Thermodynamic formalism of rational maps

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Integral means and geometric pressure

1. Integral means spectrum;
2. Quadratic JULIA sets;
3. Geometric pressure function.

Integral means spectrum

$\phi : \mathbb{D} \to \overline{\mathbb{C}}$: Univalent,

$$\phi(z) = \frac{1}{z} + b_1 z + b_2 z^2 + \cdots .$$

$$\beta_\phi(t) := \limsup_{r \to 1^-} \frac{\log\left(\int_0^{2\pi} \left|\phi'(re^{i\theta})\right|^t \, d\theta\right)}{|\log(1-r)|}.$$

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Integral means spectrum.

$$B(t) := \sup_\phi \beta_\phi(t).$$

Universal spectrum.

Conjeture: For $|t| < 2, B(t) = \frac{t^2}{4}$.

$B(1) =$ Littlewood's constant;

$\Rightarrow$ Hölder domains and Brenan's conjectures.

Littlewood's constant

$\phi : \mathbb{D} \to \overline{\mathbb{C}}$: Univalent, $\phi(z) = \frac{1}{z} + b_1 z + b_2 z^2 + \cdots$.

$$\beta_\phi(1) = \limsup_{r \to 1^-} \frac{\log \mathrm{Length}\left(\phi\left(\{z \in \mathbb{D} : |z| = r\}\right)\right)}{|\log(1-r)|}.$$

Length = Euclidean length in $\mathbb{C}$.

Theorem (Littlewood, 1925; Carleson–Jones, 1992)

For every $\phi(z) = \frac{1}{z} + b_1 z + b_2 z^2 + \cdots$,

$$|b_n| \lesssim n^{B(1)}.$$

Moreover, $B(1)$ *is the least constant with this property.*

$B(1) < 0.46$, Hedenmalm–Shimorin, 2005.

$B(1) > 0.2308$, Beliaev–Smirnov, 2010;

LITTLEWOOD's constant

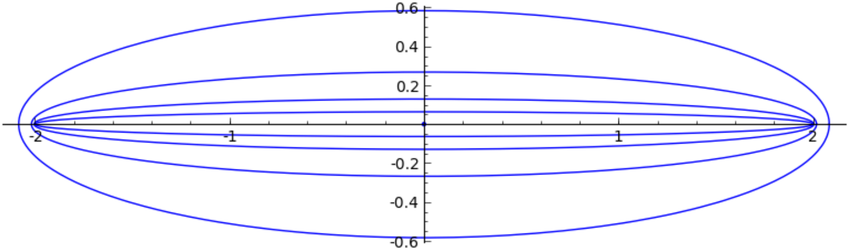

Figure : Equipotentials of $\phi(z) = \frac{1}{z} + z$, for $r = 1 - \frac{1}{2^2}, 1 - \frac{1}{2^3}, 1 - \frac{1}{2^4}$, and $1 - \frac{1}{2^5}$.

Littlewood's constant

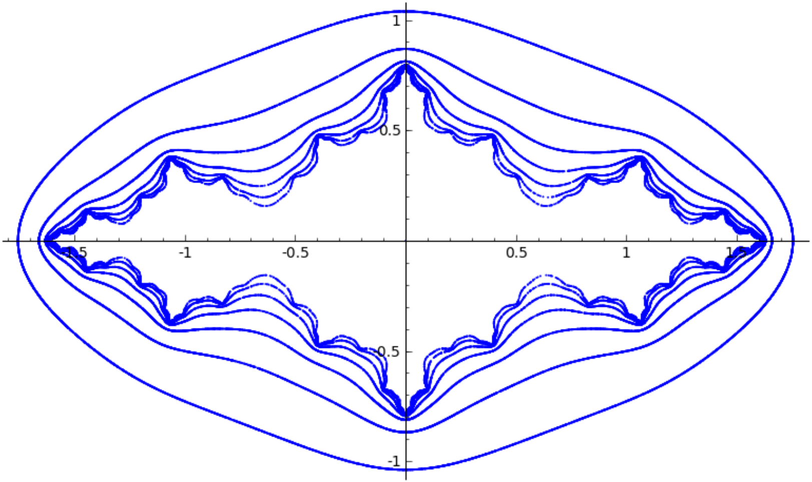

Figure : Extremal functions must have a fractal nature

Quadratic JULIA sets

For $c \in \mathbb{C}$:

$$\begin{array}{rccc} f_c: & \mathbb{C} & \to & \mathbb{C} \\ & z & \mapsto & f_c(z) := z^2 + c \end{array}$$

$$K_c := \left\{ z_0 \in \mathbb{C} : \left(f_c^n(z_0)\right)_{n \geq 1} \text{ is bounded} \right\}$$

Filled JULIA set of f_c;
= complement of the attracting basin of infinity.

$$J_c := \partial K_c$$

JULIA set of f_c.

Quadratic JULIA sets

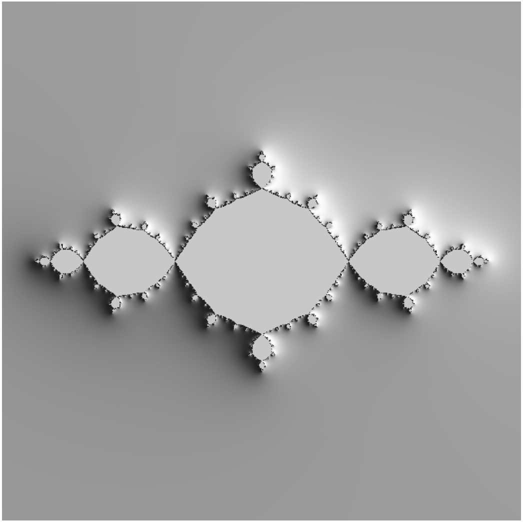

Figure : Quadratic JULIA set; from Tomoki KAWAHIRA's gallery.

Quadratic Julia sets

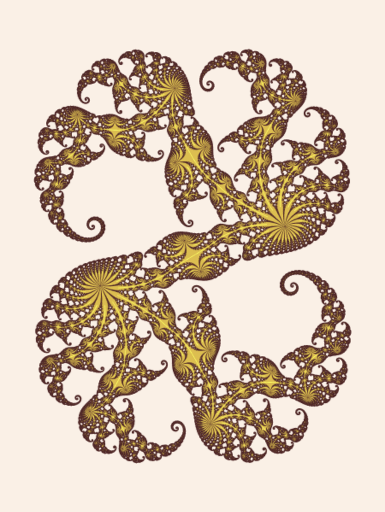

Figure : Another quadratic Julia set, from Arnaud Chéritat's gallery.

The spectrum as a pressure function

$c \in \mathbb{C}$: Such that J_c is connected;

$\phi_c : \mathbb{D} \to \overline{\mathbb{C}}$: Conformal representation of $\overline{\mathbb{C}} \setminus K_c$,

$$\phi_c(z) = \frac{1}{z} + b_1 z + b_2 z^2 + \cdots .$$

The universal spectrum can be computed with JULIA sets of arbitrary degree (BINDER, JONES, MAKAROV, SMIRNOV).

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$$P_c(t) := \left(\beta_{\phi_c}(t) - t + 1\right) \log 2;$$

Geometric pressure function of f_c.

$$= \lim_{n\to\infty} \frac{1}{n} \log \sum_{z \in f_c^{-n}(z_0)} |Df_c^n(z)|^{-t};$$

= spectral radius of the transfer operator.

Multifractal analysis

ρ_c: Harmonic measure of J_c

= Maximal entropy measure of f_c.

$$D_c(\alpha) := \mathrm{HD}\left(\{z \in J_c : \rho_c(B(z,r)) \sim r^\alpha\}\right).$$

Local dimension spectrum;
Frequently D_c is analytic (!!!).

Theorem (Sinaï, Ruelle, Bowen, 1970's)

f_c uniformly hyperbolic $\Rightarrow$ D_c and P_c are analytic and

$$D_c(\alpha) = \inf_{t \in \mathbb{R}} \left\{ t + \alpha \frac{P_c(t)}{\log 2} \right\}.$$

$\sim$ Legendre transform;
Morally: P_c is analytic $\Leftrightarrow$ D_c is analytic.

Classification of phase transitions

1. Basic properties of the geometric pressure function;
2. Negative spectrum;
3. Phase transitions are of freezing type;
4. Positive spectrum tricothomy;
5. Phase transitions at infinity.

Geometric pressure function

Variational Principle

$$P_c(t) = \sup_{\mu \text{ invariant probability on } J_c} \left(h_\mu - t \int \log|Df_c| \, d\mu \right).$$

h_μ = measure-theoretic entropy.

Definition

- Equilibrium state for the potential $-t\log|Df_c|$ $:=$ A measure μ realizing the supremum.
- Phase transition $:=$ A parameter at which P_c is not analytic.

Comparison with statistical mechanics.

Geometric pressure function

$$P_c(t) = \sup_{\mu \text{ invariant probability on } J_c} \left(h_\mu - t \int \log |Df_c| \, d\mu \right).$$

- P_c is convex, Lipschitz, and non-increasing;
- $P_c(0) = \log 2$ topological entropy of f_c;

Geometric pressure function

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- P_c is convex, LIPSCHITZ, and non-increasing;
- $P_c(0) = \log 2$ topological entropy of f_c;
- $P_c(t) \geq \max\{-t\chi_{\inf}(c), -t\chi_{\sup}(c)\}$, where

$$\chi_{\sup}(c) := \lim_{t \to +\infty} \frac{P_c(t)}{-t};$$

= Supremum of LYAPUNOV exponents.

$$\chi_{\inf}(c) := \lim_{t \to -\infty} \frac{P_c(t)}{-t}.$$

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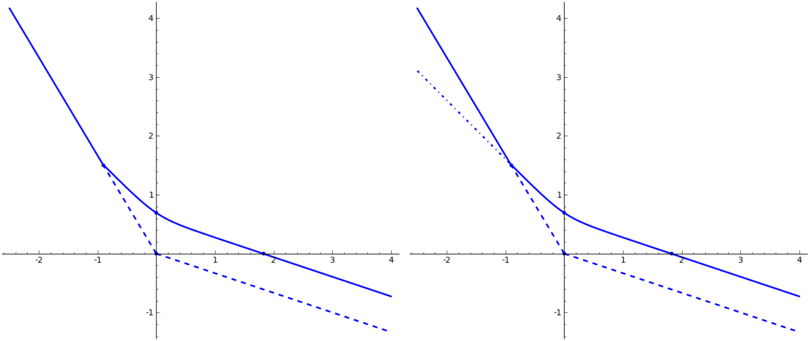
Theorem (Generalized BOWEN formula, PRZYTYCKI, 1998)

$$\inf\{t \in \mathbb{R} : P_c(t) = 0\} = \mathrm{HD}_{\mathrm{hyp}}(J_c).$$

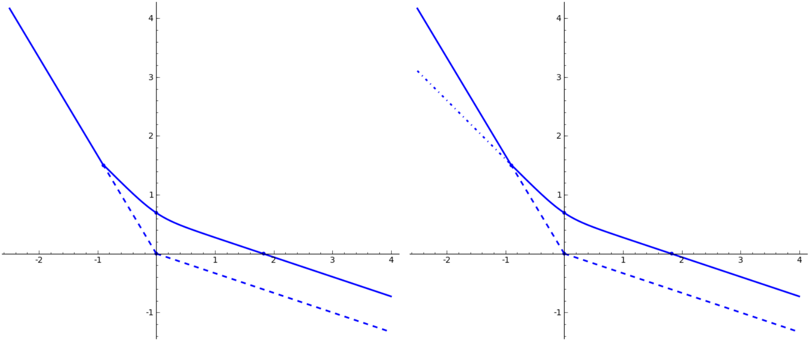
Negative spectrum



Mechanism: Gap in the LYAPUNOV spectrum.

$\Leftrightarrow$ there is a finite set Σ such that $f(\Sigma) = \Sigma$, $f^{-1}(\Sigma) \setminus \Sigma \subset \mathrm{Crit}(f)$.

Negative spectrum



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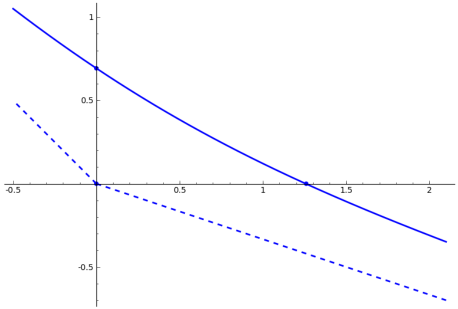
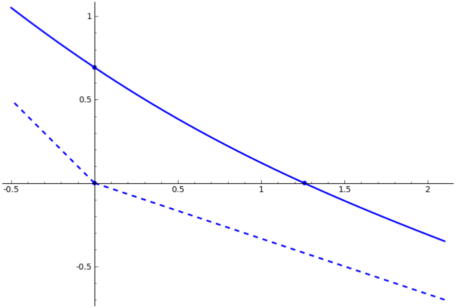
These phase transitions are removable.

MAKAROV–SMIRNOV, 2000.

Phase transitions are of freezing type



Phase transitions are of freezing type



Theorem (Pryzycki–RL, 2011)

$P_c(t_o) > \max\{-t_o\chi_{\inf}(c), -t_o\chi_{\sup}(c)\}$

$\Rightarrow$ *P_c is analytic at $t = t_o$.*

Positive spectrum tricothomy

$$\chi_{\mathrm{crit}}(c) := \liminf_{m\to+\infty} \frac{1}{m} \log|Df_c^m(c)|.$$

1. $\chi_{\mathrm{crit}}(c) < 0 \quad \Leftrightarrow \quad f_c$ is uniformly hyperbolic;

   LEVIN–PRZYTYCKI–SHEN, 2014.

2. $\chi_{\mathrm{crit}}(c) = 0 \quad \Leftrightarrow \quad$ Phase transition at the first zero of P_c;

   $\Leftrightarrow \quad \chi_{\mathrm{inf}}(c) = 0$

   PRZYTYCKI–RL–SMIRNOV (2003),

   "High-temperature phase transition"

Mechanism: Lack of expansion.

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Mechanism: Lack of expansion.

3. $\chi_{\mathrm{crit}}(c) > 0 \quad \Leftrightarrow \quad f_c$ is Collet–Eckmann

   Non-uniformly hyperbolic in a strong sense;

   Any phase transition in this case must be at "low-temperature":

   After the first zero of the geometric pressure function.

Positive spectrum tricothomy

Theorem (Coronel–RL, 2013)

There is $c \in \mathbb{R}$ such that $\chi_{\mathrm{crit}}(c) > 0$ and such that f_c has a phase transition at some $t_* > \mathrm{HD}_{\mathrm{hyp}}(J_c)$.
Moreover, c can be chosen so that the critical point of f_c is non-recurrent.

Examples show the phase transition can be of first order, or of "infinite order";
Inspired conformal Cantor of Makarov and Smirnov (2003).

Mechanism: Irregularity of the critical orbit.

Phase transitions at infinity

Theorem (Coronel–RL, 2016 (hopefully ...))

There is a quadratic-like map f such that:

- *For every $t > 0$ there is a unique equilibrium state ρ_t for $-t \log|Df|$;*
- $\lim_{t \to +\infty} \rho_t$ *does not exists.*

Theorem (Sensitive dependence of equilibria)

There is a quadratic-like map f such that, for every sequence $(t(\ell))_{\ell \geq 1}$ going to infinity, there is $\widetilde{f}$ arbitrarily close to f such that

- *For every $t > 0$ there is a unique equilibrium state $\widetilde{\rho}_t$ of $\widetilde{f}$ for $-t \log|D\widetilde{f}|$;*
- $\lim_{\ell \to +\infty} \widetilde{\rho}_{t(\ell)}$ *does not exists.*