Determinacy problem for measures

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March 10, 2016

Happy Birthday Nick

My favorite paper:
N. Makarov, A. Poltoratski. "*Meromorphic inner functions, Toeplitz kernels and the uncertainty principle.*" Perspectives in analysis. Essays in Honor of Lennart Carleson's 75th Birthday, 2005

Nick's favorite things: Model spaces, de Branges spaces, Toeplitz kernels.

Moment problem

Given a sequence $\{c_n\}$ of real numbers.

(Hamburger) moment problem:

1. (Existence) Is there a measure μ whose moments are $\{c_n\}$?

$$c_n = \int_{-\infty}^{\infty} t^n d\mu(t), n \geq 0.$$

2. (Uniqueness, determinacy) If such a measure exists, is it unique?

Moment problem

Related problems:

Let $\mathcal{P}$ be the class of all polynomials.

Given a weight W s.t. $\mathcal{P} \subseteq C_0(W)$. Find a (necessary and sufficient) condition for $\mathcal{P}$ to be dense in $C_0(W)$.

Given a measure μ with $\mathcal{P} \subseteq L^p(d\mu)$. Find a (necessary and sufficient) condition for $\mathcal{P}$ to be dense in $L^p(d\mu)$.

Moment problem

Given a measure μ, consider the set Σ_μ of all ν s.t. $\int p d\mu = \int p d\nu$, for all $p \in \mathcal{P}$.

For a fixed $z \in \mathbb{C}_+$, consider the map: $\psi_z : \Sigma_\mu \to \mathbb{C}_+$ defined by

$$\psi_z(\nu) = \int \frac{d\nu(t)}{t - z}.$$

For each $z \in \mathbb{C}_+$ the range of ψ_z is a closed disc $D_z \subset \mathbb{C}_+$ (Weyl disc).

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Thm: μ is determinate iff $\Sigma_\mu^{\mathcal{L}}$ is a singleton set.

Thm: $\text{clos}_{L^1(d\mu)}\mathcal{P} = L^1(\mu)$ iff μ is an extreme point of Σ_μ.

Thm: $\text{clos}_{L^2(d\mu)}\mathcal{P} = L^2(\mu)$ iff for some/any z, $\psi_z(\mu)$ is an extreme point of the disc D_z.

M. Riesz classical result

Def: $m(z) = \sup\{|p(z)| : p \in \mathcal{P}, \|p\|_2 \leq 1\}$.

Thm (M.Riesz): μ is indeterminate iff $m(z)$ is finite for some/all $z \in \mathbb{C} \setminus \mathbb{R}$ iff $\log^+ m \in L^1(dx/(1+x^2))$

The usefulness of the previous result depends on the possibility to estimate the majorant $m(z)$.

General moment problem

Replace $\mathcal{P}$ with a more general class of functions $\mathcal{L}$.

Conditions on $\mathcal{L}$:

1. The elements of $\mathcal{L}$ are entire functions.
2. If $F \in \mathcal{L}$, $w \in \mathbb{C}$ with $F(w) = 0$, then $F(z)/(z - w) \in \mathcal{L}$.
3. If $F \in \mathcal{L}$, then $F^*(z) = \overline{F(\bar{z})} \in \mathcal{L}$.

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$\mathcal{L}$ is said to be regular if 2. is replaced by

2'. If $F \in \mathcal{L}, w \in \mathbb{C}$, then $(F(z) - F(w))/(z - w) \in \mathcal{L}$.

Important special case

$\mathcal{L} = \mathcal{PW}_a$ = Paley-Wiener space.
$\mathcal{E}_a = \{e^{i\lambda x} : \lambda \in [-a, a]\}$

1. For a given μ, when can we say that there is no ν s.t.

$$\hat{\mu}(x) = \hat{\nu}(x), x \in [-a, a]?$$

In such case we say that μ is *a*-determinate.

2. Given a weight W s.t. $\mathcal{E}_a \subseteq C_0(W)$. Find a necessary and sufficient condition for $\mathcal{E}_a$ to be dense in $C_0(W)$.

3. Given a measure μ with $\mathcal{E}_a \subseteq L^p(d\mu)$. Find a necessary and sufficient condition for $\mathcal{E}_a$ to be dense in $L^p(d\mu)$.

Important special case

Try to solve these in the following form:

1. For a given μ, compute

$$Det(\mu) := \inf\{a \geq 0 : \mu \text{ is } a - \text{determinate}\}.$$

2. Given a weight W s.t. $\mathcal{E}_a \subseteq C_0(W)$. Compute

$$T^\infty(W) := \inf\{a \geq 0 : \mathcal{E}_a \text{ is dense in } C_0(W)\}.$$

3. Given a measure μ with $\mathcal{E}_a \subseteq L^p(d\mu)$. Compute

$$T^p(\mu) := \inf\{a \geq 0 : \mathcal{E}_a \text{ is dense in } L^p(d\mu)\}.$$

Previous results

If μ is *a*-indeterminate then there is ν s.t. signed measure $\sigma = \mu - \nu$ has a *spectral gap* $(-a, a)$, i.e., $\hat{\sigma}(x) = 0, x \in (-a, a)$.

Thm (A. Beurling, 50's): If σ is a non-zero, signed, finite measure, which is supported on a set whose complement is long, then σ has no spectral gap.

A union of disjoint open intervals $\bigcup (a_n, b_n)$ is said to be *long* if

$$\sum_n \left(\frac{b_n - a_n}{a_n} \right)^2 = \infty.$$

For a non-zero, signed measure to have a spectral gap its support must contain a sequence of positive interior Beurling-Malliavin density.

Previous results

Thm (N. Levinson, 40's): Let σ be a non-zero, signed, finite measure. If there exists a non negative function $w(t)$ on $\mathbb{R}$ which is increasing on $[0,\infty)$ and satisfies $e^{w(t)} \in L^1(|\sigma|)$ and $\int_1^\infty \frac{w(t)}{1+t^2}dt = \infty$, then σ has no spectral gap.

Thm (L. de Branges, 60's): Let σ be a non-zero, signed, finite measure. If there exists a non negative uniformly continuous function $w(t)$ on $\mathbb{R}$ satisfying $e^{w(t)} \in L^1(|\sigma|)$ and $\int \frac{w(t)}{1+t^2}dt = \infty$, then σ has no spectral gap.

Thm (A. Eremenko, D. Novikov): If σ is a nonzero, signed, finite measure with a spectral gap $(-a, a)$, then the number of sign changes $s(r,\sigma)$ of σ on the interval $(0, r)$ satisfies

$$\liminf_{r\to\infty} \frac{s(r,\sigma)}{r} \geq \frac{a}{\pi}.$$

Analog of M. Riesz classical result

Def: $m_{\mathcal{L}}(z) = \sup\{|f(z)| : f \in \mathcal{L}, \|f\|_2 \leq 1\}$.

Thm (M.M, A. Poltoratski): μ is $(\mathcal{L} \odot \mathcal{L})$-indeterminate iff $m_{\mathcal{L}}(z)$ is finite for some/all $z \in \mathbb{C} \setminus \mathbb{R}$ iff $\text{clos}_{L^2(d\mu)}\mathcal{L}$ is a de Branges space.

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In the regular case we have one more equivalent condition
iff $\log^+ m_{\mathcal{L}} \in L^1(dx/(1+x^2))$

Corollaries.

The usefulness of the previous result depends on the possibility to estimate the majorant $m_{\mathcal{L}}(z)$.

Cor: If $(\text{supp}\mu)^c$ contains a sequence of integrals $\{I_n\}$ s.t. $\sum_n \frac{|I_n|^2}{1+\text{dist}(0,I_n)^2} = \infty$, then μ is a-determinate for all $a > 0$.

Cor: If there exists a non-negative, uniformly continuous w s.t. $e^w \in L^1(\mu)$ and $\int w(x)/(1+x^2) = \infty$, then μ is a-determinate for all $a > 0$.

Cor: If there exists a non-negative w, increasing on $[1, \infty)$ s.t. $e^w \in L^1(\mu)$ and $\int_1^\infty w(x)/(1+x^2) = \infty$, then μ is a-determinate for all $a > 0$.

Better result

Thm(M.M, A. Poltoratski) Let μ be $\mathcal{L} \odot \mathcal{L}$-indeterminate and let ϕ be the phase function corresponding to the de Branges space $\text{clos}_{L^2(d\mu)}\mathcal{L}$. Then there exists $\Lambda \subset \text{supp}(\mu)$ such that $\pi n_\Lambda - \phi = \tilde{h}$ for some Poisson summable h.

In the case $\mathcal{L} = \mathcal{PW}_a, \phi(x) = ax$ one can characterize (up to ϵ) all sequences Λ such that $\pi n_\Lambda(x) - ax = \tilde{h}(x)$ for some Poisson summable h.

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Def (Poltoratski) A real sequence $\Lambda = \{\lambda_n\}$ is said to be *a*-uniform if
1. (Density condition) $\int \frac{|n_\Lambda(x) - ax|}{1+x^2} < \infty$.
2. (Energy condition) There exists a short partition $\{I_n\}$ such that $\sum_n \frac{\#(\Lambda \cap I_n)^2 \log^+ |I_n| - E_{I_n}(dn_\Lambda)}{1+\text{dist}(0,I_n)^2} < \infty$.

Corollaries

Cor: If μ is a-indeterminate, then the support of μ contains a sequence which is a/π-uniform. In other words

$$Det(\mu) = \sup\{a \geq 0 : \text{supp}\mu \text{ contains } a/\pi\text{-uniform sequence}\}.$$

Cor: $2Det(\mu) = G(\text{supp}\mu)$.

Thm: Let A and B be disjoint closed subsets of $\mathbb{R}$. Then

$$G(A, B) = \pi \sup\{d \geq 0 : \exists\{\lambda_n\}\ d\text{-uniform}, \{\lambda_{2n}\} \subset A, \{\lambda_{2n+1}\} \subset B\}.$$

Thm: (A. Poltoratski) If $1 \leq p < \infty$, then $G^p(\mu)$ is the supremum of all $a \geq 0$ such that for every weight W satisfying $\int W d\mu < \infty$ there exists an a-uniform $\Lambda = \{\lambda_n\}$ such that $\sum \frac{\log W(\lambda_n)}{1+\lambda_n^2} < \infty$.

Another application

Thm: (A. Eremenko, D. Novikov) If σ is a nonzero signed measure with spectral gap $(-a, a)$ then the number of sign changes $s(r, \sigma)$ of σ on the interval $(0, r)$ satisfies

$$\liminf_{r\to\infty} \frac{s(r,\sigma)}{r} \geq \frac{a}{\pi}.$$

Thm: (M.M., A. Poltoratski) If σ is a nonzero signed measure with spectral gap $(-a, a)$ then there exists an a/π-uniform sequence $\{\lambda_n\}$ such that σ has at least one sign change in every $(\lambda_n, \lambda_{n+1})$.

Thank you.