2D Coulomb gas on Riemann surfaces

P. Wiegmann

March 7, 2016

2D Coulomb gas: Nick's Favorite Thing

$$\boxed{Z_\beta = \int_{\mathbf{C}^N} \prod_{i>j}^N |z_i - z_j|^{2\beta} \prod_{i=1}^N e^{-\frac{\beta k}{2}|z_i|^2} dz_i d\bar{z}_i}$$

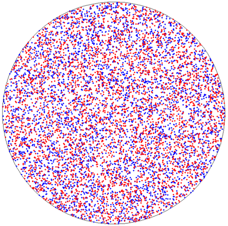


Figure: equilibrium measure $\beta = 1$ (after N.-G. Kang)

$$N \to \infty,, \quad k \to \infty, \quad N/k = \text{fixed}$$

Nick's Favorite Thing

Mathematical theory of Coulomb gas is due to

Nikolai Makarov, Yacin Ameur, Hakan Hedenmalm, Nam-Gyu Kang, Seung-Yeop Lee,...

Applications

- Statistical Mechanics,
- Quantum Hall Effect,
- Stochastic processes (Dyson diffusion),
- Hydrodynamics and Turbulence
- Conformal Field Theory and quantum gravity,
- Analytic (Quillen) anomaly of spectral determinants
- Selberg Integrals

Relation to Riemann Geometry

- The large N expansion generates objects of Riemann Geometry
- Provide a finite dimensional scheme to approximate geometry

Is Geometry - Nick's next favorite thing?

2D Coulomb gas on a Riemann surface

Joint works with T. Can, M. Laskin, S. Klevtsov, A. Zabrodin

Particles on a surface interacting through Coulomb force with a neutralizing background

$$P = \frac{1}{Z} e^{-\beta E}$$

$$E = -2\pi \sum_{i \neq j}^{N} G(\xi_i, \xi_j)$$

$$-\Delta G(\xi, \xi') = \delta^{(2)}(\xi - \xi') - \frac{1}{V}$$

$$N \to \infty, \quad V \to \infty, \quad N/V = \text{fixed}$$

Boltzmann weight in complex coordinates genus - 0

- Complex coordinates

$$ds^2 = g_{ij}\,dx^i dx^j = g_{z\bar{z}} dz d\bar{z}, \quad \sqrt{g} = g_{z\bar{z}}$$

- Volume form

$$dV = \sqrt{g}\,dz d\bar{z}$$

- Kähler potential

$$\partial_z \partial_{\bar{z}} K = \sqrt{g} - \sqrt{g}_0, \quad \sqrt{g}_0 = \frac{1}{\left(1 + \frac{|z_i|^2}{4r^2}\right)^2}$$

Round sphere

$$\boxed{Z_\beta = \int_{\mathbf{C}^N} \prod_{i>j} |z_i - z_j|^{2\beta} \prod_i \frac{e^{-\frac{k\beta}{2} K(z,\bar{z})}}{\left(1+\frac{|z_i|^2}{4r^2}\right)^k} \sqrt{g} dz_i d\bar{z}_i}$$

Geometric functional

The integral

$$Z_N[g] = \int_{\mathbf{C}^N} \prod_{i>j} |z_i - z_j|^{2\beta} \prod_i e^{-\frac{\beta k}{4} K(z_i, \bar{z}_i)} \sqrt{g} dz_i d\bar{z}_i$$

converges if

$$\boxed{N \leq \beta^{-1} k + \frac{\chi}{2}}$$

- If $N < \frac{k}{\beta} + \frac{\chi}{2}$, the support of the equilibrium distribution has a boundary,
- If $N = \frac{k}{\beta} + \frac{\chi}{2}$, the support of the equilibrium distribution is the entire surface,

$$\boxed{\text{The functional } Z_N[g] \text{ depends only on geometry: curvature } R \text{ and moduli}}$$

Analog of Riemann-Roch theorem

$$e^{-\beta E} = |\Psi_\beta|^2 e^{-\frac{\beta k}{2}\sum_i^N K(z_i,\bar{z}_i)},$$
$$\Psi_\beta(z_1,...,z_N) \propto \prod_{i>j}(z_i - z_j)^\beta \,,$$

The "wave-function" is the degree of the determinant

$$\Psi_\beta(z_1,\ldots,z_N) = \left(\det[s_n(z_i)]\right)^\beta$$

of holomorphic sections of line bundle L^k

$$\partial_{\bar{z}} s_i = 0, \quad \langle s_i, s_j\rangle = \int s_i(z)\overline{s_j(z)} e^{-\frac{k}{2}K(z,\bar{z})}\sqrt{g}dzd\bar{z}.$$

The number of Holomorphic sections is

$$\boxed{N = k + \frac{\chi}{2}}$$

Geometric functionals and variations

- Geometric functionals

$$Z_N[g] = \int_M |\Psi|^2 dV$$

- Consecutive variation over metric

$$T_{ij}\delta g^{ij} = \delta \log Z_N[g]$$

produces all interesting correlation functions

Equilibrium measure: Density as a function of curvature

Can, Laskin, PW, 2014

$$\rho(z) = \int |\Psi(z, z_2, \dots, z_N)|^2 dV_2 \dots dV_N$$

$$\boxed{\rho = \frac{k}{\beta} + \frac{1}{8\pi} R + \frac{1}{k} \frac{c}{48} \Delta R + \dots}$$

$$\boxed{c = 1 - 3\beta}$$

$$S(q) = \int \langle \rho(r) \rho(0) \rangle e^{-iqr} d^2 r = \frac{q^2}{2} - \left(1 - \frac{\beta}{2}\right) \left(\frac{q^2}{2}\right)^2 + \left[\left(1 - \frac{\beta}{2}\right)^2 + \frac{\beta}{12}\right] \left(\frac{q^2}{2}\right)^3 + \dots$$

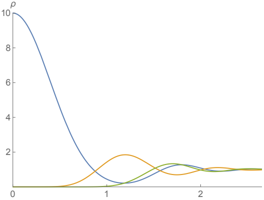
Expansion of the Bergmann kernel $\beta = 1$

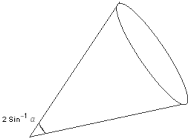
$$\beta = 1 : \ \partial_{\bar z} s(z) = 0, \quad \rho = e^{-\frac{k}{2}K(z,\bar z)} \sum_{n=0}^{N-1} |s_n(z)|^2$$

Ferrari, Klevtsov, Zelditch (2012);

$$\begin{aligned}\rho =& k + \frac{1}{8\pi} R + \frac{k^{-1}}{24\pi}\Delta R + \frac{k^{-2}}{16\pi}\left(\frac{1}{8}\Delta^2 R - \frac{5}{48}\Delta(R^2)\right) + \\ & \frac{k^{-3}}{32\pi}\left(\frac{29}{720}\Delta(R^3) - \frac{7}{160}\Delta^2(R^2) + \frac{1}{30}\Delta^3 R - \frac{11}{120}\Delta(R\Delta R)\right) + \ldots\end{aligned}$$

Emergent conformal symmetry on singular surfaces





conical metric $g_{z\bar{z}} = z^{-\alpha}$

Density is singular: $k \to \infty \quad m_{2n} = k^{2n} \int r^{2n} \left(\rho(r) - \rho_\infty\right) dV$

$$\begin{cases} m_0 = \frac{\alpha}{2} \\ m_2 - m_0 = -\Delta_\alpha = \frac{c}{24}(\gamma^{-1} - \gamma), \\ \gamma = 1 - \alpha, \quad c = 1 - 3\beta \end{cases}$$

Δ_α is conformal dimension of the "twist" operator in CFT

GEOMETRIC FUNCTIONALS: GRAVITATIONAL "WESS-ZUMINO"

$$Z[g] = \int_\Sigma |\Psi|^2 dV$$

Large N expansion generates geometric invariants

$$\log Z[g] = p_2\, k^2 A^{(2)}[g] + p_1\, k A^{(1)}[g] + p_0\, A^{(0)}[g] + p_{-1} \overbrace{\int R^2 dV}^{\text{local in } R}$$

Geometric functional with non-trivial co-cycle property

$$A^{(2)} = \frac{\pi}{2V^2} \int K\, dV - \text{Aubin-Yau}, \qquad p_2 = -\frac{1}{\beta}$$

$$A^{(1)} = \frac{2}{V} \int (-\Delta)^{-1} R\, dV, -\text{Mabuchi}, \qquad p_1 = -\frac{1}{4}$$

$$A^{(0)} = \int R(-\Delta)^{-1} R\, dV - \text{Polyakov}, \qquad p_0 = \frac{c}{48\pi}$$

Polyakov's Liouville action of quantum gravity

$$Z[g] = e^{\frac{2\pi}{\beta}k^2 A^{(2)}[g] - \frac{k}{4}A^{(1)}[g]} \left[\det\left(-\Delta_g\right)\right]^{\frac{c}{2}}$$

$$c = 1 - 3\beta$$

$$\log\det\left(-\Delta_g\right) = -\frac{1}{48\pi}\int R(-\Delta)^{-1}R\,dV$$

Scaling on a cone $g \to \lambda g$

$$\det\left(-\Delta_{\lambda g}\right) = \lambda^{-\Delta_\alpha}\det\left(-\Delta_g\right)$$

$$\boxed{\Delta_\alpha = -\tfrac{c}{24}(\gamma^{-1} - \gamma)}$$

Moduli space $\mathscr{M}_g$, or Teichmülcer space: $3g-3$ complex parameters

$$\tau_1, \ldots, \tau_{3g-3}, \quad g > 1.$$

The tangent space of $\mathscr{M}_g$ is represented by the Beltrami holomorphic differentials μ: that is area preserving transformation of the metric

$$|dz|^2 \to |dz + \delta\mu d\bar{z}|^2, \quad \nabla_{\bar{z}}\delta\mu = 0.$$

Weil-Petersson form on the moduli space: invariant 2-form with respect to a coordinate choice of the moduli space

$$\Omega_{WP} = \mathrm{i}\int (\delta\mu \wedge \delta\bar{\mu}) - \text{Weil-Petersson form}$$

The wave-function on $\mathscr{M}_g$

$$\prod_{i>j}(z_i - z_j)^\beta \longrightarrow \Psi(z_1, \dots, z_N | \tau_1, \dots \tau_{3g-3})$$

yields the 2-closed (curvature) form on L^k

$$\boxed{\Omega = \mathrm{i}\langle \delta_{\bar\mu} | \delta_\mu \rangle = \mathrm{i} \int_\Sigma \delta_\mu \Psi^*(z_1, \dots, z_N | \tau) \wedge \delta_{\bar\mu} \Psi(z_1, \dots, z_N | \tau)}$$

$\log Z$ is Kähler potential on L^k $\boxed{\Omega = \int_\Sigma \delta_{\bar\mu} \delta_\mu \log Z}$

$\boxed{\Omega = \left(\frac{k}{4} + \frac{1-3\beta}{12}\right) \Omega_{WP} - \text{Weil-Petersson form}}$ S. Klevtsov & P. W.

$$\text{Chern numbers} = \oint_{2-cycle} \Omega = \left(\frac{k}{4} + \frac{1-3\beta}{12}\right) \oint_{2-cycle} \Omega_{WP}$$