Inner Functions and Inverse Spectral theory

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I_n disjoint intervals is *short* if $\sum_n \frac{|I_n|^2}{1 + dist(0, I_n)} < \infty$. It is *long* otherwise.

$$D^*(\Lambda) = \sup\{d : \exists \text{ long } \{I_n\} \text{ s.t. } \#(\Lambda \cap I_n) \geq d|I_n|\}$$

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Theorem (Beurling & Malliavin)

$$R(\Lambda) = D^*(\Lambda)$$

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(Uniqueness set: $F, G \in PW_a$ and $F = G$ on $\Lambda \Rightarrow F \equiv G$.)

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Paley-Wiener Space

$$PW_a = S^{-a}[\mathcal{H}^2 \ominus S^{2a}\mathcal{H}^2],$$

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What about uniqueness sets of K_Θ?

Meromorphic Inner Functions (MIFs)

Definition

An MIF Θ is a bounded analytic function on $\mathbb{C}_+$, with a meromorphic continuation on $\mathbb{C}$ such that $|\Theta| = 1$ on $\mathbb{R}$.

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Also a spectrum

$\{x \in \mathbb{R} | \Theta(x) = e^{i\alpha}\}$

Schrödinger Operators

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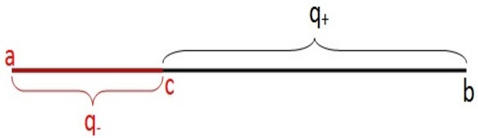
$$\cos(\alpha)u(a) + \sin(\alpha)u'(a) = 0 \quad (1)$$

$$\cos(\beta)u(b) + \sin(\beta)u'(b) = 0. \quad (2)$$

Let $L = (q, \alpha, \beta)$.



The data $(q_-, \sigma(L))$ determines L if for any Schrödinger operator $\tilde{L}$ with potential $\tilde{q}$,

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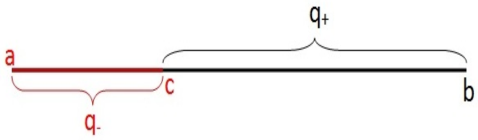

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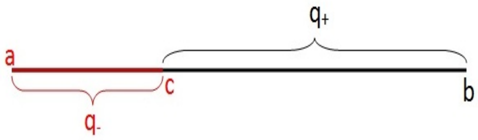

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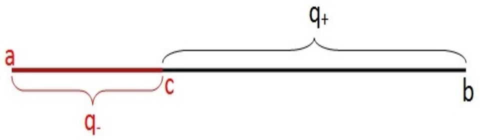

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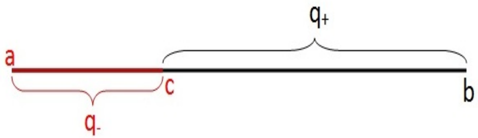

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3. Del-Rio, Gesztezy, Simon:
   (i) c=$\frac{3}{4}(a+b)$ & half the spectrum is enough to determine L.
   (ii) c=$\frac{1}{4}(a+b)$ and $1\frac{1}{4}$ spectrum is enough to determine L.

Fix β. For $\lambda \in \mathbb{C}$, let u_λ be a solution to $-u'' + qu = \lambda u$ and (2).

$$m(\lambda) \quad := \quad \frac{u'_\lambda(a)}{u_\lambda(a)} \text{ ... (Weyl m-function)}$$

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Remark 1: The Weyl inner function determines the potential.
Remark 2: $\sigma(L) = \sigma(\Theta)$.

- $\sigma(\Theta_-\Theta_+) = \sigma(L)$
- $q_- = \tilde{q}_- \Rightarrow \Theta_- = \tilde{\Theta}_+$
- $[q_-, \sigma(L)]$ determines q if $[\Theta_-, \sigma(\Theta_-\Theta_+)]$ determine Θ_+
  (i.e. for $\tilde{\Theta} = \Theta_-\tilde{\Theta}_+$, if $\sigma(\Theta) = \sigma(\tilde{\Theta}) \Rightarrow \Theta_+ = \tilde{\Theta}_+$)

Q: Let $\Theta = \Phi\Psi$. Does $[\Phi, \sigma(\Theta)]$ determine Ψ? i.e., can there exist a different $\Theta_2 = \Phi\Psi_2$, with $\sigma(\Theta) = \sigma(\Theta_2)$?

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Toy problem

Let $b(z) = \frac{z-i}{z+i}$, $\Theta = b^2$. $\sigma(\Theta) = \{0, \infty\}$.
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Q: Let $\Theta = \Phi\Psi$. $[\Phi, \sigma(\Theta)]$ determines $\Psi \Leftrightarrow \sigma(\Theta)$ determines Ψ.

Let $\Lambda \subset \mathbb{R}$. Does Λ *define* Ψ? i.e. if there is another MIF Ψ_2 with $\arg \Psi = \arg \Psi_2$ on $\Lambda \Leftrightarrow \Psi \equiv \Psi_2$?

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Theorem (Mitkovski, R.)

The sequence Λ uniquely defines Φ(in the class of Schrödinger Weyl inner functions in $L^2(a, b)$) if and only if the set $\Lambda \cup \{x\}$ is a uniqueness set for $K^2_{\Phi^2}$, for all $x \in \mathbb{R} \setminus \Lambda$.

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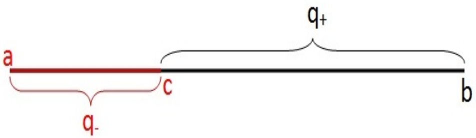

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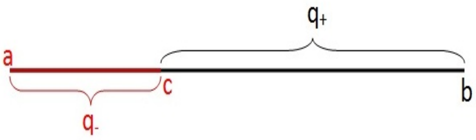

Theorem (Mitkovski, R.)

If $D^*(\Lambda) > 2(c-b)$, then $[q_-, \Lambda]$ determines q. If $D^*(\Lambda) < 2(c-b)$, then there is a different potential $\tilde{q}$ such that $q_- = \tilde{q}_-$ that has the same spectral data on Λ.

Corollary: Results of Del Rio, Gesztesy, Simon.

Thank you!