# Norm Control for Inverses of Convolutions and Large Matrices

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- Motivation: Effective Inversions

(constructive, algorithmic, norm controlled)

- Example 1: Convolution T: $f \mapsto f \star S$ on a group G as a map on a Banach function space X

- The visible spectrum of T: the range of the Fourier transform $S\hat{}(G\hat{})$. A necessary condition for inversion:

$$\delta =: \inf | S\hat{} | > 0$$

«Well posed inversion»: $\|T^{-1}\| \leq c(\delta), \delta > 0$.

- Motivation: Effective Inversions

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- Example 2: Condition Numbers of Matrices T, n×n: $CN(T)= \|T\| \cdot \|T^{-1}\|$ .

- The visible spectrum of T: eigenvalues $\lambda_i(T)$, $i= 1,\ldots,n$; an invertibility condition:

$$\delta =: \inf|\lambda_i(T)| > 0$$

«Well posed inversion»:

$$CN(T)= \|T\| \cdot \|T^{-1}\| \leq c(\delta/\|T\|), \delta>0.$$

# Enough motivations?..

Finally, let him who has never used a convolution or a large matrix cast the first stone…

# My goal in effective inversions is to understand:

- Relations «Full Spectrum»/ «Visible Spectrum» (the Wiener-Pitt phenomenon)
- «Invisible» but Numerically Detectable Spectrum ($c(\delta)= \infty$)

# Plan for today:

1. Convolutions/Fourier multipliers
2. Large Matrices
3. Some Integration Operators

# 1. Convolutions/Fourier Multipliers

**Notation:**

- $G$ a Locally Compact Abelian Group
- $X$ a function space on $G$

- $S$ a distribution which acts on $X$ as a bounded convolution operator

$$S : X \longrightarrow X, Sf = S * f$$

- $\hat{S} = \mathcal{F}S$ Fourier transform of $S$ (on the dual group $\hat{G}$), $\hat{S} \in L^\infty(\hat{G})$

**More notation:**

Given $0 < \delta \leq 1$, we define

- *the best possible upper bound for inverses* with a given lower bound $\delta$ of $\hat{S}$ on $\hat{G}$:

$$c_1(\delta) = \ sup\{\|S^{-1}\|_{X \longrightarrow X} : \ \delta \leq |\hat{S}(t)| \leq \ \|S\| \leq \ 1 \ (\forall t \in \hat{G})\},$$

- a *"critical constant"*

$$\delta_1 = \ inf\{\delta : \ 0 < \delta \leq 1, \ c_1(\delta) < \ \infty\}.$$

**Characteristic property of $\delta_1$:**

- for $\delta_1 < \delta \leq 1$, there is an estimate for $\delta$-flat $S$: $\|S^{-1}\| \leq \ c_1(\delta) < \ \infty$,
- for $0 < \delta < \delta_1$, there is NO estimate of $\|S^{-1}\|$: $c_1(\delta) = \ \infty$.

*The problem is* to find/bound $\delta_1$ and $c_1(\delta)$ for $\delta_1 < \delta \leq 1$.

## Classical $L^p$ Fourier multipliers

- **M.Riesz:** $Mult(L^1(G)) = \mathcal{M}(G) =$ *all complex measures on G*
- **N.Wiener and H.Pitt, 1938:** $\delta_1(\mathcal{M}(\mathbb{R})) > 0$, and moreover, there exists an "*invisible spectrum*":

$$\exists \mu \in \mathcal{M}(\mathbb{R}),\ inf_{t\in\mathbb{R}}|\hat{\mu}(t)| > 0 \text{ but } \mu \text{ is NOT invertible } (\mu * \nu \neq \delta_0,\ \forall \nu)$$

- **E.Hewitt/W.Rudin, 1958:** the same for *every nondiscrete LCAG G*
- **S.Igari/M.Zafran, 1976:** the same for all $Mult(L^p(G))$, $p \neq 2$
- **J.Stafney, 1967:** $\delta_1(l^1(\mathbb{Z})) > 0$ (*NO Wiener-Pitt phenomenon!*)

**Y.Katznelson-H.Shapiro conjecture, 1975:** $\delta_1(l^1(\mathbb{Z})) = 1/2$ (or, $\delta_1(l^1(\mathbb{Z}_+)) = 1/2$?...)

**The case of a discrete group:**

$\mathfrak{M}(\mathcal{M}(\mathbb{Z})) = \hat{\mathbb{Z}} = \mathbb{T}$, $\mathfrak{M}(\mathcal{M}(G)) = \hat{G}$ (NO Wiener-Pitt phenomenon: $f \in \mathcal{FM}(\mathbb{Z})$, $f(\zeta) \neq 0 \Rightarrow 1/f \in \mathcal{FM}(\mathbb{Z})$)

$\mathcal{FM}(\mathbb{Z}_+) = \{f = \Sigma_{k\geq 0}\hat{f}(k)z^k : \Sigma_{k\geq 0}|\hat{f}(k)| < \infty\}$ the *holomorphic Wiener algebra*; $\mathfrak{M}(W_+) = \bar{\mathbb{D}}$ the closed unit disc.

- **(N.N., 1995) (1)** $\delta_1(\mathcal{M}(\mathbb{Z}_+)) = 1/2$ and $c_1(\delta) = \frac{1}{2\delta-1}$ for $1/2 < \delta \leq 1$.

**(2)** $1/2 \leq \delta_1(\mathcal{M}(\mathbb{Z})) \leq 1/\sqrt{2}$ and $c_1(\delta) \leq (2\delta^2 - 1)^{-1}$ for $1/\sqrt{2} < \delta \leq 1$.

- **(O.El-Fallah, N.N., M.Zarrabi, 1995)** Let $G$ be an *infinite* LCAG.

**(1)** $1/2 \leq \delta_1(\mathcal{M}(G)) \leq 1/\sqrt{2}$ and $c_1(\delta) \leq (2\delta^2 - 1)^{-1}$ for $1/\sqrt{2} < \delta \leq 1$.

**(2)** If $G_+ \subset G$ is a "true" sub-semigroup of $G$ and $\hat{G}_+$ its dual semigroup (of bounded semicharacters), then $\delta_1(\mathcal{M}(G_+)) = 1/2$.

**(3)** If $w = (w_n)_n$ is "regularly varying" weight on $\mathbb{Z}$ such that $w_n \longrightarrow \infty$ (as $n \longrightarrow \infty$) such that the weighted Beurling-Sobolev space

$$A =: l^p(\mathbb{Z}, w_n) = \{x = (x_k) : (\sum_n |x_n w_n|^p)^{1/p} < \infty\}$$

is a convolution algebra, then

$$\delta_1(A) = 0, \textit{ and hence } c_1(\delta) < \infty \textit{ for every } 0 < \delta \leq 1.$$

**Multipliers (convolutions) on weighted $L^p$ spaces:**

$$L^p(\mathbb{T}, w) = \{f : \int_{\mathbb{T}} |f|^p w dm < \infty\};\ w \geq 0,\ w \in L^1(\mathbb{T})$$

$$Mult(L^p(\mathbb{T}, w)) =$$

$$= \{(\lambda_j)_{j\in\mathbb{Z}} : \textit{the map } T_\Lambda : \sum c_j e^{ijx} \longmapsto \sum \lambda_j c_j e^{ijx} \textit{ bdd on } L^p(w)\}.$$

**The "visible spectrum":** eigenvalues $T_\Lambda(e^{ijx}) = \lambda_j e^{ijx}$, $j \in \mathbb{Z}$.
**Problem:** estimate $\|T_\Lambda^{-1}\|$ in terms of $\delta_\Lambda = inf_j |\lambda_j| > 0$.

**Observation:** if the eigenfunctions $(e^{ijx})_{j\in\mathbb{Z}}$ form an *unconditional basis* ($\Leftrightarrow$ $p = 2$ AND $w$ is bounded from above and from below), then $Mult(L^p(\mathbb{T}, w)) = l^\infty(\mathbb{Z})$ and $\|T_\Lambda^{-1}\| \leq \frac{const}{\delta}$.

**- (N.N., 2012):** For every $p$, $1 < p < \infty$, and every $\epsilon > 0$, there exists a Muckenhoupt weight $w \in (A_p)$ ($\Leftrightarrow$ $(e^{ijx})_{j\in\mathbb{Z}}$ is *a Schauder basis* in $L^p(w)$) and a $\Lambda = (\lambda_j)_{j\in\mathbb{Z}} \in Mult(L^p(w))$ such that $1 - \epsilon < \delta_\Lambda$, $\|T_\Lambda\| \leq 1$ and $1/\Lambda \notin Mult(L^p(w))$; in particular,

$$\delta_1(Mult(L^p(w))) \geq 1 - \epsilon.$$

## Weights with norm control of inverses

**- (I.Verbitsky, N.N., 2015):** Let $w(\zeta) = w_1(\zeta\overline{\alpha}_1)...w_n(\zeta\overline{\alpha}_n), \zeta \in \mathbb{T}$, where $|\alpha_j| = 1$ and $\zeta \longmapsto w_j(\zeta) > 0$ (bounded) have the only singular point at $\zeta = 1$ vanishing "regularly" ("power-like") when $\zeta \longrightarrow 1$. Then,

**(1)** *If* $1/w \notin L^1(\mathbb{T})$, *then* $Mult(L^2(w))$ *is finite dimensional.*

**(2)** *If* $1/w \in L^1(\mathbb{T})$, *then multipliers* $(\lambda_j) \in Mult(L^2(w))$ *can be completely described in terms of capacitary inequalities for differences* $\lambda_j\alpha_k^j - \lambda_{j+1}\alpha_l^{j+1}$ *(*$1 \leq k, l \leq n$*); always,*

$$\delta_1 = 0 \text{ and } c_1(\delta) \leq c/\delta^2, 0 < \delta \leq 1.$$

**Comments: 1)** $w_j(e^{it}) \approx |t|^{\gamma_j}$, $0 < \gamma_j < 2$ are OK.
**2)** Complete description of admissible weights: $w_j(e^{it}) = \sum_{k\geq 1} c_k Sin^2 \frac{kt}{2}$, $0 < \sum c_k < \infty$ (Lévy-Khinchin-Schoenberg *(LKS)* weights).
**3)** No inclusions between *(LKS)* and $(A_2)$.

# II. Large Matrices

- A is an $n \times n$ matrix
- $\|A\| \leq 1 \Rightarrow \|A^{-1}\| \leq 1/|\det(A)| \leq 1/\delta^n$ where

  $\delta = \min|\sigma(A)|$
- For a Banach normed $\mathbb{C}^n, |\cdot|$ it is $\sqrt{n}$ times worth: $\|A^{-1}\| \leq \sqrt{en}/|\det(A)|$ - J.Schäffer (1970); sharpness – E.Gluskin, M.Meyer, A.Pajor; J.Bourgain; H.Queffelec (1993)
- My subject below: Matrices commuting with a given A (or, just functions of A)

# Condition numbers of matrices commuting with a given matrix

• Given a subset $\sigma \subset \mathbb{D}$, consider the set $\mathcal{A}_\sigma$ of matrices $A$ having $\|A\| \leq 1$ and $\sigma(A) \subset \sigma$, as well as all matrices commuting with such an $A$.

• For $\mathcal{A}_\sigma$ and $\delta$ $(0 < \delta < 1)$, define $c_1(\delta) = sup\{\|T^{-1}\| : T \in \mathcal{A}_\sigma, |\lambda_j(T)| \geq \delta\}$, and the critical constant $\delta_1$ as above.

**- (P.Gorkin, R.Mortini, N.N. - 2008): (1)** If $\delta_1 < 1$ then $\sigma$ is a sequence $\sigma = (\lambda_j)$ satisfying the Blaschke condition.

**(2)** $\delta_1 = 0 \Leftrightarrow \sigma = (\lambda_j)$ satisfies the following Weak Embedding Property (WEP): $\forall \epsilon > 0\ \exists C(\epsilon) > 0\ s.t.$

$$\sum_j \frac{1 - |\lambda_j|^2}{|1 - \bar{\lambda}_j z|^2} \leq \frac{C(\epsilon)}{1 - |z|^2} \text{ for } z \in \mathbb{D} \setminus \cup_j \{|b_{\lambda_j}(z)| < \epsilon\}.$$

**Comments: (1)** The Carleson Embedding Property (CEP) for $\mu = \Sigma_j(1-|\lambda_j|^2)\delta_{\lambda_j}$ is equivalent to $sup_{\epsilon>0}C(\epsilon) < \infty$.
**(2)** (WEP) does not imply (CEP) but this is the case for Stolz angular sequences (with a participation of S.Treil and V.Vasyunin).
**(3)** $c_1(\delta)$ and $\delta_1$ for $\mathcal{A}_\sigma$ are the same as for the trace algebra $H^\infty|\sigma$.
**(4)** $(WEP)$ is also equivalent to a "corona theorem" for $H^\infty|\sigma$ ($(WEP) \Leftrightarrow \sigma$ is dense in $\mathfrak{M}(H^\infty|\sigma)$).

**- (V.Vasyunin, N.N. - 2011):** There exist $\sigma$'s with a given in advance value $\delta_1 = \Delta$, $0 \leq \Delta < 1$.

**- (A.Borichev - 2014):** There exist $\sigma$'s with $\delta_1 = 0$ ($\sigma \in (WEP)$) and an arbitrarily fast growth of $c_1(\delta) \uparrow \infty$ as $\delta \downarrow 0$.

# III. Algebras Generated by Integration Operators

• Let $\mu$ be Borel probability measure on $[0,1]$ and $J_\mu$ *an integration operator*

$$J_\mu f(x) = \int_{[0,x>} f d\mu,\ 0 \le x \le 1,$$

on the spaces $L^p([0,1],\mu)$.
• $[0, x >$ can be $[0,x)$ or $[0,x]$, or - which is better for a symmetry reason between $J_\mu$ and $J_\mu^*$ - an arithmetic mean of these two:

$$J_\mu f(x) = \int_{[0,x>} f d\mu = \int_{[0,x)} f d\mu + \frac{1}{2}\mu(\{x\})f(x),\ x \in [0,1].$$

• Of course, if $\mu$ is continuous (in general, $\mu = \mu_c + \mu_d$, $\mu_d = \Sigma_{y\in[0,1]}\, \mu(\{y\})\delta_y$ is a discrete component of $\mu$ and $\mu_c$ continuous), then

$$J_\mu f(x) = \int_0^x f d\mu.$$

The case $d\mu = dx$ corresponds to the standard Volterra operator.

• **The subject of Part III of the talk** - the algebras

$$A_{\mu,p} = \ alg_{L^p(\mu)}(J_\mu)$$

generated by $J_\mu : L^p(\mu) \longrightarrow L^p(\mu)$, $1 \leq p \leq \infty$ (the norm closure of polynomials in $J_\mu$, $J_\mu^0 =: \ id$).

• **The same questions as above:** to find (or to estimate) $\delta_1$ and $c_1(\delta)$ for $\delta_1 < \ \delta \leq \ 1$.

And so to decide whether there exist an "invisible" but numerically detectable spectrum.

• I know **the answers** for two following cases only:

- $p = \ 1$ or $\infty$ AND $\mu = \ \mu_c$ (continuous measure),
- $p = \ 2$, $\mu$ arbitrary.

• **One of the reasons** why $p \neq \ 1, 2, \infty$ are more complicated: $J_\mu$ is a kind of convolution operator (the compression to $[0, 1]$ of a convolution on $[0, \infty)$) and on $L^p$ spaces with $p \neq 1, 2, \infty$ very little is known on the spectrum of convolutions.

• **The visible spectrum** of an element $T \in A_{\mu,p} = alg_{L^p(\mu)}(J_\mu)$ is defined, of course, as the whole spectrum $\sigma(T)$: $\sigma(J_\mu : L^p(\mu) \longrightarrow L^p(\mu))$ does not depend on $p$ and consists of $\{0\}$ and the eigenvalues $\frac{1}{2}\mu(\{y\})$, $y \in [0,1]$; next, for $T = f(J_\mu)$ - by the spectral mapping theorem.

• **The case** $p = 1, \ \mu = \mu_c$**:**

$$\delta_1(A) = 1/2, \ c_1(A, \delta) = \frac{1}{2\delta - 1} \ \textbf{for} \ 1/2 < \delta \leq 1.$$

• **The case** $p = 2$**:** *The following alternative holds.*
*(1) Either,* $\mu_c = 0$ and $\sigma(J_\mu)$ *is a (finite) union of* $N$ *geometrically decreasing sequences, and then*

$$\delta_1(A_{\mu,2}) = 0 \text{ and } c_1(\delta) \leq a \frac{log \frac{1}{\delta}}{\delta^{2N}}, \ 0 < \delta < 1,$$

*where* $a > 0$ *depends on* $N$ *and ratios of geometric sequences in* $\sigma(J_\mu)$.

*(2) Or, this is not the case, and then* $\delta_1(A_{\mu,2}) = 1$ *(so that,* $c_1(\delta, A) = \infty$ *for every* $0 < \delta < 1$*).*

• **The proof** depends on Sarason-Sz.-Nagy-Foias model theory and the analysis of the algebra $H^\infty/\theta H^\infty$ from Part II of the talk.

# Summary

- The nature of «invisible spectrum» is different (characters measurable wrt «thin σ-algebras», or forced holomorphic extensions, or complex homomorphisms in fibers over the boundary).
- The «invisible» but numerically detectable spectrum comes from a «true invisible spectrum» and its discontinuity wrt to a weak approximation.

The End

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# Thank you!

And Happy Birthday

to N.G.M.!!