Rank one perturbations of unitary operators and Clark's model in general situation

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Rank one perturbations

- For a unitary $U = U_1$ let

$$U_\gamma := U + (\gamma - 1) b b_1^*, \qquad \|b\| = 1, \quad b_1 := U^* b, \quad \gamma \in \mathbb{C}.$$

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- WLOG: b is cyclic, so $U = M_\xi$ in $L^2(\mu)$, $\mu(\mathbb{T}) = 1$; $b \equiv \mathbf{1}$, therefore $b_1(\xi) = \overline{\xi}$.
- If $|\gamma| < 1$, U_γ is a *completely non-unitary* (c.n.u.) contraction with defect indices 1-1,

$$\operatorname{rank}(I - U_\gamma^* U_\gamma) = \operatorname{rank}(I - U_\gamma U_\gamma^*) = 1.$$

Models for U_γ

- If $|\gamma| = 1$ then U_γ is unitary, so $U_\gamma \cong M_z$,

$$M_z : L^2(\mu_\gamma) \to L^2(\mu_\gamma), \qquad M_z f(z) = z f(z).$$

- If $|\gamma| < 1$ then U_γ is a c.n.u. contraction and admits the functional model, $U_\gamma \cong \mathcal{M}_\theta$,

$$\mathcal{M}_\theta : \mathcal{K}_\theta \to \mathcal{K}_\theta, \qquad \mathcal{M}_\theta = P_{\mathcal{K}_\theta} M_z \bigm| \mathcal{K}_\theta;$$

here $\theta \in H^\infty$, $\|\theta\|_\infty \le 1$ is the *characteristic function* of U_γ, and $\mathcal{K}_\theta$ is the *model space*

Goal: Want to describe unitary operators intertwining U_γ and its model.

Characteristic function

For detail see Sz.-Nagy–Foiaș [9].

- For T, $\|T\| \le 1$ let

$$D_T = (I - T^*T)^{1/2}, \qquad D_{T^*} := (I - TT^*)^{1/2}$$
$$\mathfrak{D} = \mathfrak{D}_T := \operatorname{clos}\operatorname{Ran} D_T, \qquad \mathfrak{D}_* = \mathfrak{D}_{T^*} := \operatorname{clos}\operatorname{Ran} D_{T^*}.$$

- Characteristic function $\theta \in H^\infty(\mathbb{D}; B(\mathfrak{D}; \mathfrak{D}_*))$ is defined as

$$\theta_T(z) = \left(-T + zD_{T^*}(I - zT^*)^{-1}D_T\right)\Big|_{\mathfrak{D}}, \qquad z \in \mathbb{D}.$$

  Note that $\|\theta\|_\infty \le 1$.
- Usually θ is defined up to constant unitary factors (choice of bases in $\mathfrak{D}$ and $\mathfrak{D}_*$); spaces $E \cong \mathfrak{D}$ and $E_* \cong \mathfrak{D}_*$ are used.

Functional model(s)

Following Nikolskii–Vasyunin [4] the functional model is constructed as follows:

1. For a contraction $T : \mathcal{K} \to \mathcal{K}$ consider its *minimal* unitary dilations $\mathcal{U} : \mathcal{H} \to \mathcal{H}$, $\mathcal{K} \subset \mathcal{H}$,

$$T^n = P_{\mathcal{K}} \mathcal{U}^n \Big| \mathcal{K}, \qquad n \geq 0.$$

2. Pick a spectral representation of $\mathcal{U}$
3. Work out formulas in this spectral representation
4. Model subspace $\mathcal{K} = \mathcal{K}_\theta$ is usually a subspace of a weighted space $L^2(E \oplus E_*, W)$, $E \cong \mathfrak{D}$, $E_* \cong \mathfrak{D}_*$ with some operator-valued weight.

Specific representations give us a *transcription* of the model.

Among common transcriptions are: the Sz.-Nagy–Foiaş transcription, the de Branges–Rovnyak transcription, Pavlov transcription.

Sz.-Nagy–Foiaș and de Branges–Rovnyak transcriptions

- **Sz.-Nagy–Foiaș:** $\mathcal{H} = L^2(E \oplus E_*)$ (non-weighted, $W \equiv I$).

$$\mathcal{K}_\theta := \begin{pmatrix} H^2_{E_*} \\ \operatorname{clos} \Delta L^2_E \end{pmatrix} \ominus \begin{pmatrix} \theta \\ \Delta \end{pmatrix} H^2_E,$$

  where $\Delta(z) := (1 - \theta(z)^*\theta(z))^{1/2}$, $z \in \mathbb{T}$.

- **de Branges–Rovnyak:** $\mathcal{H} = L^2(E \oplus E_*, W_\theta^{[-1]})$, where

$$W_\theta(z) = \begin{pmatrix} I & \theta(z) \\ \theta(z)^* & I \end{pmatrix}$$

  and $W_\theta^{[-1]}$ is the Moore–Penrose inverse of W_θ. $\mathcal{K}_\theta$ is given by

$$\left\{ \begin{pmatrix} g_+ \\ g_- \end{pmatrix} : g_+ \in H^2(E_*),\ g_- \in H^2_-(E),\ g_- - \theta^* g_+ \in \Delta L^2(E) \right\}.$$

Characteristic function and defects for U_γ

Recall: $U_\gamma = U_1 + (\gamma - 1)bb_1^*$, $b_1 = U_1^* b$, $|\gamma| < 1$.

- $\mathfrak{D}_{U_\gamma}$ and $\mathfrak{D}_{U_\gamma^*}$ are spanned by the vectors b_1 and b respectively.
- Characteristic function θ_T of a contraction T is defined as

$$\theta_T(z) = \left(-T + zD_{T^*}(I - zT^*)^{-1}D_T\right)\Big|_{\mathfrak{D}}, \qquad z \in \mathbb{D}.$$

- To compute it use Rank one inversion formula (Sherman–Morrison formula)

$$(I - bc^*)^{-1} = I + \frac{1}{d}bc^*, \qquad d = (b, c) = c^*b.$$

- $I - zU_\gamma^*$ is a rank one perturbation of $I - zU_1^* = I - zM_{\overline{\xi}}$;
- The inverse of $I - zM_{\overline{\xi}}$ is multiplication by $(1 - z\overline{\xi})^{-1}$, so Cauchy integrals appear.

Characteristic function and defects for U_γ

- Define Cauchy integrals

$$R_1\tau(\lambda) := \int_{\mathbb{T}} \frac{\overline{\xi}\lambda d\tau(\xi)}{1-\overline{\xi}\lambda}, \qquad R_2\tau(\lambda) := \int_{\mathbb{T}} \frac{1+\overline{\xi}\lambda}{1-\overline{\xi}\lambda} d\tau(\xi).$$

- Characteristic function θ_γ of U_γ in the bases b_1, b:

$$\theta_\gamma(\lambda) = -\gamma + \frac{(1-|\gamma|^2)R_1\mu(\lambda)}{1+(1-\overline{\gamma})R_1\mu(\lambda)} = \frac{(1-\gamma)R_2\mu(\lambda)-(1+\gamma)}{(1-\overline{\gamma})R_2\mu(\lambda)+(1+\overline{\gamma})},$$

- Note that $\theta_\gamma(0) = -\gamma$, because $R_1\mu(0) = 0$
- For $\gamma = 0$

$$\theta_0(\lambda) = \frac{R_1\mu(\lambda)}{1+R_1\mu(\lambda)} = \frac{R_2\mu(\lambda)-1}{R_2\mu(\lambda)+1}, \qquad \lambda \in \mathbb{D}.$$

"Model" case of unitary perturbations

Recall: $U_\alpha = U_1 + (\alpha - 1)bb_1^*$, $|\alpha| = 1$

$$U_1 = M_\xi \text{ in } L^2(\mu), \quad \mu(\mathbb{T}) = 1, \qquad b \equiv 1, \quad b_1 = U_1^* b \equiv \overline{\xi}$$

- Let μ_α be the spectral measure of U_α corresponding to the vector b.
- Want to find a unitary operator $\mathcal{V}_\alpha : L^2(\mu) \to L^2(\mu_\alpha)$ such that $\mathcal{V}_\alpha b = \mathbf{1} \in L^2(\mu_\alpha)$ and such that

$$\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha.$$

Case of self-adjoint perturbations was treated earlier by Liaw–Treil in [2]. This case is treated similarly.

Pretending to be a physysist

Let $\mathcal{V}_\alpha$ be an integral operator with kernel $K(z,\xi)$.

- $U_\alpha = M_\xi + bb_1^*$, so we can rewrite the relation $\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha$ as

$$\mathcal{V}_\alpha M_\xi = M_z \mathcal{V}_\alpha - (1-\alpha)\mathcal{V}_\alpha b b_1^*.$$

- We know that $\mathcal{V}_\alpha b = 1$, $b_1 = \overline{\xi}$, so $\mathcal{V}_\alpha b b_1^*$ is an integral operator with kernel ξ

$$K(z,\xi)\xi = zK(z,\xi) - (\alpha-1)\xi.$$

- Solving for K we get

$$K(z,\xi) = (1-\alpha)\frac{\xi}{\xi - z} = (1-\alpha)\frac{1}{1-\overline{\xi}z}$$

First representation for $\mathcal{V}_\alpha$

Theorem (Repesentation of $\mathcal{V}_\alpha$)

The unitary operator $\mathcal{V}_\alpha : L^2(\mu) \to L^2(\mu_\alpha)$ *such that* $\mathcal{V}_\alpha b = \mathbf{1} \in L^2(\mu_\alpha)$ *and such that*

$$\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha .$$

is given by

$$\mathcal{V}_\alpha f(z) = f(z) + (1-\alpha) \int_{\mathbb{T}} \frac{f(\xi) - f(z)}{1 - \bar{\xi} z} \, d\mu(\xi)$$

for $f \in C^1(\mathbb{T})$

Idea of the proof

- Recalling that $U_\alpha = U_1 + (\alpha - 1)bb_1^*$ rewrite $\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha$ as

$$\mathcal{V}_\alpha U_1 = M_z \mathcal{V}_\alpha + (1-\alpha)(\mathcal{V}_\alpha b)b_1^*.$$

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- Recalling that $U_\alpha = U_1 + (\alpha - 1) b b_1^*$ rewrite $\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha$ as

$$\mathcal{V}_\alpha U_1 U_1 = M_z \mathcal{V}_\alpha U_1 + (1-\alpha)(\mathcal{V}_\alpha b) b_1^* U_1 .$$

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- Right multiplying by U_1 and applying the above “black” identity to $\mathcal{V}_\alpha U_1$ in the right hand side, we get

$$\mathcal{V}_\alpha U_1^2 = M_z^2 \mathcal{V}_\alpha + (1-\alpha)\left[(M_z \mathcal{V}_\alpha b) b_1^* + (\mathcal{V}_\alpha b) b_1^* U_1\right]$$

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- By induction we get

$$\mathcal{V}_\alpha U_1^n = M_z^n \mathcal{V}_\alpha + (1-\alpha) \sum_{k=1}^{n} M_z^{k-1} (\mathcal{V}_\alpha b) b_1^* U_1^{n-k} .$$

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- Applying to $b \equiv 1$ and summing geometric progression we get the formula for $f(\xi) = \xi^n$, $n \geq 0$.

Idea of the proof, continued

- To get the formula for $\overline{\xi}^n$ we use $\mathcal{V}_\alpha U_\alpha^* = M_{\overline{z}}\mathcal{V}_\alpha$, which is obtained by taking adjoint in $\mathcal{V}_\alpha U_\alpha = M_z\mathcal{V}_\alpha$.
- Extend the formula from trig. polynomials to $f \in C^1$ by standard approximation reasoning.

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- Extend the formula from trig. polynomials to $f \in C^1$ by standard approximation reasoning.

A general statement

Rank one commutation relations like

$$\mathcal{V} M_\xi = M_z \mathcal{V} + c b_1^*$$

usually give singular integral representations for $\mathcal{V}$.

Singular integral operators

Recall that $\mathcal{V}_\alpha f(z) = f(z) + (1-\alpha)\int_{\mathbb{T}} \frac{f(\xi)-f(z)}{1-\bar{\xi}z}\,d\mu(\xi)$

Theorem (Regularization of the weighted Cauchy transform)

The integral operators $T_r = T_r^\alpha : L^2(\mu) \to L^2(\mu_\alpha)$ with kernels $1/(1-r\bar{\xi}z)$, $r \in \mathbb{R}_+ \setminus \{1\}$ are uniformly bounded.

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- Let $Tf(z) := \int_{\mathbb{T}} \frac{f(\xi)}{1-\bar{\xi}z} d\mu(\xi)$; well defined for $z \notin \operatorname{supp} f$

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- Let $Tf(z) := \int_{\mathbb{T}} \frac{f(\xi)}{1-\bar{\xi}z} d\mu(\xi)$; well defined for $z \notin \operatorname{supp} f$
- Since $\mathcal{V}_\alpha$ is bounded, we get for $f, g \in C^1$, $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$

$$(Tf, g)_{L^2(\mu_\alpha)} \le C\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu_\alpha)}$$

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- By a theorem of Liaw–Treil [3] this implies uniform boundedness of the regularizations T_r if the measures μ and μ_α do not have common atoms (U_1 and U_α do not have common eigenvalues).

Singular integral operators

- Uniform boundedness of T_r together with μ_α-a.e. convergence of $T_r f$ imply existence of w.o.t.-limits $T^\alpha_\pm = \text{w.o.t.-}\lim_{r\to 1\mp} T_r$.
- Using $T^\alpha_\pm$ we can rewrite the representation

$$\mathcal{V}_\alpha f(z) = f(z) + (1-\alpha)\int_{\mathbb{T}} \frac{f(\xi)-f(z)}{1-\bar{\xi}z}\, d\mu(\xi)$$

as

$$\mathcal{V}_\alpha f = [\mathbf{1} - (1-\alpha)T^{1,\alpha}_\pm \mathbf{1}]f + (1-\alpha)T^{1,\alpha}_\pm f.$$

- $(\mu_\alpha)_{\mathrm{a}}$-a.e. convergence follows from classical results about jumps of Cauchy transform; $(\mu_\alpha)_{\mathrm{s}}$-a.e. convergence can be obtained from Poltoratskii's theorem about boundary values of the normalized Cauchy transform, see [7].
- For the weak convergence it is enough to have μ_α-a.e. convergence of $T_r f$ for $f \in C^1$, which can be proved using elementary methods.

Adjoint Clark operator, freedom of choice

- Let $U_\gamma = U_1 + (\gamma - 1) b b_1^*$, $|\gamma| < 1$;
- $U_\gamma \cong \mathcal{M}_{\theta_\gamma}$, $\mathcal{M}_{\theta_\gamma} := P_{\mathcal{K}_{\theta_\gamma}} M_z \big|_{\mathcal{K}_{\theta_\gamma}}$

Adjoint Clark operator: a unitary $\Phi_\gamma^* : L^2(\mu) \to \mathcal{K}_{\theta_\gamma}$ such that

$$\Phi_\gamma^* U_\gamma = \mathcal{M}_{\theta_\gamma} \Phi_\gamma^* \qquad (*)$$

- Defect spaces $\mathfrak{D}_{U_\gamma}$ and $\mathfrak{D}_{U_\gamma^*}$ are spanned by the vectors $b_1 \equiv \overline{\xi}$ and $b \equiv \mathbf{1}$ respectively.
- Let $\mathfrak{D}_{\mathcal{M}_{\theta_\gamma}}$ and $\mathfrak{D}_{\mathcal{M}_{\theta_\gamma}^*}$ be spanned by c_1^γ and c^γ, $\|c_1^\gamma\| = \|c^\gamma\| = 1$.
- Relation $(*)$ implies that $\Phi_\gamma^* b = \alpha c^\gamma$, $\Phi_\gamma^* b_1 = \beta c_1^\gamma$, $|\alpha| = |\beta| = 1$.
- Except for the case $\gamma = 0$ and $\mu = |dz|/2\pi$, β is uniquely defined by α.

Defect vectors of $\mathcal{M}_{\theta_\gamma}$ in the Sz.-Nagy–Foiaș transcription

- Defect subspaces $\mathfrak{D}_{\mathcal{M}_\theta}$ and $\mathfrak{D}_{\mathcal{M}_\theta^*}$ are spanned by c_1 and c, $\|c\| = \|c_1\| = 1$,

$$c(z) := \left(1 - |\theta(0)|^2\right)^{-1/2} \begin{pmatrix} 1 - \overline{\theta(0)}\theta(z) \\ -\overline{\theta(0)}\Delta(z) \end{pmatrix},$$

$$c_1(z) := \left(1 - |\theta(0)|^2\right)^{-1/2} \begin{pmatrix} z^{-1}\left(\theta(z) - \theta(0)\right) \\ z^{-1}\Delta(z) \end{pmatrix},$$

- Vectors c^γ and c_1^γ agree, i.e. $\Phi_\gamma^* b = c^\gamma$ implies $\Phi_\gamma^* b_1 = c_1^\gamma$ (not considering the exceptional case $\gamma = 0$, $\mu = |dz|/2\pi$)

Theorem (A "universal" representation formula)

Let θ_γ be a characteristic function of U_γ, $|\gamma| < 1$. Assume that the vectors $c^\gamma \in \mathfrak{D}_{\mathcal{M}^*_{\theta_\gamma}}$, $c_1^\gamma \in \mathfrak{D}_{\mathcal{M}_{\theta_\gamma}}$ $\|c^\gamma\| = \|c_1^\gamma\| = 1$ agree. Let $\Phi^* = \Phi^*_\gamma : L^2(\mu) \to \mathcal{K}_{\theta_\gamma}$ be a unitary operator satisfying

$$\Phi^*_\gamma U_\gamma = \mathcal{M}_{\theta_\gamma} \Phi^*_\gamma,$$

and such that $\Phi^*_\gamma b = c^\gamma$ (so $\Phi^*_\gamma b_1 = c_1^\gamma$).
Then for all $f \in C^1(\mathbb{T})$

$$\Phi^*_\gamma f(z) = A_\gamma(z) f(z) + B_\gamma(z) \int \frac{f(\xi) - f(z)}{1 - \bar{\xi} z} \, d\mu(\xi), \qquad z \in \mathbb{T},$$

where $A_\gamma(z) = c^\gamma(z)$, $B_\gamma(z) = c^\gamma(z) - z c_1^\gamma(z)$.

This theorem works in any transcription of the model.

Idea of the proof

- Write, denoting $c_2^\gamma(z) := zc_1^\gamma(z)$,

$$\begin{aligned}\mathcal{M}_{\theta_\gamma} &= M_z - c_2^\gamma(c_1^\gamma)^* - \theta_\gamma(0)c^\gamma(c_1^\gamma)^* \\ &= M_z + (\gamma c^\gamma - c_2^\gamma)(c_1^\gamma)^*.\end{aligned}$$

Rank one perturbation of M_z! Should get at most rank 2 commutation relation.

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  Rank one perturbation of M_z! Should get at most rank 2 commutation relation.

- Using this identity rewrite $\Phi_\gamma^* U_\gamma = \mathcal{M}_{\theta_\gamma} \Phi_\gamma^*$ as

$$\Phi_\gamma^* U_1 + (\gamma - 1) c^\gamma b_1^* = M_z \Phi_\gamma^* + (\gamma c^\gamma - c_2^\gamma) b_1^*$$

  or equivalently

$$\Phi_\gamma^* U_1 = M_z \Phi_\gamma^* + (c^\gamma - c_2^\gamma) b_1^*.$$

  We got rank one commutation relation!

- Commutation relations imply integral representation.

Idea of the proof, difficulties

- Formally the right side of

$$\Phi_\gamma^* U_1 = M_z \Phi_\gamma^* + (c^\gamma - c_2^\gamma) b_1^*. \qquad (*)$$

acts from $L^2(\mu)$ to outside of $\mathcal{K}_\theta$.

- To get $\Phi_\gamma^* \overline{\xi}^n$ we use the commutant relation

$$\begin{aligned}\Phi_\gamma^* U_1^* &= M_{\overline{z}} \Phi_\gamma^* + (c_1^\gamma - M_{\overline{z}} c^\gamma) b^* \\ &= M_{\overline{z}} \Phi_\gamma^* - M_{\overline{z}} (c^\gamma - c_2^\gamma) b^*,\end{aligned}$$

which cannot be obtained by taking the adjoint of $(*)$.

- It is a miracle that the formulas for $\Phi_\gamma^* \xi^n$ and $\Phi_\gamma^* \overline{\xi}^n$ agree.

Cauchy type operators and regularizations

- For $f \in L^2(\mu)$ let

$$Rf\mu(z) = \int_{\mathbb{T}} \frac{f(\xi)}{1-\bar{\xi}z} d\mu(\xi)$$

  and let $T_+ f$ be the non-tangential boundary values of $Rf\mu(z)$, $|z|<1$.
- Let $T_r : L^2(\mu) \to L^2(v)$, $v = |B|^2$ be the integral operators with kernel $1/(1-r\bar{\xi}z)$, $r \in \mathbb{R}_+ \setminus \{1\}$.
- Operators $T_r : L^2(\mu) \to L^2(v)$ (equivalently $M_B T_r : L^2(\mu) \to L^2$) are uniformly in r bounded.
- $T_+ =$ w.o.t.- $\lim\limits_{r\to 1^-} T_r$ (as operators $L^2(\mu) \to L^2(v)$); equivalently, $M_B T_+ =$ w.o.t.- $\lim\limits_{r\to 1^-} M_B T_r$ (as operators $L^2(\mu) \to L^2$)

Theorem

The vector $g := (1-|\gamma|^2)^{1/2}\Phi_\gamma^* f$ *can be represented in the Sz.-Nagy–Foiaș transcription as*

$$g = \begin{pmatrix} 0 \\ (\overline{\gamma} - (\overline{\gamma}-1)T_+\mathbf{1})\Delta_\gamma \end{pmatrix} f + \begin{pmatrix} \frac{1+\overline{\gamma}\theta_\gamma}{T_+\mathbf{1}} \\ (\overline{\gamma}-1)\Delta_\gamma \end{pmatrix} T_+ f$$

$$= \begin{pmatrix} 0 \\ \frac{1-\overline{\gamma}\theta_0}{|1-\overline{\gamma}\theta_0|} T_+\mathbf{1}\cdot\Delta_0 \end{pmatrix} f + \begin{pmatrix} \frac{1-|\gamma|^2}{1-\overline{\gamma}\theta_0}\cdot\frac{1}{T_+\mathbf{1}} \\ (\overline{\gamma}-1)\frac{(1-|\gamma|^2)^{1/2}}{|1-\overline{\gamma}\theta_0|}\Delta_0 \end{pmatrix} T_+ f$$

for $f \in L^2(\mu)$.

- Since $\frac{1}{T_+\mathbf{1}} = 1-\theta_0$, the top floor g_1 is in the Hardy space H^2.
- For $\gamma = 0$ we get

$$\Phi_0^* f = \begin{pmatrix} 0 \\ (T_+\mathbf{1})\Delta_0 \end{pmatrix} f + \begin{pmatrix} 1/T_+\mathbf{1} \\ -\Delta_0 \end{pmatrix} T_+ f$$

Idea of the proof

- Take the representation

$$\Phi_\gamma^* f(z) = A_\gamma(z) f(z) + B_\gamma(z) \int \frac{f(\xi) - f(z)}{1 - \bar{\xi} z} \, d\mu(\xi), \qquad z \in \mathbb{T},$$

for $f \in C^1$.

Idea of the proof

- Take the representation

$$\Phi_\gamma^* f(z) = A_\gamma(z) f(z) + B_\gamma(z) \int \frac{f(\xi) - f(z)}{1 - r\overline{\xi} z} \, d\mu(\xi), \qquad z \in \mathbb{T},$$

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- Substituting expressions for A_γ and B_γ we get the result.

Recall: de Branges–Rovnyak transcription

- $\mathcal{K}_\theta$ in the de Branges–Rovnyak transcription is given by

$$\left\{ \begin{pmatrix} g_+ \\ g_- \end{pmatrix} : \ g_+ \in H^2, \ g_- \in H^2_-, \ g_- - \overline{\theta} g_+ \in \Delta L^2 \right\}.$$

- Recall that in the Sz.-Nagy–Foiaș transcription $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathcal{K}_\theta$ iff

$$g_1 = g_+ \in H^2, \quad g_2 \in \operatorname{clos} \Delta L^2, \qquad g_- := \overline{\theta} g_1 + \Delta g_2 \in H^2_-;$$

the last inclusion means that $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \perp \begin{pmatrix} \theta \\ \Delta \end{pmatrix} H^2$.

- $g_1 = g_+$ and g_- are exactly the same as in the de Branges–Rovnyak transcription.

Representation in the de Branges–Rovnyak transcription

- We had

$$g_1 = g_+ = (1-|\gamma|^2)^{-1/2}(1+\overline{\gamma}\theta_\gamma)\,\frac{T_+ f}{T_+\mathbf{1}} = \frac{(1-|\gamma|^2)^{1/2}}{1-\overline{\gamma}\theta_0}\,\frac{T_+ f}{T_+\mathbf{1}}$$

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- For $g_- = g_-^\gamma$ we get

$$g_-^\gamma = (1-|\gamma|^2)^{-1/2}\left(\overline{\theta}_\gamma+\overline{\gamma}\right)\frac{T_-f}{T_-\mathbf{1}} = \frac{(1-|\gamma|^2)^{1/2}\overline{\theta}_0}{1-\gamma\overline{\theta}_0}\cdot\frac{T_-f}{T_-\mathbf{1}}$$
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Clark operator Φ_γ

- the non-tangential boundary values of the function

$$z \mapsto \frac{1-\overline{\gamma}}{(1-|\gamma|^2)^{1/2}} g_1(z), \qquad z \in \mathbb{D}$$

exist and coincide with f_s μ_s-a.e. on $\mathbb{T}$.

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Follows from representation

$$g_1 = \frac{(1-|\gamma|^2)^{1/2}}{1-\overline{\gamma}\theta_0} \frac{T_+ f}{T_+ \mathbf{1}}$$

and Poltoratskii's theorem that boundary values of $Rf\mu(z)/R\mu(z)$, $z \in \mathbb{D}$ exist and equal f $\mu_{\rm s}$-a.e.; also uses $\theta(z) = 1$ $\mu_{\rm s}$-a.e.

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- For the "absolutely continuous" part $f_{\rm a}$ of f

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a.e. on $\mathbb{T}$; here, recall, $g_- := g_1 \overline{\theta}_\gamma + \Delta_\gamma g_2 \in H^2_-$.

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Proof uses representation for g_1 and g_- and standard jump formulas for Cauchy integrals.

Bounds on the normalized Cauchy transform

- A. Poltoratskii, [7]: the normalized Cauchy transform $f \mapsto \frac{T_+ f}{T_+ \mathbf{1}}$ acts $L^2(\mu) \to L^2$.
- Equivalently: $T_+ : L^2(\mu) \to L^2(v)$, $v = 1/|T_+\mathbf{1}|^2 = |1-\theta_0|^2$. (because $1/T_+\mathbf{1} = 1-\theta_0$).
- Follows from our result: $T_+ : L^2(\mu) \to L^2(v_0)$,

$$v_0 = |B_0|^2 = |1-\theta_0|^2 + \Delta_0^2 = 2\,\mathrm{Re}(1-\theta_0).$$

v_0 can be much bigger than v: $v \asymp v_0^2$ when $\theta_0(z) \to 1$ non-tangentially.

Exterior normalized Cauchy transform

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  Indeed, $1/T_- \mathbf{1} = \overline{\theta}_0/(\overline{\theta}_0 - 1)$, so

$$\frac{T_- f}{T_- \mathbf{1}} = \frac{\overline{\theta}_0 - 1}{\overline{\theta}_0} T_- f.$$

If θ_0 is small near i, so $1/\theta_0 \notin L^2$ there, and $\mu(E) > 0$ in a small neighborhood $E \ni 1$, then $|T_- \mathbf{1}_E| \geq \delta > 0$ near i, so $\frac{T_- f}{T_- \mathbf{1}} \notin L^2$.

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- The operator

  $$f \mapsto \overline{\theta}_0 \frac{T_- f}{T_- \mathbf{1}}$$

  acts $L^2(\mu) \to L^2$.
  Is it correct "exterior" normalized Cauchy transform?

Comparison with Clark model

- D. Clark started with model operator K_θ, (θ inner $\iff$ μ is purely singular) and considered it all unitary rank one perturbations.
- In our model it corresponds considering operator $U_\gamma = U_1 + (\gamma - 1)bb_1^*$, $\gamma = -\theta(0)$, then all unitary rank one perturbations are exactly the operators U_α, $|\alpha| = 1$.
- Clark measures $\widetilde{\mu}_\alpha$ are the spectral measures of the operators U_α.
- If $\theta(0) = 0$ them $\widetilde{\mu}_\alpha = \mu_\alpha$ and the Clark operators coincide with ours.
- If $\theta(0) \neq 0$ $\widetilde{\mu}_\alpha$ is a multiple μ_α, and the operators differ by a factor $c(\gamma)$.
- In Clark model $\widetilde{\mu}_\alpha$ is not a probability measure, $|c(\gamma)|$ compensate for that.

Comparison with Sarason's model

- D. Sarson in [8] presented a unitary operator between $H^2(\mu) = \overline{\text{span}}\{z^n : n \in \mathbb{Z}_+\}$ and the de Branges space $\mathcal{H}(\theta)$; like Clark, he started with a model operator in $\mathcal{K}_\theta$
- The space $\mathcal{H}(\theta) \subset H^2$ is defined as a range $(I - T_\theta T_{\theta^*})^{1/2} H^2$ endowed with the *range norm* (the minimal norm of the preimage); $T_\varphi : H^2 \to H^2$ is a Toeplitz opearator, $T_\varphi f = P_{H^2}(\varphi f)$.
- If θ is an extreme point of the unit ball in H^∞ $(\int_\mathbb{T} \ln(1-\theta|^2)|dz| = -\infty \iff \int_\mathbb{T} \ln w|dz| = -\infty$, w density of $\mu)$ then $\mathcal{H}(\theta)$ is canonically isomorphic to the model space $\mathcal{K}_\theta$ in the de Branges–Rovnyak transcription, see [6].
- His measure μ coinsides with the Clark measure $\widetilde{\mu}_\alpha$,

$$\alpha = \frac{1+\gamma}{1+\overline{\gamma}};$$

the formulas are the same as Clark's.

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