Rectifiability of harmonic measure

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1. Main theorem

Theorem

Let $n \geq 1$ and $\Omega \subsetneq \mathbb{R}^{n+1}$ be an open connected set and let $\omega := \omega^p$ be the harmonic measure in Ω where p is a fixed point in Ω. Suppose that there exists $E \subset \partial\Omega$ with Hausdorff measure $0 < \mathcal{H}^n(E) < \infty$ and that the harmonic measure $\omega|_E$ is absolutely continuous with respect to $\mathcal{H}^n|_E$. Then $\omega|_E$ is n-rectifiable, in the sense that ω-almost all of E can be covered by a countable union of n-dimensional (possibly rotated) Lipschitz graphs.

2. A brief history

The metric properties of harmonic measure attracted attention of many mathematicians. Fundamental results of Makarov establish that if $n+1=2$ then the Hausdorff dimension $\dim_{\mathcal{H}} \omega = 1$ if the set $\partial\Omega$ is connected (and $\partial\Omega$ is not a point of course). The topology is somehow felt by harmonic measure, and for a general domain Ω on the Riemann sphere whose complement has positive logarithmic capacity there exists a subset of $E \subset \partial\Omega$ which supports harmonic measure in Ω and has Hausdorff dimension at most 1, by a very subtle result of Jones and Wolff. In particular, the supercritical regime becomes clear on the plane: if $s \in (1,2)$, $0 < \mathcal{H}^s(E) < \infty$, then ω is always singular with respect to $\mathcal{H}^s|_E$). However, in the space $(n+1>2)$ the picture is murkier. Bourgain proved that the dimension of harmonic measure always drops: $\dim_{\mathcal{H}} \omega < n+1$. But even for connected $E = \partial\Omega$ it can be strictly bigger than n by the result of Wolff.

3. A brief history

In 1916 F. and M. Riesz proved that for a simply connected domain in the complex plane, with a rectifiable boundary, harmonic measure is absolutely continuous with respect to arclength measure on the boundary. More generally, if only a portion of the boundary is rectifiable, Bishop and Jones have shown that harmonic measure is absolutely continuous with respect to arclength on that portion. They have also proved that the result of may fail in the absence of some topological hypothesis (e.g., simple connectedness). The higher dimensional analogues of BJ include absolute continuity of harmonic measure with respect to the Hausdorff measure for Lipschitz graph, and more generally non-tangentially accessible (NTA) domains: Dahlberg, David–Jerison, Semmes. A_∞ property: Lavrent'ev. Also Badger, Lewis, Hofmann–Martell, Azzam–Mourgoglou–Tolsa, Toro. On the other hand, some counterexamples show that some topological restrictions, even stronger than in the planar case, are needed for the absolute continuity of ω with respect to $\mathcal{H}^n$, Wu, Ziemer.

4. Converse to BJ: necessity of rectifiability of ω

In the present paper we attack the converse direction. We establish that rectifiability is *necessary* for absolute continuity of the harmonic measure. This is a free boundary problem. However, the departing assumption, absolute continuity of the harmonic measure with respect to the Hausdorff measure of the set, is essentially the weakest meaningfully possible from a PDE point of view, putting it completely out of the realm of more traditional work, e.g., that related to minimization of functionals. At the same time, absence of any a priori topological restrictions on the domain (porosity, flatness, suitable forms of connectivity) notoriously prevents from using the conventional PDE toolbox.

The fact is that this necessity is true *always*: any dimension, no topological restrictions.

Given a signed Radon measure ν in $\mathbb{R}^{n+1}$ we consider the n-dimensional Riesz transform

$$\mathcal{R}\nu(x) = \int \frac{x-y}{|x-y|^{n+1}}\, d\nu(y),$$

whenever the integral makes sense. For $\varepsilon > 0$, its ε-truncated version is given by

$$\mathcal{R}_\varepsilon\nu(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}}\, d\nu(y).$$

For $\delta \geq 0$ we set $\mathcal{R}_{*,\delta}\nu(x) = \sup_{\varepsilon>\delta} |\mathcal{R}_\varepsilon\nu(x)|$. We also consider the maximal operator

$$\mathcal{M}^n_\delta\nu(x) = \sup_{r>\delta} \frac{|\nu|(B(x,r))}{r^n},$$

In the case $\delta = 0$ we write $\mathcal{R}_*\nu(x) := \mathcal{R}_{*,0}\nu(x)$ and $\mathcal{M}^n\nu(x) := \mathcal{M}^n_0\nu(x)$.

6. Notations, first main lemma

For a bounded open set, we may write the Green function exactly: for $x, y \in \Omega$, $x \neq y$, define

$$G(x,y) = \mathcal{E}(x-y) - \int_{\partial\Omega} \mathcal{E}(x-z)\, d\omega^y(z). \qquad (1)$$

Here $\mathcal{E}$ denotes the fundamental solution for the Laplace equation in $\mathbb{R}^{n+1}$, so that $\mathcal{E}(x) = c_n\,|x|^{1-n}$ for $n \geq 2$, and $\mathcal{E}(x) = -c_1 \log|x|$ for $n = 1$, $c_1, c_n > 0$.

Lemma

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^{n+1}$ be a bounded open connected set. Let $B = B(x_0, r)$ be a closed ball with $x_0 \in \partial\Omega$ and $0 < r < \mathrm{diam}(\partial\Omega)$. Then, for all $a \geq 4$,

$$\omega^x(aB) \gtrsim \inf_{z\in 2B\cap\Omega} \omega^z(aB)\, r^{n-1}\, G(y,x) \quad \forall x \in \Omega\setminus 2B, y \in B\cap\Omega, \qquad (2)$$

with the implicit constant independent of a.

Lemma

There is $a_0 > 1$ *depending only on* $n \geq 1$ *so that the following holds for* $a \geq a_0$*. Let* $\Omega \subsetneq \mathbb{R}^{n+1}$ *be a bounded domain,* $n-1 < s \leq n+1$, $\xi \in \partial\Omega$, $r > 0$, *and* $B = B(\xi, r)$*. Then*

$$\omega^z(aB) \gtrsim_{n,s} \frac{\mathcal{H}^s_\infty(\partial\Omega \cap B)}{r^s} \quad \text{for all } z \in B \cap \Omega.$$

We fix a point $p \in \Omega$ far from the boundary. To prove that $\omega^p|_E$ is rectifiable we will show that any subset of positive harmonic measure of E contains another subset G of positive harmonic measure such that $\mathcal{R}_*\omega^p(x) < \infty$ in G. Applying a theorem due to Nazarov, Treil and Volberg, one deduces that G contains yet another subset G_0 of positive harmonic measure such that $\mathcal{R}_{\omega^p|_{G_0}}$ is bounded in $L^2(\omega^p|_{G_0})$. Then from the results of Nazarov, Tolsa and Volberg, it follows that $\omega^p|_{G_0}$ is n-rectifiable. This suffices to prove the full n-rectifiability of $\omega^p|_E$.
One of the difficulties of Theorem 1 is due to the fact that the non-Ahlfors regularity of $\partial\Omega$ makes it difficult to apply some usual tools from potential of theory. In our proof we solve this issue by applying some stopping time arguments involving the harmonic measure and a suitable Frostman measure.

fix a point $p \in \Omega$, and consider the harmonic measure ω^p of Ω with pole at p. The reader may think that p is point deep inside Ω. Let $g \in L^1(\omega^p)$ be such that

$$\omega^p|_E = g\,\mathcal{H}^n|_{\partial\Omega}.$$

Given $M > 0$, let $E_M = \{x \in \partial\Omega : M^{-1} \leq g(x) \leq M\}$. Take M big enough so that $\omega^p(E_M) \geq \omega^p(E)/2$, say. Consider an arbitrary compact set $F_M \subset E_M$ with $\omega^p(F_M) > 0$. We will show that there exists $G_0 \subset F_M$ with $\omega^p(G_0) > 0$ which is n-rectifiable. Clearly, this suffices to prove that $\omega^p|_{E_M}$ is n-rectifiable, and letting $M \to \infty$ we get the full n-rectifiability of $\omega^p|_E$.
Let μ be an n-dimensional Frostman measure for F_M. That is, μ is a non-zero Radon measure supported on F_M such that

$$\mu(B(x,r)) \leq C\,r^n \qquad \text{for all } x \in \mathbb{R}^{n+1}.$$

Further, by renormalizing μ, we can assume that $\|\mu\| = 1$. Of course the constant C above will depend on $\mathcal{H}^n_\infty(F_M)$. Notice that $\mu \ll \mathcal{H}^n|_{F_M} \ll \omega^p$.

Lemma

Let $\mu(O) \le \tau = \tau\mu(F_M)$ with sufficiently small positive τ. Then $\omega^p(F_M \setminus O) \ge \frac{1}{2CM}\omega^p(F_M)$.

Proof.

Just put $\tau = \frac{1}{2}$. Then $\frac{1}{2}\mu(F_M) \le \mu(F_M \setminus O)$. Then

$$\frac{1}{2}\omega^p(F_M) \le \frac{1}{2} = \frac{1}{2}\mu(F_M) \le \mu(F_M \setminus O)$$
$$\le C\mathcal{H}^n_\infty(F_M \setminus O) \le C\mathcal{H}^n(F_M \setminus O) \le CM\omega^p(F_M \setminus O).$$

What is O? To build this exceptional set we need David–Mattila cells and a special *stopping time*.

□

11. David–Mattila cells

Now we will consider the dyadic lattice of "cubes" with small boundaries of David-Mattila associated with ω^p. This lattice has been constructed by David-Mattila (with ω^p replaced by a general Radon measure). Its properties are summarized in the next lemma.

Lemma of David–Mattila. Consider two constants $C_0 > 1$ and $A_0 > 5000\, C_0$ and denote $W = \mathrm{supp}\omega^p$. Then there exists a sequence of partitions of W into Borel subsets Q, $Q \in \mathcal{D}_k$, with the following properties:

- For each integer $k \geq 0$, W is the disjoint union of the "cubes" Q, $Q \in \mathcal{D}_k$, and if $k < l$, $Q \in \mathcal{D}_l$, and $R \in \mathcal{D}_k$, then either $Q \cap R = \varnothing$ or else $Q \subset R$.
- The general position of the cubes Q can be described as follows. For each $k \geq 0$ and each cube $Q \in \mathcal{D}_k$, there is a ball $B(Q) = B(z_Q, r(Q))$ such that

$$z_Q \in W, \qquad A_0^{-k} \leq r(Q) \leq C_0\, A_0^{-k},$$

$$W \cap B(Q) \subset Q \subset W \cap 28\, B(Q) = W \cap B(z_Q, 28 r(Q)),$$

- the balls $5B(Q)$, $Q \in \mathcal{D}_k$, are disjoint.
- The cubes $Q \in \mathcal{D}_k$ have small boundaries. That is, for each $Q \in \mathcal{D}_k$ and each integer $l \geq 0$, set

$$N_l^{ext}(Q) = \{x \in W \setminus Q : \operatorname{dist}(x, Q) < A_0^{-k-l}\},$$

$$N_l^{int}(Q) = \{x \in Q : \operatorname{dist}(x, W \setminus Q) < A_0^{-k-l}\},$$

and

$$N_l(Q) = N_l^{ext}(Q) \cup N_l^{int}(Q).$$

Then

$$\omega^p(N_l(Q)) \leq (C^{-1} C_0^{-3d-1} A_0)^{-l}\, \omega^p(90B(Q)). \qquad (3)$$

- Denote by $\mathcal{D}_k^{db}$ the family of cells $Q \in \mathcal{D}_k$ for which

$$\omega^p(100B(Q)) \leq C_0\, \omega^p(B(Q)). \tag{4}$$

We have that $r(Q) = A_0^{-k}$ when $Q \in \mathcal{D}_k \setminus \mathcal{D}_k^{db}$ and

$$\omega^p(100B(Q)) \leq C_0^{-l}\, \omega^p(100^{l+1}B(Q)) \tag{5}$$

for all $l \geq 1$ such that $100^l \leq C_0$ and $Q \in \mathcal{D}_k \setminus \mathcal{D}_k^{db}$.

We use the notation $\mathcal{D} = \bigcup_{k\geq 0} \mathcal{D}_k$. Given $Q \in \mathcal{D}_k$, we denote $J(Q) = k$. We denote $\mathcal{D}^{db} = \bigcup_{k\geq 0} \mathcal{D}_k^{db}$. Note that, in particular, it follows that

$$\omega^p(3B_Q) \leq \omega^p(100B(Q)) \leq C_0\, \omega^p(Q) \qquad \text{if } Q \in \mathcal{D}^{db}. \tag{6}$$

Lemma

If C_0, A_0 are large, then for any given cell $R \in \mathcal{D}$ there exists a family of doubling cells $\{Q_i\}_{i\in I} \subset \mathcal{D}^{db}$, with $Q_i \subset R$ for all i, such that their union covers ω^p-almost all R.

Lemma

Let $R \in \mathcal{D}$ and let $Q \subset R$ be a cell such that all the intermediate cells S, $Q \subsetneq S \subsetneq R$ are non-doubling (i.e. belong to $\mathcal{D} \setminus \mathcal{D}^{db}$). Then

$$\omega^p(100B(Q)) \leq A_0^{-10n(J(Q)-J(R)-1)} \omega^p(100B(R)). \tag{7}$$

15. Density lemma for "non-doubling wells"

Given a ball $B \subset \mathbb{R}^{n+1}$, we consider its n-dimensional density:

$$\Theta_\omega(B) = \frac{\omega^p(B)}{r(B)^n}.$$

From the preceding lemma we deduce:

Lemma

Let $Q, R \in \mathcal{D}$ be as in Lemma 6. Then

$$\Theta_\omega(100B(Q)) \leq C_0\, A_0^{-9n(J(Q)-J(R)-1)}\, \Theta_\omega(100B(R))$$

and

$$\sum_{S \in \mathcal{D}: Q \subset S \subset R} \Theta_\omega(100B(S)) \lesssim_{A_0, C_0} \Theta_\omega(100B(R)).$$

Now we need to define a family of bad cells. We say that $Q \in \mathcal{D}$ is bad and we write $Q \in \text{Bad}$, if $Q \in \mathcal{D}$ is a maximal cell satisfying one of the conditions below:

(a) $\mu(Q) \leq \tau\, \omega^p(Q)$, where $\tau > 0$ is a small parameter to be fixed below, or

(b) $\omega^p(3B_Q) \geq A\, r(B_Q)^n$, where A is some big constant to be fixed below.

The existence maximal cubes is guaranteed by the fact that all the cubes from $\mathcal{D}$ have side length uniformly bounded from above (since $\mathcal{D}_k$ is defined only for $k \geq 0$). If the condition (a) holds, we write $Q \in \text{LM}$ (little measure μ) and in the case (b), $Q \in \text{HD}$ (high density). On the other hand, if a cube $Q \in \mathcal{D}$ is not contained in any cube from Bad, we say that Q is good and we write $Q \in \text{Good}$.

17. Choice of exceptional set O from slide 10

$$O := (\cup_{Q \text{is Bad}} Q) \cup \left(\cup_{Q \in \mathcal{D} \setminus \mathcal{D}_0^{db}} Q\right).$$

Notice that

$$\sum_{Q \in \mathrm{LM} \cap \mathrm{Bad}} \mu(Q) \le \tau \sum_{Q \in \mathrm{LM} \cap \mathrm{Bad}} \omega^p(Q) \le \tau \, \|\omega\| = \tau = \tau \, \mu(F_M).$$

As $\mathcal{H}^n(E) < \infty$ the same can be said about HD cells if A is large. If the constant of doubling is sufficiently large then the same can be said about

$$\sum_{Q \text{ not inside some } R \in \mathcal{D}_0^{db}} \omega^p(Q) \le small \|\omega^p\| = small \, .$$

By Lemma on slide 10

$$\omega^p(F_M \setminus O) \ge \kappa \omega^p(F_M), \; \kappa > 0.$$

18. Outside exceptional set O

Notice that for the points $x \in F_M \setminus \bigcup_{Q \in \text{Bad}} Q$, from the condition (b) in the definition of bad cubes, it follows that

$$\omega^p(B(x,r)) \lesssim A\, r^n \qquad \text{for all } 0 < r \leq 1.$$

Trivially, the same estimate holds for $r \geq 1$, since $\|\omega^p\| = 1$. So we have

$$\mathcal{M}^n \omega^p(x) \lesssim A \quad \text{for } \omega^p\text{-a.e. } x \in F_M \setminus \bigcup_{Q \in \text{Bad}} Q. \tag{8}$$

Lemma (Key lemma)

Let $Q \in$ Good be contained in some cube from the family $\widetilde{\mathcal{D}}_0^{db}$, and $x \in Q$. Then we have

$$\left|\mathcal{R}_{r(B_Q)}\omega^p(x)\right| \leq C(A, M, T_{db}, \tau, \mathrm{dist}(p, \partial\Omega)). \tag{9}$$

Let $\varphi : \mathbb{R}^d \to [0,1]$ be a radial $\mathcal{C}^\infty$ function which vanishes on $B(0,1)$ and equals 1 on $\mathbb{R}^d \setminus B(0,2)$, and for $\varepsilon > 0$ and $z \in \mathbb{R}^{n+1}$ denote $\varphi_\varepsilon(z) = \varphi\left(\frac{z}{\varepsilon}\right)$ and $\psi_\varepsilon = 1 - \varphi_\varepsilon$. We set

$$\widetilde{\mathcal{R}}_\varepsilon \omega^p(z) = \int K(z-y)\,\varphi_\varepsilon(z-y)\,d\omega^p(y),$$

where $K(\cdot)$ is the kernel of the n-dimensional Riesz transform. Consider a ball $\widetilde{B}_Q$ centered at some point from $Q \cap \partial\Omega$ with $r(\widetilde{B}_Q) = \frac{1}{a_0}\, r(B_Q)$ and so that $\mu(\widetilde{B}_Q) \gtrsim \mu(B_Q)$, with the implicit constant depending on a_0, which is from Hall's lemma on slide 7. Note that, for every $x, z \in B_Q$, by standard Calderón-Zygmund estimates

$$\left|\widetilde{\mathcal{R}}_{r(\widetilde{B}_Q)}\omega^p(x) - \mathcal{R}_{r(B_Q)}\omega^p(z)\right| \leq C(\delta)\,\mathcal{M}^n_{r(\widetilde{B}_Q)}\omega^p(z),$$

and $\mathcal{M}^n_{r(\widetilde{B}_Q)}\omega^p(z) \leq C(\delta, A)$ for all $z \in B_Q$, since Q being good implies that Q and all its ancestors are not from HD.

Thus, to prove the Key lemma it suffices to show that

$$\left|\widetilde{\mathcal{R}}_{r(\widetilde{B}_Q)}\omega^p(x)\right| \le C(\delta, A, M, T, \tau, d(p)) \qquad \text{for the center } x \text{ of } \widetilde{B}_Q. \tag{10}$$

To shorten notation, in the rest of the proof we will write $r = r(\widetilde{B}_Q)$, so that $\widetilde{B}_Q = B(x, r)$. For a fixed $x \in Q \subset \partial\Omega$ and $z \in \mathbb{R}^{n+1} \setminus \left[\mathrm{supp}(\varphi_r(x-\cdot)\,\omega^p) \cup \{p\}\right]$, consider the function

$$u_r(z) = \mathcal{E}(z-p) - \int \mathcal{E}(z-y)\,\varphi_r(x-y)\,d\omega^p(y),$$

so that,

$$G(z,p) = u_r(z) - \int \mathcal{E}(z-y)\,\psi_r(x-y)\,d\omega^p(y) \qquad \text{for } m\text{-a.e. } z \in \mathbb{R}^{n+1}. \tag{11}$$

$$\nabla u_r(z) = c_n\, K(z-p) - c_n\, \mathcal{R}(\varphi_r(\cdot - x)\, \omega^p)(z).$$

In the particular case $z = x$ we get

$$\nabla u_r(x) = c_n\, K(x-p) - c_n\, \widetilde{\mathcal{R}}_r \omega^p(x),$$

and thus

$$|\widetilde{\mathcal{R}}_r \omega^p(x)| \lesssim \frac{1}{d(p)^n} + |\nabla u_r(x)|. \tag{12}$$

Left to estimate $\nabla u_r(x) \lesssim \frac{1}{r} \fint_{B(x,r)} |u_r(y)|\, dm(y)$, as u_r is harmonic in $B(x,r)$. Now see slide 20, (11): Only the estimate $\frac{1}{r}|G(y,p)| \lesssim ?$, $y \in B(x,r)$ is left to get.
By Lemmas on slides 6 and 7

$$\omega^p(a_0 B) \gtrsim \inf_{z \in 2B\cap\Omega} \omega^z(a_0 B)\, r^{n-1}\, G(y,p) \geq \frac{\mu(Q)}{r^n} r^{n-1}\, G(y,p).$$

22. Proof of Key lemma

Thus,

$$y \in B(x,r) \Rightarrow \frac{1}{r}|G(y,p)| \le \frac{\omega^p(B_Q)}{\mu(Q)} \le T_{db}\frac{\omega^p(Q)}{\mu(Q)} \le T_{db}\tau^{-1},$$

if $Q \in \mathcal{D}^{db}$. And we are done if $Q \in \mathcal{D}^{db}$.
If $Q \notin \mathcal{D}^{db}$ but lies inside $Q' \in \mathcal{D}_0^{db}$, let R be the first doubling ancestor. Then we have the well of non-doubling cells between Q and R. By Lemma on slide 15 we have

$$|\widetilde{\mathcal{R}}_{r(B_Q)}\omega^p(x) - \widetilde{\mathcal{R}}_{r(B_R)}\omega^p(x)| \lesssim \frac{\omega^p(B_R)}{r(B_R)^n} \lesssim \mathcal{M}\omega^P(x) \lesssim A\,.$$

But R is doubling, so the display formula at the top of this slide

$$|\widetilde{\mathcal{R}}_{r(B_R)}\omega^p(x)| \le T_{db}\tau^{-1}.$$

We are done for $n \ge 2$.

23. NTV theorems

Let $G := F_m \setminus O$.

Theorem (Nazarov–Treil–Volberg)

Let σ be a Radon measure with compact support on $\mathbb{R}^{n+1}$ and consider a σ-measurable set G with $\sigma(G) > 0$ such that

$$G \subset \{x \in \mathbb{R}^d : \mathcal{M}^n\sigma(x) < \infty \text{ and } \mathcal{R}_*\sigma(x) < \infty\}.$$

Then there exists a Borel subset $G_0 \subset G$ with $\sigma(G_0) > 0$ such that $\sup_{x\in G_0} \mathcal{M}^n\sigma|_{G_0}(x) < \infty$ and $\mathcal{R}_{\sigma|_{G_0}}$ is bounded in $L^2(\sigma|_{G_0})$.

Theorem (Nazarov–Tolsa–Volberg; this is a solution of David–Semmes problem.)

Let μ be n-dimensional measure in $\mathbb{R}^{n+1}$. And let $\mathcal{R}_\mu$ is bounded in $L^2(\mu)$. Then μ is n-rectifiable.

Corollary

Suppose that $\Omega \subset \mathbb{R}^{n+1}$ *is a connected domain,* $p \in \Omega$*, and* $E \subset \partial\Omega$ *is a set such that* $0 < \mathcal{H}^n(E) < \infty$ *and* $\mathcal{H}^n \ll \omega$ *on* E*. Then* E *is n-rectifiable in the sense that it may be covered up to a set of* $\mathcal{H}^n$*-measure zero by Lipschitz graphs.*