Valence of Harmonic Polynomials and Topology of Quadrature Domains,

Everything is Complex
Saas-Fee, March 2016

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Harmonic polynomials

$$h(z) = p(z) + \overline{q(z)}, \qquad z \in \mathbb{C}.$$

What is the maximal valence of $h : \mathbb{C} \to \mathbb{C}$ given $(\deg p, \deg q) = (n, m)$?

Examples:

$$h(z) = p_4(z) + \overline{z} \;:\; 10 \text{ zeros.}$$

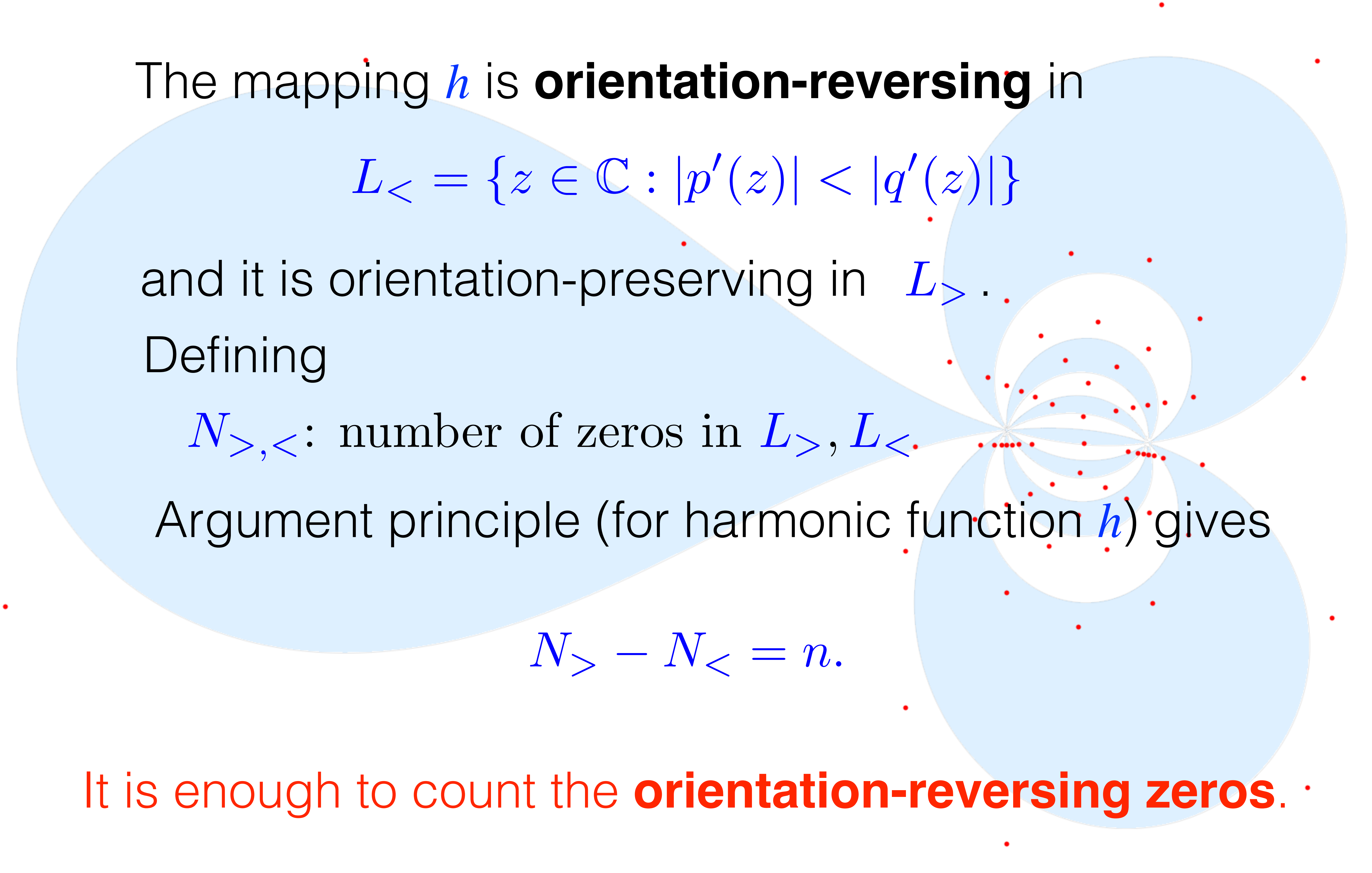
$$h(z) = p_4(z) + \overline{z}^2 \;:\; 12 \text{ zeros?}$$

Known Bounds

$(\deg p, \deg q)$	maximal valence
(n, m)	$\geq m^2 + m + n$
$(n, n-1)$	n^2
$(n, n-2)$	$\geq n^2 - (1.47052)n + \mathcal{O}(1)$
$(n, n-3)$	$\geq n^2 - 3n + \mathcal{O}(1)$
$(n, 1)$	$3n - 2$

(Wilmshurst's conjecture: m(m-1)+3n-2)

(Khavinson, Swiatek, Geyer, Lundberg, Lerario, Lee, Saez, …)



The mapping h is **orientation-reversing** in

$$L_< = \{z \in \mathbb{C} : |p'(z)| < |q'(z)|\}$$

and it is orientation-preserving in $L_>$.

Defining

$N_{>,<}$: number of zeros in $L_>, L_<$

Argument principle (for harmonic function h) gives

$$N_> - N_< = n.$$

It is enough to count the **orientation-reversing zeros**.

Counting local minima

The orientation-reversing zeros are the local minima of the following potential field.

$$Q(z) = |q(z)|^2 + 2\mathrm{Re}\left(\int^z p(w)q'(w)dw\right).$$

$$\Big(\ Q'(z) = \left(p(z) + \overline{q(z)}\right) q'(z).\ \Big)$$

Local droplet (filling in Coulomb gas): support of μ s.t.

$$0 = Q'(z) - \frac{1}{\pi}\int \frac{d\mu(w)}{z-w}, \quad z \in \operatorname{supp}\mu.$$

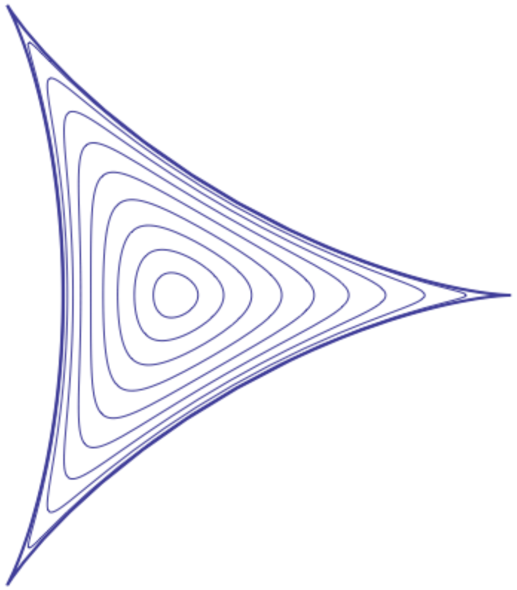
(Generalized) Quadrature Domains

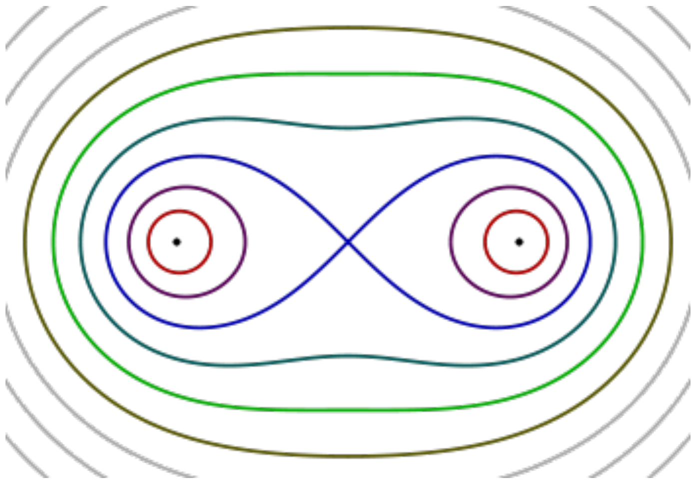
Then $\operatorname{Ext}(\operatorname{supp}\mu)$ is the (union of) “quadrature domains”.

$$\int_\Omega f(z)|q'(z)|^2 dA(z) = \sum c_k f^{(k)}(a_k).$$

Equivalently, given the Schwarz function S of the domain, $q(S(z))$ is a meromorphic function of the domain.

Deltoid



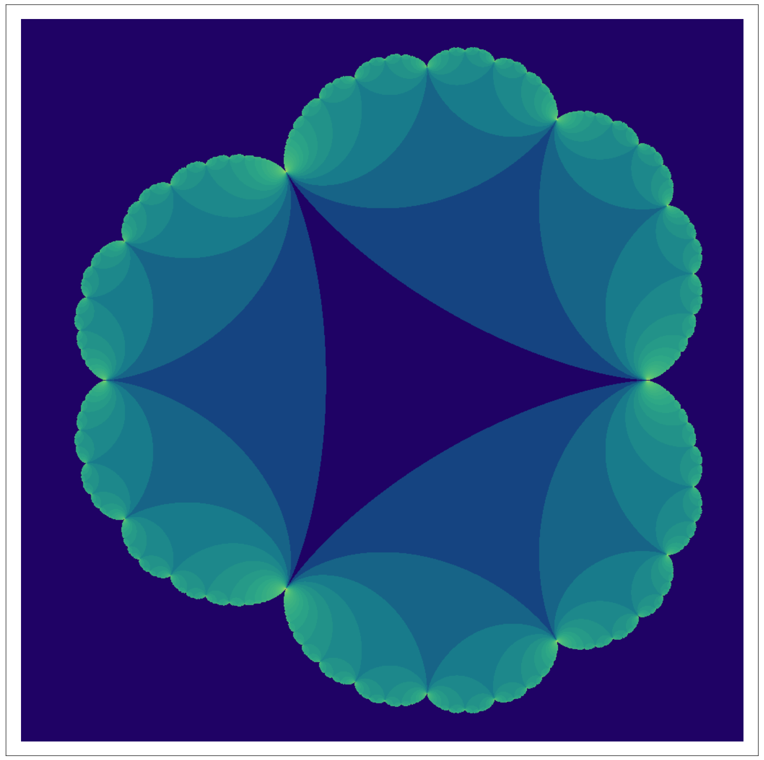


Cassini’s oval

Harmonic polynomial with **k orientation-reversing zeros**
gives
an unbounded quadrature domain with **k holes.**

From QDs to Harmonic Polynomials

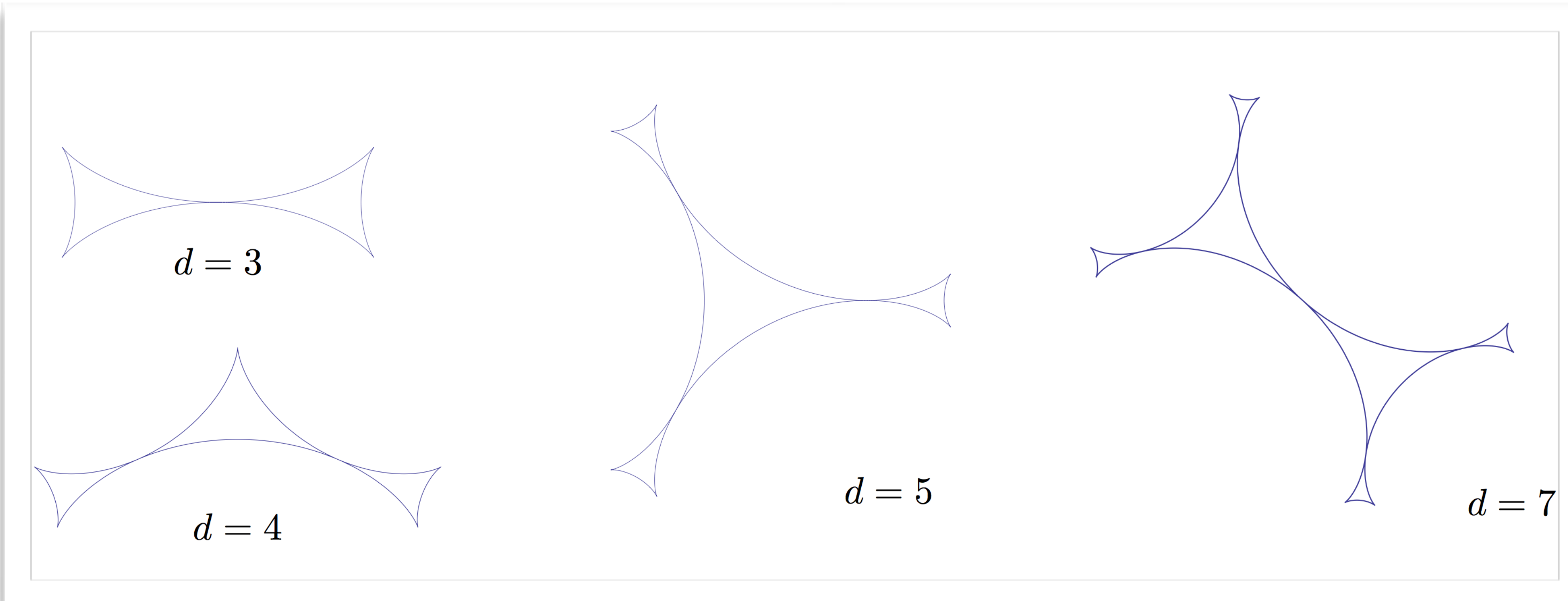
Theorem 1 (L-Makarov) A (meromorphic) Schwarz function of the quadrature domain is quasi-conformally equivalent to a rational function.



For $\deg q = 1$, the existence of certain unbounded QD with **k** holes implies the existence of the harmonic polynomials with orient.-rev. **k** zeros.

Q: Find a QD with maximal number of “holes”.

Suffridge curves

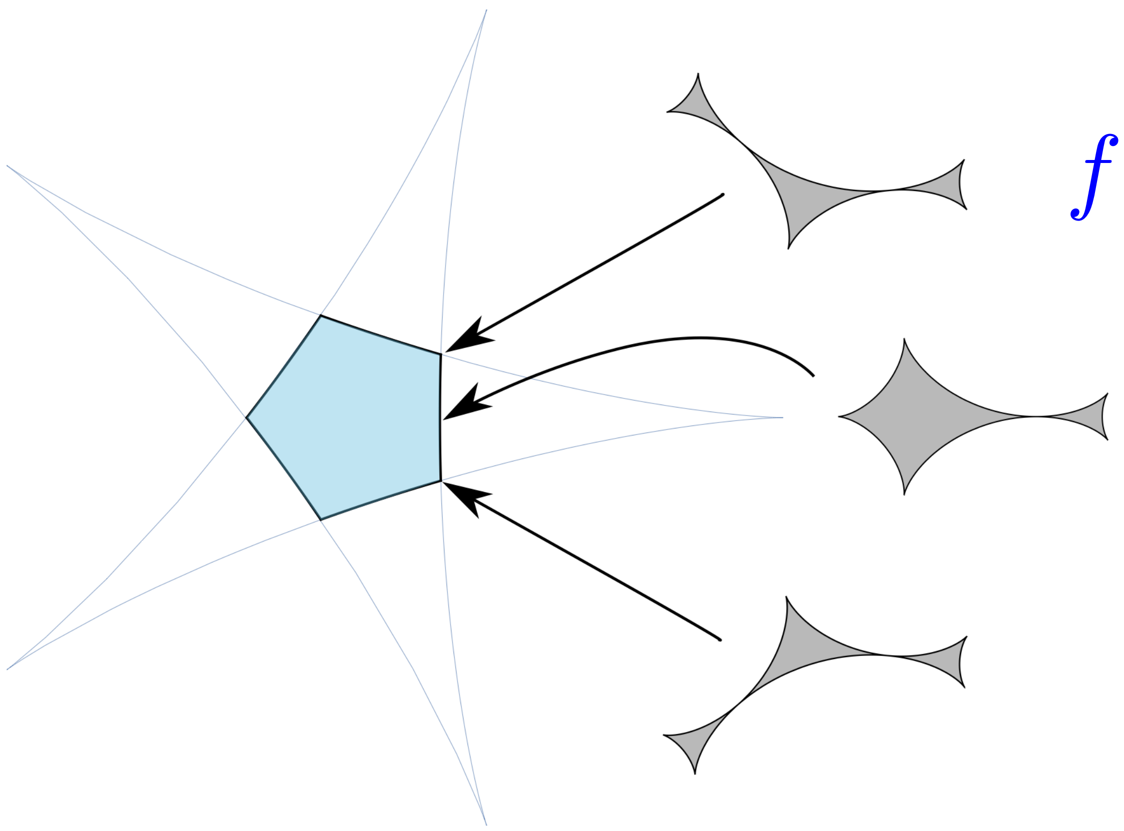


The conformal image of the exterior unit disk under the mapping:

$$f(z) = z + \frac{a_1}{z} + \cdots + \frac{a_{d-1}}{(d-1)z^{d-1}} - \frac{1}{dz^d}.$$

Idea of the proof.

$\mathfrak{S}$: The space of all univalent f.

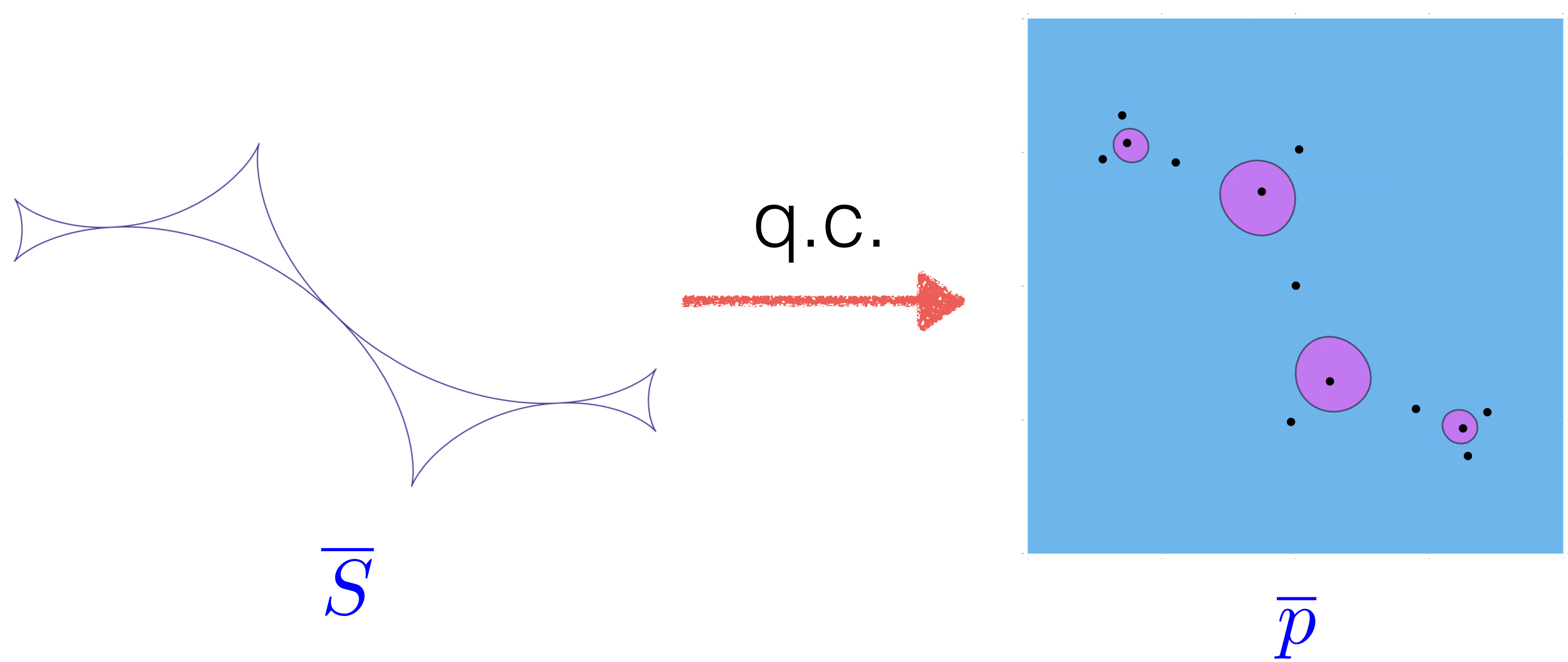


$$f(z) = z + \frac{a+ib}{z} + \frac{a-ib}{2z^2} - \frac{1}{4z^4}.$$

Theorem 2 (L-Makarov).
Extreme points of $\mathfrak{S}$ give Suffridge curves.

Proof of **Theorem 1:** From QDs to Harm. Polynomials.

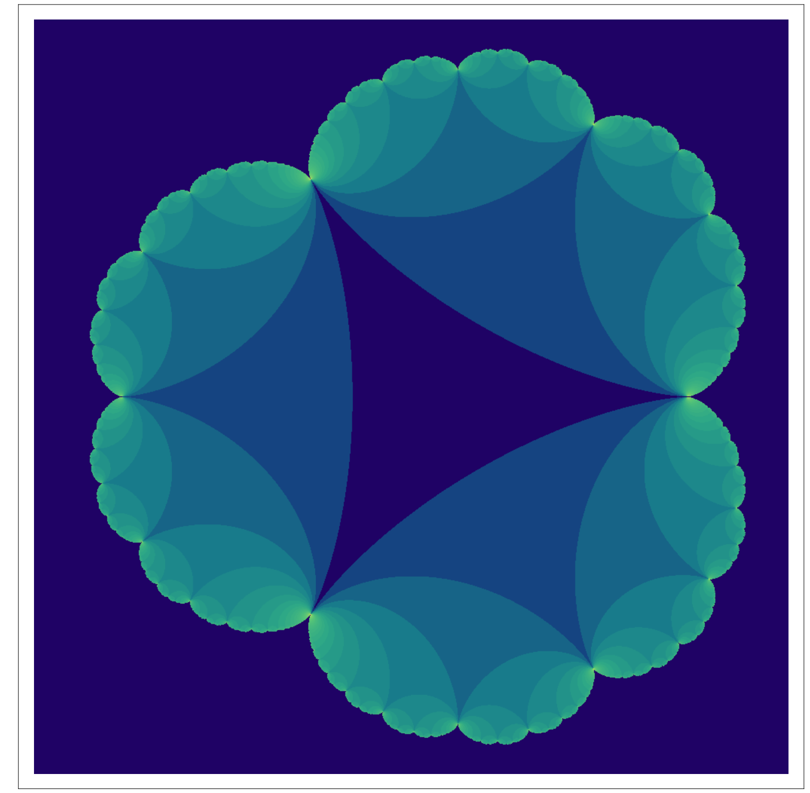
The Schwarz reflection is **anti-analytic in the QD**. (This is our version of “polynomial-like mapping” in the Douady-Hubbard straightening theorem.) We want to find a q.c. homeomorphism such that the **Schwarz refl. is q.c.-conjugated to a (anti-analytic) polynomial mapping**.

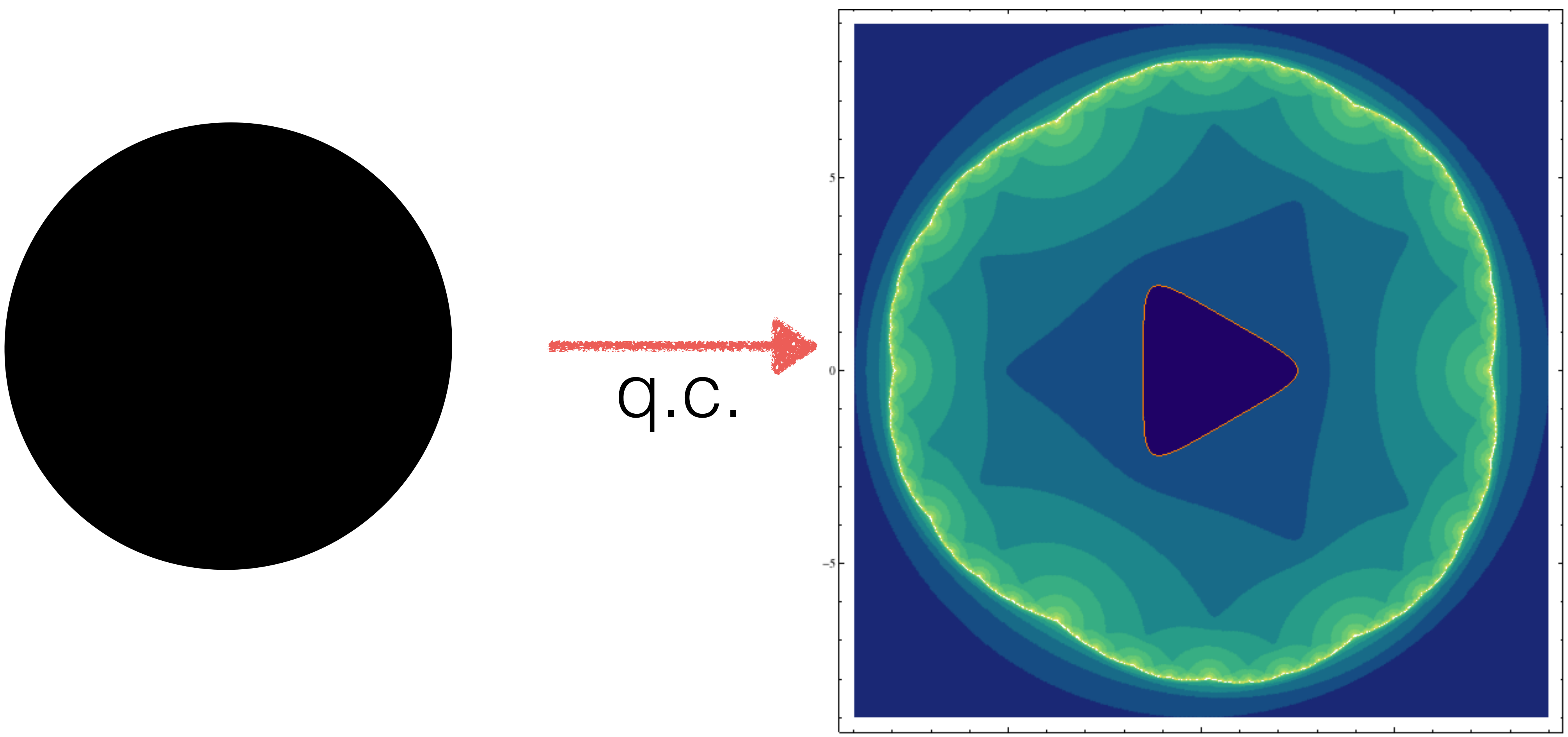


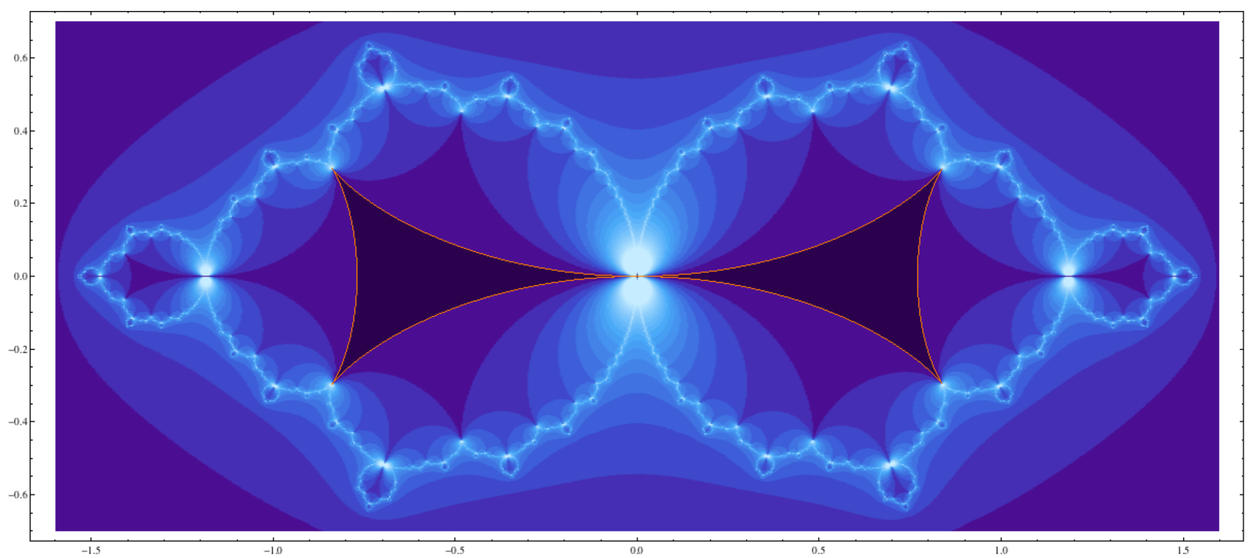
The region of q.c. distortion is

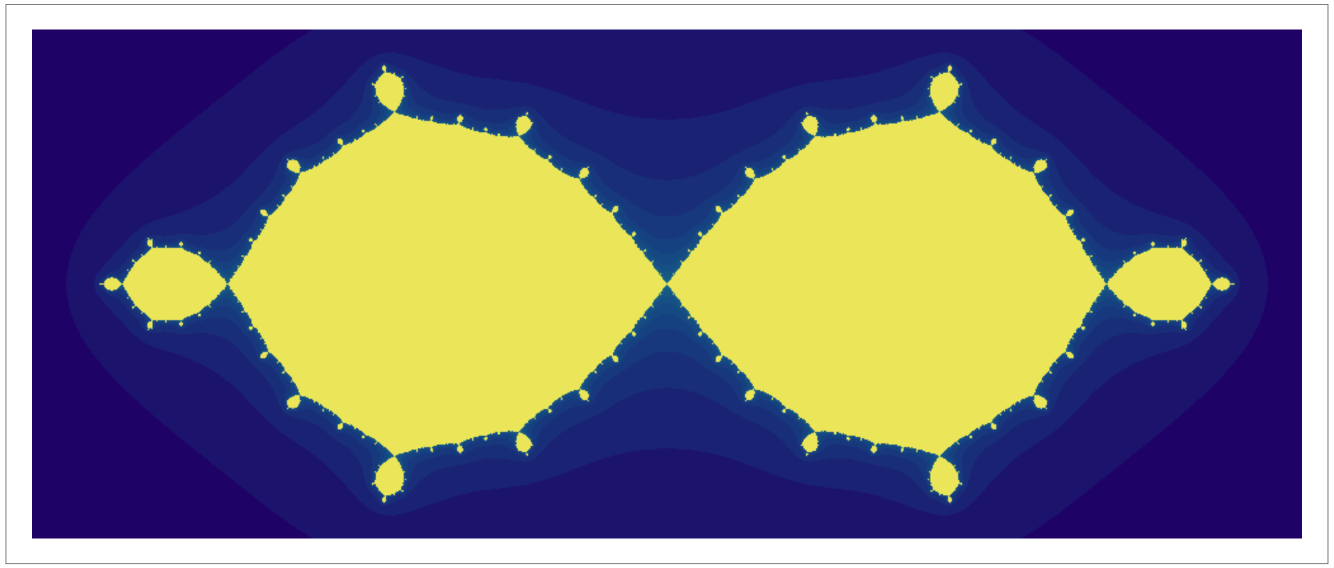
$$\bigcup_{j=1}^{\infty} \overline{S}^{-j}(\text{deltoid})$$

(the filled Julia set of the Schwarz reflection.)









q.c.

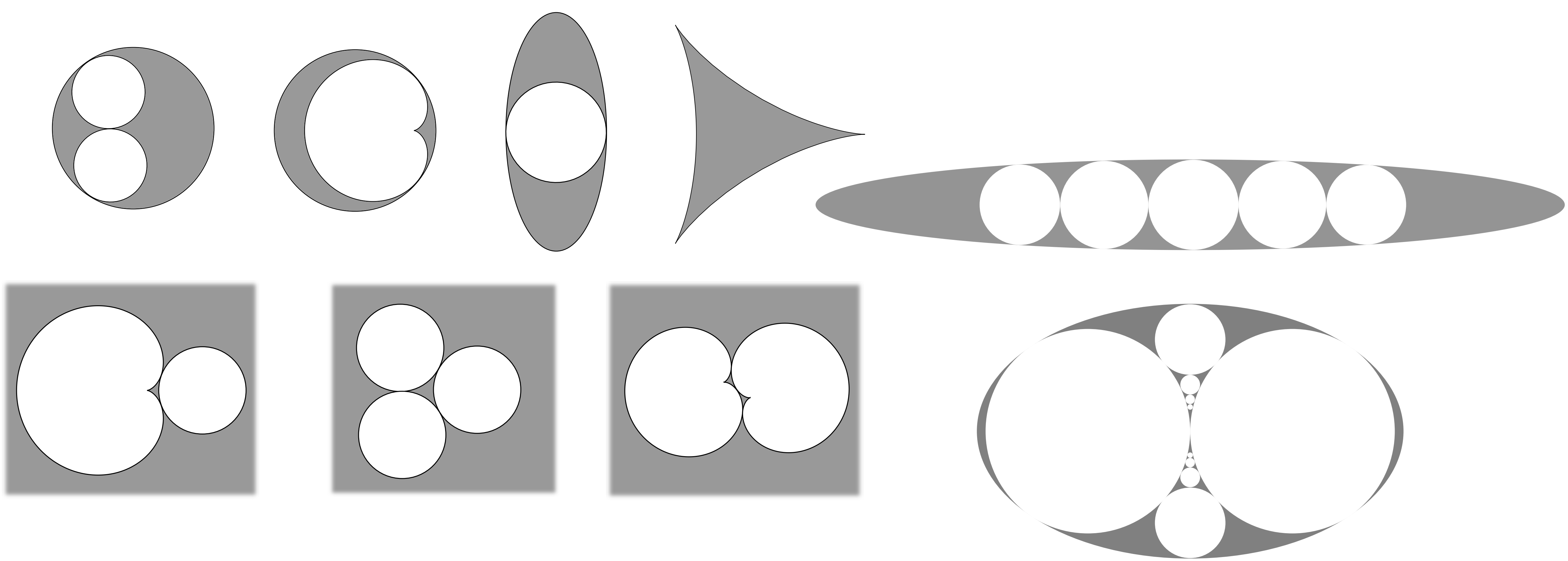
Rational case

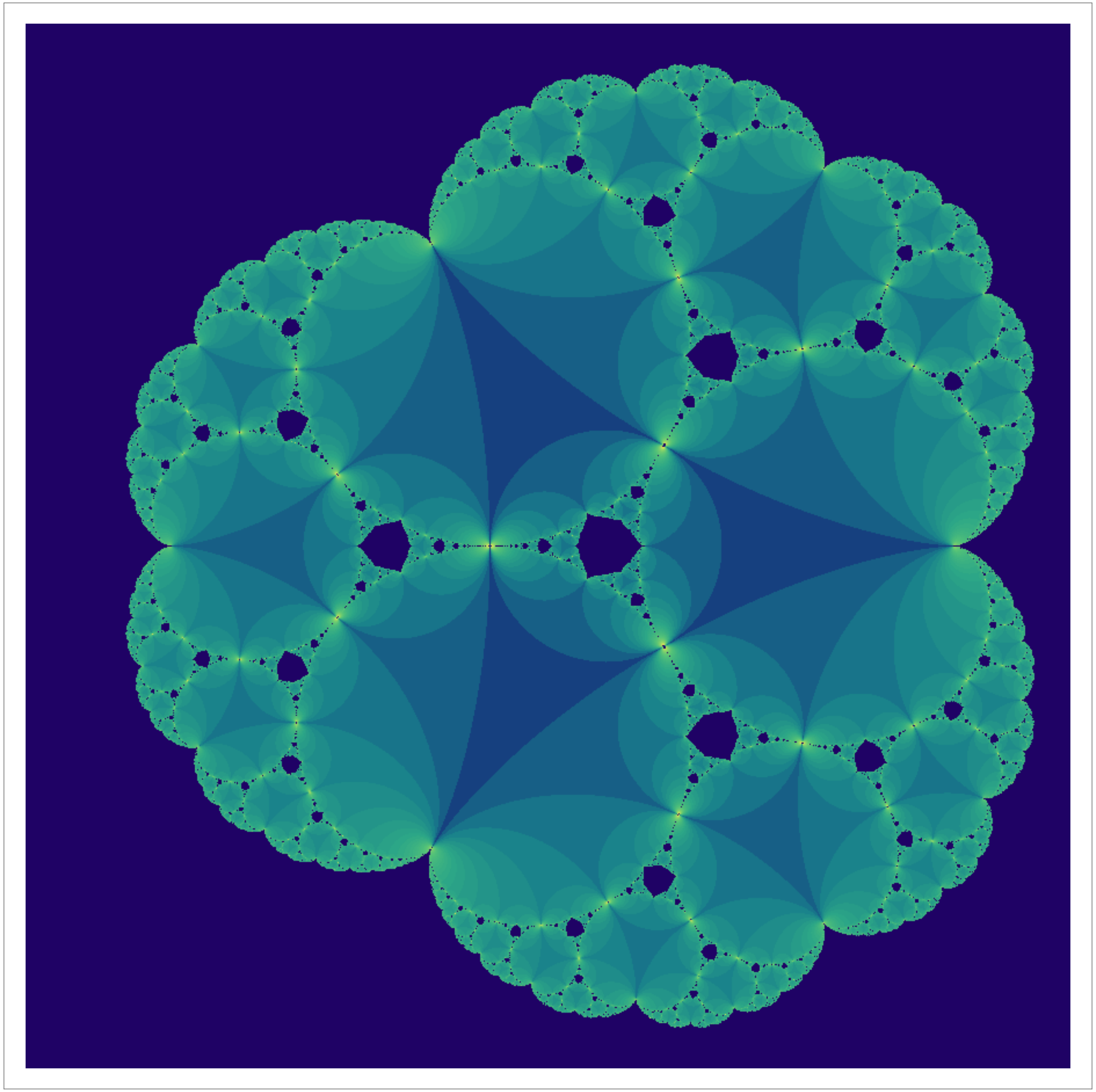
Theorem. There exists a rational function r such that

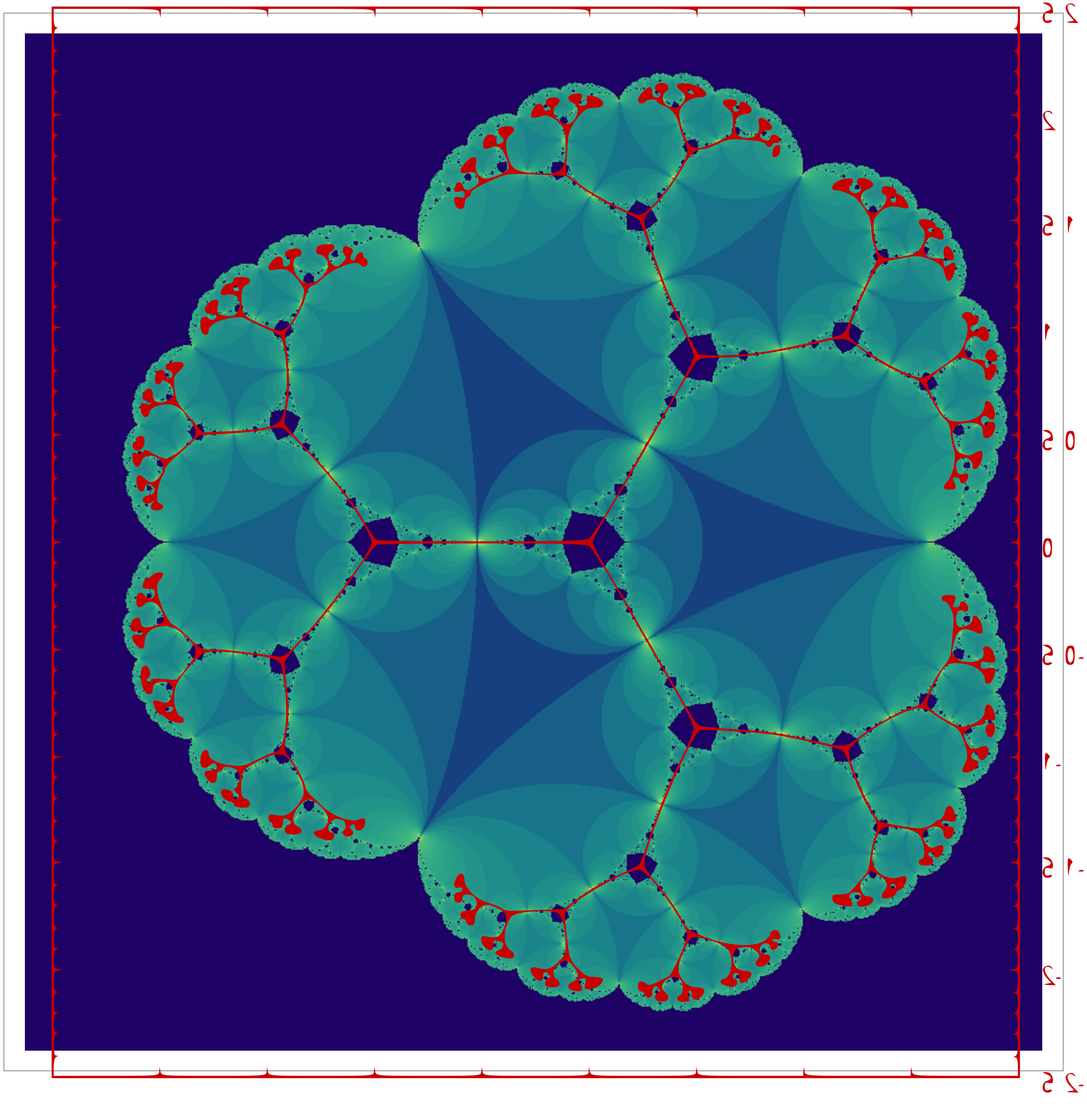
$$r(z) + \overline{z} = 0$$

has the maximal number of roots that is given by

$$\min(d + n - 1, 2d - 2)$$







(Steffen Rhode,
Brent Werness)