f_t is well defined on $\mathbb{H}$, and maps $\mathbb{H}$ conformally onto $\mathbb{H} \setminus L_t$ for some $\mathbb{H}$-hull L_t. We have $\text{hcap}(L_t) = 2t$ and $f_t = f_{L_t}$. But (L_t) may not be an increasing family. Instead, it satisfies that $L_{t_1} \prec L_{t_2}$ if $t_1 \le t_2$. To describe other properties of (L_t), we need the notation of quotient Loewner chain.

A family of $\mathbb{H}$-hulls $(L_t)_{0\le t<T}$ is called a quotient $\mathbb{H}$-Loewner chain if it satisfies that $L_0=\emptyset$, $L_{t_1}\prec L_{t_2}$ when $t_1<t_2$, and

$$\bigcap_{0<\varepsilon<t}\overline{L_t:L_{t-\varepsilon}}=\{\lambda_t\},\quad 0<t<T,$$

for some real continuous function λ_t, $0\le t<T$. Here $L_t:L_{t-\varepsilon}$ is decreasing in ε. We say (L_t) is normalized if $\mathrm{hcap}(L_t)=2t$ for each t. Every quotient $\mathbb{H}$-Loewner chain can be normalized by applying a time-change.

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Proposition

(L_t) are backward chordal Loewner hulls driven by some continuous function λ iff it is a normalized quotient $\mathbb{H}$-Loewner chain. Moreover, when the above holds, we have $\bigcap_{0<\varepsilon<t}\overline{L_t:L_{t-\varepsilon}}=\{\lambda_t\}$, $0\le t<T$.

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Let's first observe how a forward $\mathbb{H}$-Loewner chain is transformed by a conformal map. The technique was used to study the locality of SLE_6 ([LSW01]) and restriction of $\mathrm{SLE}_{8/3}$ ([LSW02]).

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We call a domain $\mathbb{R}$-symmetric if it is invariant under the conjugate map $z \mapsto \overline{z}$. We call a conformal map $\mathbb{R}$-symmetric if its definition domain is $\mathbb{R}$-symmetric, it commutes with the conjugate map, and its derivatives on $\mathbb{R}$ are positive. For example, g_K and f_K are $\mathbb{R}$-symmetric after extensions.

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Let (K_t) be an $\mathbb{H}$-Loewner chain, and W an $\mathbb{R}$-symmetric conformal map whose domain Ω contains every K_t^{doub}. Then $(W(K_t))$ is an increasing family of $\mathbb{H}$-hulls. We will see in the next slide that $(W(K_t))$ is an $\mathbb{H}$-Loewner chain. This is obvious if (K_t) is generated by an $\mathbb{H}$-simple curve.

Let $W_t = g_{W(K_t)} \circ W \circ f_{K_t}$. Then W_t is a conformal map defined on a neighborhood of S_{K_t} minus S_{K_t}. By Schwarz reflection principle, W_t extends to a conformal map on the neighborhood of S_{K_t}. From $g_{W(K_t)} \circ W = W \circ g_{K_t}$, we get

$$W(K_{t+\varepsilon})/W(K_t) = W_t(K_{t+\varepsilon}/K_t), \quad \varepsilon > 0.$$

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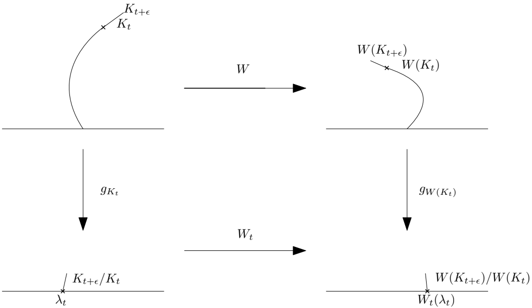

From $\bigcap_{\varepsilon>0}\overline{K_{t+\varepsilon}/K_t}=\{\lambda_t\}$ we get $\bigcap_{\varepsilon>0}\overline{W(K_{t+\varepsilon})/W(K_t)}=\{W_t(\lambda_t)\}$. So $(W(K_t))$ is also an $\mathbb{H}$-Loewner chain.

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If (K_t) are chordal Loewner hulls driven by λ, then (K_t) is normalized but $(W(K_t))$ may not be normalized. Let $u(t)=\mathrm{hcap}(W(K_t))/2$ be the time-change function. Then $(W(K_{u^{-1}(t)}))$ are chordal Loewner hulls driven by $W_{u^{-1}(t)}(\lambda_{u^{-1}(t)})$.

We tried to develop a similar theory for quotient $\mathbb{H}$-Loewner chain. Let (L_t) be a quotient $\mathbb{H}$-Loewner chain. Let W be an $\mathbb{R}$-symmetric conformal map. Then $(W(L_t))$ may not be a quotient $\mathbb{H}$-Loewner chain because $L_{t_1} \prec L_{t_2}$ does not imply that $W(L_{t_1}) \prec W(L_{t_2})$. This means that we can not define $(W(L_t))$ as the conformal transformation of (L_t) under W. Instead, we want to find a continuous family of conformal maps (W^{L_t}) such that $W^{L_0} = W$ and $(W^{L_t}(L_t))$ is a quotient $\mathbb{H}$-Loewner chain. We need the following theorem.

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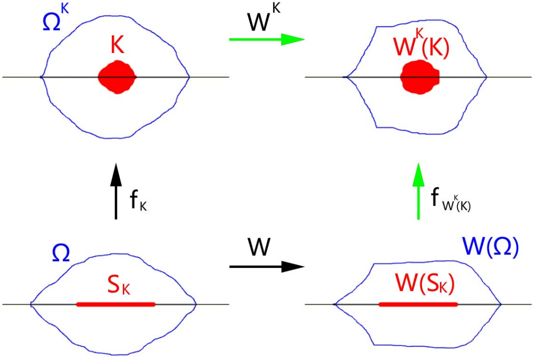
Theorem 1.

Let K be an $\mathbb{H}$-hull. Let W be an $\mathbb{R}$-symmetric conformal map, whose domain Ω contains S_K. Then there is a unique conforml map W^K defined on $\Omega^K := f_K(\Omega \setminus S_K) \cup K^{\text{doub}}$ such that $W = g_{W^K(K)} \circ W^K \circ f_K$.



It is easy to get W from W^K using Schwarz reflection principle, but non-trivial to get W^K from W.

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We now explain how Theorem 1 is applied. Suppose (L_t) is a quotient $\mathbb{H}$-Loewner chain, and W is an $\mathbb{R}$-symmetric conformal map, whose domain contains S_{L_t} for every t. Let (W^{L_t}) be given by the theorem. For $t_1 < t_2$, from $L_{t_1} \prec L_{t_2}$ we can conclude that $W^{L_{t_1}}(L_{t_1}) \prec W^{L_{t_2}}(L_{t_2})$. In fact, we have $W^{L_{t_2}}(L_{t_2}) : W^{L_{t_1}}(L_{t_1}) = W^{L_{t_2}}(L_{t_2} : L_{t_1})$. Thus, if $\bigcap_{\varepsilon>0} \overline{L_t : L_{t-\varepsilon}} = \{\lambda_t\}$, then

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$$\bigcap_{\varepsilon>0} \overline{W^{L_t}(L_t) : W^{L_{t-\varepsilon}}(L_{t-\varepsilon})} = \{W^{L_t}(\lambda_t)\}.$$

So $(W^{L_t}(L_t))$ is a quotient $\mathbb{H}$-Loewner chain, and we define it to be the transformation of (L_t) under W. If (L_t) are backward chordal Loewner hulls driven by λ, then we may normalize $(W^{L_t}(L_t))$ to get a backward Loewner process using the function $u(t) := \mathrm{hcap}(W^{L_t}(L_t))/2$. Sometimes we refer the normalization of $(W^{L_t}(L_t))$ as the conformal transformation of (L_t) via W.

One nice property of the conformal transformation is that it preserves the welding map. If in a quotient $\mathbb{H}$-Loewner chain (L_t), every L_t is the image of an $\mathbb{H}$-simple curve (which is the case for backward SLE$_\kappa$ with $\kappa \in (0,4]$), then each f_{L_t} extends continuously to $\overline{\mathbb{H}}$, and maps S_{L_t} onto the two sides of L_t. Such f_{L_t} induces a welding map $\phi_t : S_{L_t} \to S_{L_t}$, which is an orientation-reversed map, such that $f_{L_t} \circ \phi_t = f_{L_t}$, i.e., x and $\phi_t(x)$ have the same f_{L_t}-image on L_t. Moreover, if $t_1 < t_2$, then $\phi_{t_1} = \phi_{t_2}|_{S_{L_{t_1}}}$. Thus, the quotient $\mathbb{H}$-Loewner chain (L_t) induces a welding map ϕ on $\bigcup S_{L_t}$ such that $\phi|_{S_{L_t}} = \phi_t$ for each t.

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Suppose $(W^{L_t}(L_t))$ is a conformal transformation of (L_t), which induces another welding map ϕ^W. Then we have $\phi^W = W \circ \phi \circ W^{-1}$. This holds because if $f_{L_t}(x) = f_{L_t}(y)$, then $f_{W^{L_t}(L_t)}(W(x)) = f_{W^{L_t}(L_t)}(W(y))$, which follows from $f_{W^{L_t}(L_t)} \circ W = W^{L_t} \circ f_{L_t}$.

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The backward SLE$(\kappa;\underline{\rho})$ processes can be defined similarly as forward SLE$(\kappa;\underline{\rho})$ processes. Following the argument in [SW05], we derived the coordinate change rule for backward SLE$(\kappa;\underline{\rho})$ process: if $\sum \rho_j = -\kappa - 6$, the conformal transformation of a backward SLE$(\kappa;\underline{\rho})$ process under a Möbius transformation is still a backward SLE$(\kappa;\underline{\rho})$ process. This suggests that a backward SLE$_\kappa$ may be viewed as SLE with a negative parameter: $-\kappa$.

Theorem 1 also makes it possible to define the commutation coupling of two backward SLEs. Let me first recall the commutation coupling between two forward SLE$(\kappa;\underline{\rho})$ processes. Roughly speaking, an SLE$(\kappa^1;\underline{\rho}^1)$ process (K_t^1) commutes with an SLE$(\kappa^2;\underline{\rho}^2)$ process (K_t^2) if the two processes are defined on the same probability space, and

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Here we only consider those K_t^1 before T_τ^1, because we want $\overline{K_t^1}$ to be contained in the domain of $g_{K_\tau^2}$, and so that $(g_{K_\tau^2}(K_t^1))_{0\le t<T_\tau^1}$ is an $\mathbb{H}$-Loewner chain.

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Here we only consider those L_t^1 before T_τ^1, because we want $S_{L_t^1}$ to be contained in the domain of $f_{L_\tau^2}$, which is $\mathbb{C}\setminus S_{L_\tau^2}$, and so that the conformal transformation of the quotient Loewner chain $(L_t^1)_{0\le t<T_\tau^1}$ via $f_{L_\tau^2}$ is well defined.

A stochastic coupling technique was developed earlier to construct commutation couplings between forward $\text{SLE}(\kappa;\underline{\rho})$ processes, which was then used to prove the reversibility of chordal SLE_κ for $\kappa \le 4$ and the duality of SLE for $\kappa > 4$.

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In the joint work, we used the stochastic coupling technique to construct commutation couplings between two backward SLE processes, and proved that, for $\kappa \le 4$, the random welding map ϕ induced by a backward chordal SLE$_\kappa$ processes satisfies the time-reversal symmetry: $h \circ \phi \circ h \sim \phi$, where $h(z) = 1/z$. Later, this symmetry result was combined with the conformal removability of SLE$_\kappa$ for $\kappa \in (0,4)$ ([JS00], [RS05]), to prove the reversibility of a whole-plane SLE($\kappa;\kappa+2$) curve stopped at a fixed capacity time.

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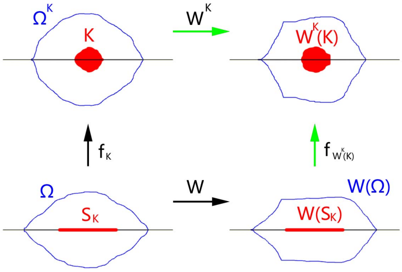
Sketch proof

Theorem 1.

Let K be an $\mathbb{H}$-hull. Let W be an $\mathbb{R}$-symmetric conformal map, whose domain Ω contains S_K. Then there is a unique conforml map W^K defined on $\Omega^K := f_K(\Omega \setminus S_K) \cup K^{\mathrm{doub}}$ such that $W = g_{W^K(K)} \circ W^K \circ f_K$.



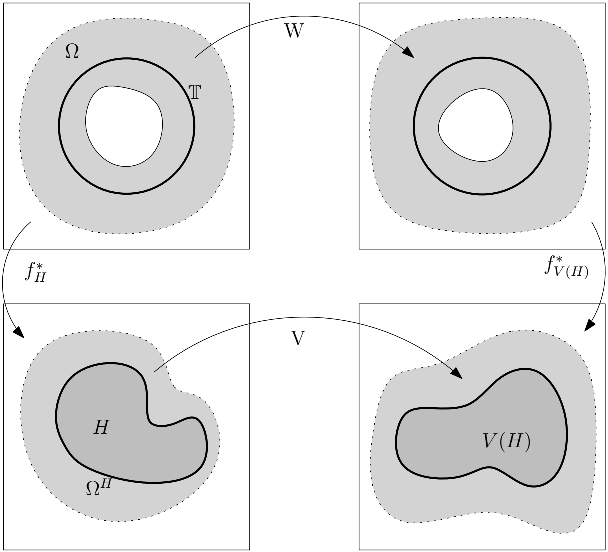
We transform the above theorem to a similar problem. We say that H is a $\mathbb{C}$-hull if H is a connected compact subset of $\mathbb{C}$ such that $\operatorname{diam}(H) > 0$ and $\mathbb{C} \setminus H$ is connected. For a $\mathbb{C}$-hull H, there is a unique $g_H^* : \mathbb{C} \setminus F \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{D}^* := \{z : |z| > 1\}$ such that $g_H^*(\infty) = \infty$ and $(g_H^*)'(\infty) > 0$. Let $f_H^* = (g_H^*)^{-1}$. These maps are closely related with the g_K and f_K for $\mathbb{H}$-hull K: if K is a nonempty $\mathbb{H}$-hull such that K^{doub} is connected, then K^{doub} and S_K are $\mathbb{C}$-hulls, and $g_K = g_{S_K}^* \circ f_{K^{\text{doub}}}^*$.

Let V be a conformal map, whose domain Ω contains a $\mathbb{C}$-hull H. Then $V(H)$ is also a $\mathbb{C}$-hull. The composition $W := g^*_{V(H)} \circ V \circ f_H$ is a conformal map defined on $\Omega^+_H := g^*_H(\Omega \setminus H)$, which is a subset of $\mathbb{D}^*$ and contains $\{1 < |z| < R\}$ for some $R > 1$. By Schwarz reflection principle, W extends conformally across $\mathbb{T} := \{|z| = 1\}$, maps $\mathbb{T}$ onto $\mathbb{T}$, and preserves the orientation of $\mathbb{T}$. Theorem 1 follows from Theorem 2 below, which tells us that we can recover V from W.

Let V be a conformal map, whose domain Ω contains a $\mathbb{C}$-hull H. Then $V(H)$ is also a $\mathbb{C}$-hull. The composition $W := g^*_{V(H)} \circ V \circ f_H$ is a conformal map defined on $\Omega^+_H := g^*_H(\Omega \setminus H)$, which is a subset of $\mathbb{D}^*$ and contains $\{1 < |z| < R\}$ for some $R > 1$. By Schwarz reflection principle, W extends conformally across $\mathbb{T} := \{|z| = 1\}$, maps $\mathbb{T}$ onto $\mathbb{T}$, and preserves the orientation of $\mathbb{T}$. Theorem 1 follows from Theorem 2 below, which tells us that we can recover V from W.

Theorem 2.

Let H be as above. Let W be a conformal map, whose domain Ω contains $\mathbb{T}$, such that W maps $\mathbb{T}$ onto $\mathbb{T}$, and preserves the orientation of $\mathbb{T}$. Then there is a conformal map V defined on $\Omega^H := \psi_H(\Omega \cap \mathbb{D}^*) \cup H$ such that $W = g^*_{V(H)} \circ V \circ f^*_H$.


W
Ω
$\mathbb{T}$
f_H^*
$f_{V(H)}^*$
V
H
Ω^H
$V(H)$

Sketch proof of Theorem 2.

By Carathéodory kernel theorem, we may assume that ∂H is an analytic Jordan curve β. Let $f_H^{\#}$ be a conformal map from $\mathbb{D} := \{|z| < 1\}$ onto the interior of β. Both f_H^* and $f_H^{\#}$ extend continuously to $\mathbb{T}$, and the welding $\phi := (f_H^*)^{-1} \circ f_H^{\#}$ is an analytic automorphism of $\mathbb{T}$, and so is $\phi^W := W \circ \phi$. From the quasiconformal theory of conformal welding, ϕ^W is the conformal welding associated with some analytic Jordan curve γ. This means that, there is a conformal map $f_L^{\#}$ from $\mathbb{D}$ onto the interior of γ such that $\phi^W = (f_L^*)^{-1} \circ f_L^{\#}$, where L is the $\mathbb{C}$-hull bounded by γ.

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Define $V = f_L^{\#} \circ (f_H^{\#})^{-1}$. Then V maps the interior of β conformally onto the interior of γ. Since β and γ are analytic curves, V extends analytically across β, and maps β onto γ. Since

$$(f_L^*)^{-1} \circ f_L^{\#} = W \circ \phi = W \circ (f_H^*)^{-1} \circ f_H^{\#} \quad \text{on } \mathbb{T},$$

we get $V = f_L^* \circ W \circ (f_H^*)^{-1}$ on β, which should also hold outside β. Thus, $W = g_{V(H)}^* \circ V \circ f_H$ outside $\mathbb{T}$, as desired. $\square$

As a byproduct, we obtain the following corollary with a simple proof.

Corollary

If ϕ is a conformal welding of $\mathbb{T}$, and W is an analytic orientation-preserving automorphism of $\mathbb{T}$, then $\phi \circ W$ and $W \circ \phi$ are conformal weldings of $\mathbb{T}$.

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Proof.

We may assume that $\phi = (f_\beta^*)^{-1} \circ f_\beta^\#$, where f_β^* and $f_\beta^\#$ map $\mathbb{D}^*$ and $\mathbb{D}$ conformally onto the exterior and the interior, respectively of a Jordan curve β. From Theorem 2, there is a conformal map V, whose domain contains β and its interior, such that $W = (f_\gamma^*)^{-1} \circ V \circ f_\beta^*$, where $\gamma = V(\beta)$ is a Jordan curve, and f_γ^* map $\mathbb{D}^*$ conformally onto the exterior of γ. Then $V \circ f_\beta^\#$ maps $\mathbb{D}$ conformally onto the interior of γ, and

$$(f_\gamma^*)^{-1} \circ (V \circ f_\beta^\#) = W \circ (f_\beta^*)^{-1} \circ f_\beta^\# = W \circ \phi.$$

Thus, $W \circ \phi$ is a conformal welding. Since $\phi \circ W = (W^{-1} \circ \phi^{-1})^{-1}$, $\phi \circ W$ is also a conformal welding. □

Thank you!

Happy Birthday, Nick!