Harmonic measure on random trees

Nicolas Curien and Jean-François Le Gall

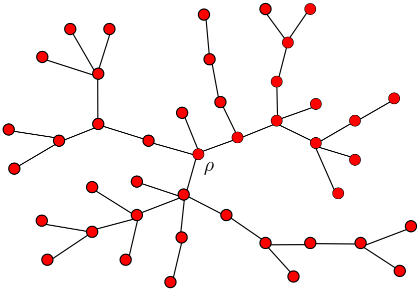
Université Paris-Sud

Saas Fee, March 2016

Makarov's theorem: The harmonic measure of a simply connected plane domain is supported on a set of Hausdorff dimension 1, regardless of the dimension of the boundary.

$\longrightarrow$ "dimension drop" phenomenon

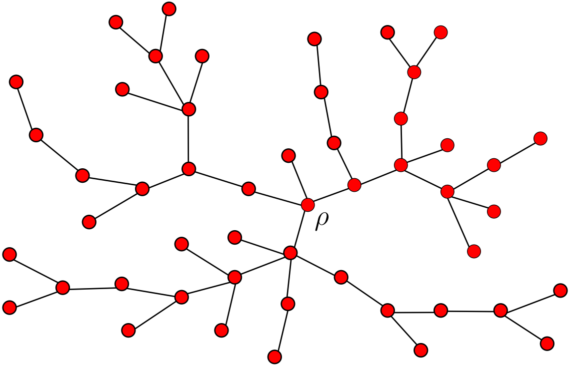
Goal of the lecture:
Similar phenomenon for "typical" **large** discrete trees.



For a (large) ball in a large discrete tree, most of harmonic measure is supported on a small subset of the boundary.

(in the associated continuous model, the dimension of harmonic measure is $\beta \approx 0.78$ whereas the dimension of the boundary is 1)

Consider a large (rooted) discrete tree



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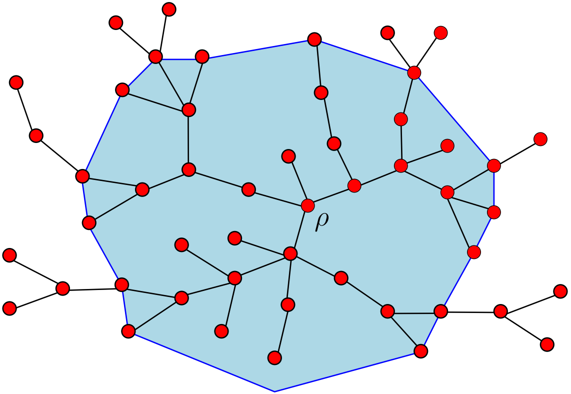

Ball of radius k = {vertices at graph distance $< k$ from ρ} ($k = 4$ here)

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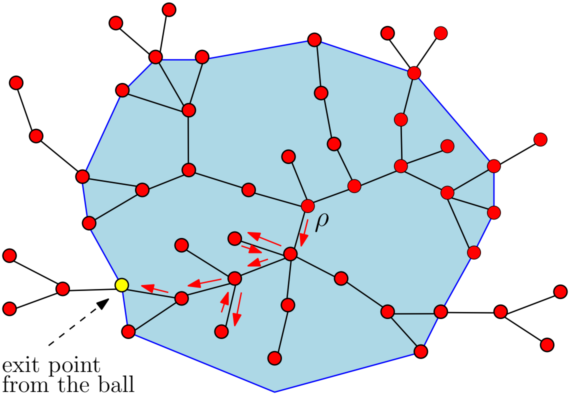

Ball of radius k = {vertices at graph distance $< k$ from ρ} ($k = 4$ here)

Harmonic measure: distribution of exit point from ball by simple random walk starting from ρ

Let $\mathbf{T}(N)$ be chosen uniformly at random in the set of all (rooted) plane trees with N vertices. Set

$$\mathbf{T}_k(N) = \{\text{vertices at distance } k \text{ from root}\}$$

$\mu_k^{(N)}$ harmonic measure of ball of radius k (supported on $\mathbf{T}_k(N)$)

Theorem (A)

There a constant $\beta \approx 0.78$ such that, for every $\varepsilon > 0$ and $\delta > 0$,

$$\lim_{\substack{k,N\to\infty \\ k=o(\sqrt{N})}} \mathbb{P}\Big(\mu_k^{(N)}(\{v : k^{-\beta-\varepsilon} \leq \mu_k^{(N)}(v) \leq k^{-\beta+\varepsilon}\}) > 1-\delta\Big) = 1.$$

Interpretation: A typical point chosen according to harmonic measure has harmonic measure approximately $k^{-\beta}$.

It is known that $\#\mathbf{T}_k(N)$ is of order k, so (since $\beta < 1$) this means that harmonic measure is essentially supported on a small subset of the boundary.

Note. The condition $k = o(\sqrt{N})$ ensures that w.h.p. there are vertices outside the ball of radius k, so that harmonic measure is well defined!

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Consequences.

- For $\delta > 0$, there exists with probab. $\to 1$ a subset A of the boundary s.t.
$$\#A \leq k^{\beta+\varepsilon} \qquad \text{and} \qquad \mu_k^{(N)}(A) > 1 - \delta$$
(recall that the boundary has of order k points)
- Conversely, $$\sup_{A\,:\,\#A \leq k^{\beta-\varepsilon}} \mu_k^{(N)}(A) \xrightarrow[k\to\infty]{} 0$$

Similar result for other types of combinatorial trees (binary trees, Cayley, etc.), with the same constant β.

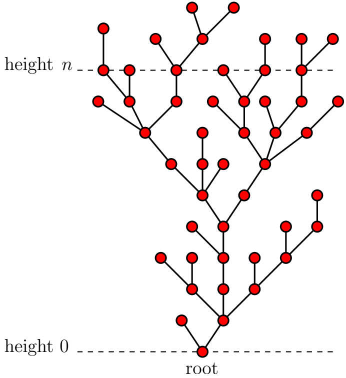
The proof relies on the fact that combinatorial trees can be interpreted as (conditioned) Galton-Watson trees.

1. Galton-Watson trees and reduced trees

Let θ be a probability measure on $\{0, 1, \ldots\}$, such that $\theta(1) < 1$ and

$$\sum_{j=0}^{\infty} j\theta(j) = 1 \text{ (critical)}, \qquad \sum_{j=0}^{\infty} j^2\theta(j) < \infty \text{ (finite variance)}$$

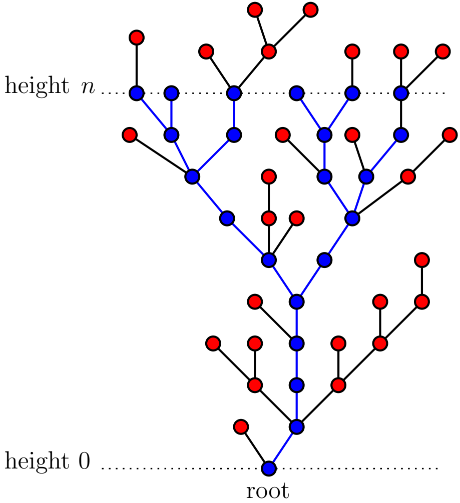
The Galton-Watson tree with offspring distribution θ is the genealogical tree of a population starting with one ancestor or root, where each individual has j children with probability $\theta(j)$. This tree is finite a.s.



$\mathcal{T}_n$ Galton-Watson tree conditioned to have height at least n

The harmonic measure of the ball of radius n is supported on $\mathcal{T}_n[n] := \{\text{vertices at height } n\}$.

Key idea : consider reduced trees.



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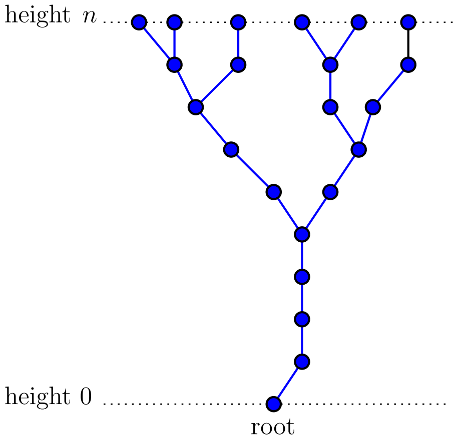

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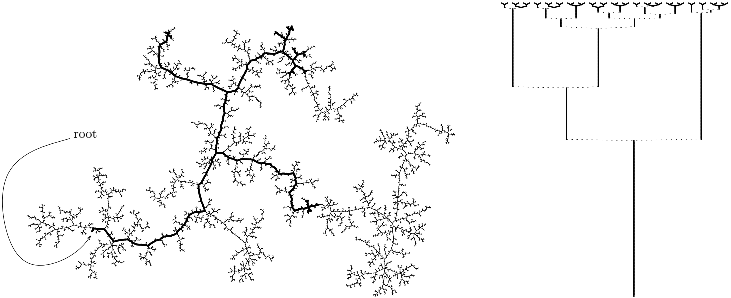
The hitting distribution of the set of vertices at height n is the same for SRW on $\mathcal{T}_n^*$ as for SRW on $\mathcal{T}_n$

A large reduced tree

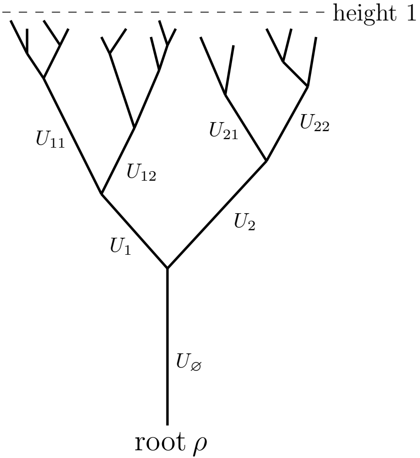


A large Galton-Watson tree and the corresponding reduced tree.

2. The continuous limit of reduced trees

$d_{\rm gr}$ graph distance on $\mathcal{T}_n^*$

Fact: $(\mathcal{T}_n^*, \frac{1}{n} d_{\rm gr}) \xrightarrow[n\to\infty]{(d)} (\mathcal{T}_\infty^*, D)$ in the Gromov-Hausdorff sense.



The tree $\mathcal{T}_\infty^*$.

$U_\varnothing$ uniform on $[0,1]$

U_1, U_2 uniform on $[0, 1 - U_\varnothing]$

U_{11}, U_{12} uniform on $[0, 1 - U_\varnothing - U_1]$

and so on.

D is the tree metric on the (completion of the) union of the segments, denoted by $\mathcal{T}_\infty^*$.

By definition $\partial \mathcal{T}_\infty^* := \{x \in \mathcal{T}_\infty^* : D(\rho, x) = 1\}$.

Harmonic measure on $\partial\mathcal{T}^*_\infty$

Let Γ_t be Brownian motion on $\mathcal{T}^*_\infty$
$\longrightarrow$ easy to define up to $T := \inf\{t \geq 0 : \Gamma_t \in \partial\mathcal{T}^*_\infty\}$
(at each branching point, Brownian motion chooses with equal probabilities each of the three possible directions)

Let μ be the law of Γ_T (harmonic measure on $\partial\mathcal{T}^*_\infty$)

Theorem (B)

A.s., $\mu(dx)$ a.e.,

$$\lim_{r\to 0} \frac{\log \mu(B_D(x,r))}{\log r} = \beta$$

In particular, $\dim \mu = \beta$.

Note: $\dim(\partial\mathcal{T}^*_\infty) = 1$ a.s. (dimension drop as in Makarov's theorem)

This theorem is a key ingredient of the proof of the discrete results.
(also explains why β is universal in the discrete setting)

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3. The Yule tree

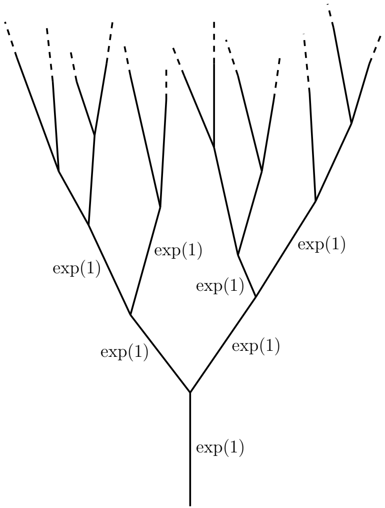
Scale the heights in $\mathcal{T}_\infty^*$ with

$$h(r) = -\log(1-r)$$

Then $\mathcal{T}_\infty^*$ is transformed in the Yule tree $\mathbb{T}$
= genealogical tree of population with

- binary branching
- exp(1) lifetimes

Γ (Brownian motion on $\mathcal{T}_\infty^*$) is transformed (up to time change) in
W Brownian motion on $\mathbb{T}$ with drift $\frac{1}{2}$ upwards



The Yule tree $\mathbb{T}$

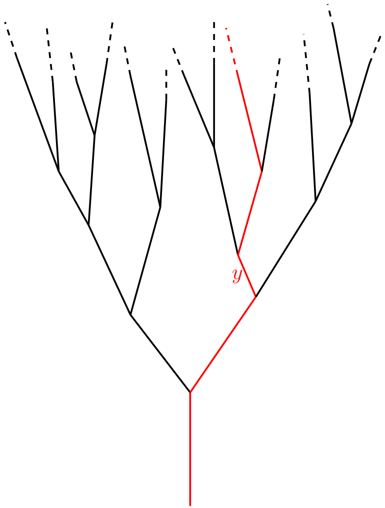
The boundary of the Yule tree

$\mathbb{T}$ Yule tree
The boundary $\partial\mathbb{T}$ is

$$\partial\mathbb{T} := \{\text{geodesic rays}\}$$

One can define W_∞
the exit ray of
the Brownian motion W

(unique ray visited by W
at arbitrarily large times)



The Yule tree $\mathbb{T}$
and a particular geodesic ray y

Asymptotics for the law of the exit ray

Recall: W_∞ exit ray of Brownian motion (with drift $\frac{1}{2}$) on the Yule tree $\mathbb{T}$
Set:

$$\nu = \text{law of } W_\infty \text{ (harmonic measure on } \partial\mathbb{T})$$

For $y \in \partial\mathbb{T}$ and $r > 0$, let

$$\mathcal{B}(y,r) = \{\text{geodesic rays that coincide with } y \text{ up to height } r\}$$

An equivalent form of Theorem (B) is:

Theorem (C)

A.s., $\nu(\mathrm{d}y)$ a.e.,

$$\lim_{r\to\infty} \frac{1}{r} \log \nu(\mathcal{B}(y,r)) = -\beta.$$

End of the lecture: ideas for the proof of Theorem (C).

4. Ergodic theory

$\Omega = \{\text{Yule-type trees}\}$

$\Omega^* = \{(\mathcal{T}, y) : \mathcal{T} \in \Omega, y \text{ ray of } \mathcal{T}\}$

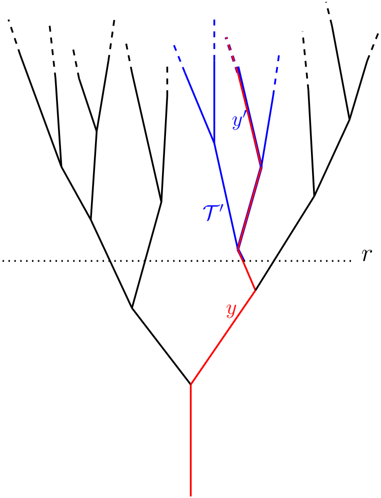
Shifts on Ω^*:

$\tau_r(\mathcal{T}, y) = (\mathcal{T}', y')$, where

- $\mathcal{T}'$ is the subtree above level r "containing" y
- y' is the ray in $\mathcal{T}'$ corresponding to y

$\Theta^*(\mathrm{d}\mathcal{T}\,\mathrm{d}y)$ = law of $(\mathbb{T}, W_\infty)$
($\mathbb{T}$ Yule tree, W_∞ exit ray for BM)

Θ^* is NOT invariant under the shifts, BUT one can find an absolutely continuous measure Λ^* which is invariant (and ergodic)



A pair $(\mathcal{T}, y) \in \Omega^*$ and the shift
$(\mathcal{T}', y') = \tau_r(\mathcal{T}, y)$
$\mathcal{T}'$ is the blue subtree

Applying Birkhoff's ergodic theorem

For $(\mathcal{T}, y) \in \Omega^*$ (= pairs consisting of a tree + a geodesic ray), set

$$F_r(\mathcal{T}, y) = -\log \nu_{(\mathcal{T})}(\mathcal{B}_{(\mathcal{T})}(y, r))$$

where

- $\nu_{(\mathcal{T})} =$ harmonic measure on $\partial\mathcal{T}$ (law of exit ray for Brownian motion on $\mathcal{T}$)
- $\mathcal{B}_{(\mathcal{T})}(y, r) =$ rays of $\mathcal{T}$ that coincide with y up to height r

Then, for every $r, s \geq 0$,

$$F_{r+s} = F_r + F_s \circ \tau_r$$

(conditionally on the event that Brownian motion escapes in a subtree $\mathcal{T}'$ above height r, the law of the exit ray is given by the harmonic measure of $\mathcal{T}'$)

By Birkhoff's theorem,

$$\frac{1}{r} F_r \underset{r\to\infty}{\longrightarrow} \beta := \Lambda^*(F_1)$$

Λ^* a.s. hence also Θ^* a.s. (recall Θ^* has a density w.r.t. Λ^*)

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The last convergence

$$\frac{1}{r}F_r \underset{r\to\infty}{\longrightarrow} \beta := \Lambda^*(F_1), \quad \Theta^* \text{ a.s.}$$

gives Theorem (C) (recall that Θ^* is the law of $(\mathbb{T}, W_\infty)$).

There is an explicit formula for β in terms of integrals with respect to the distribution of the conductance of the continuous reduced tree (the latter distribution is itself determined by a fixed point equation).

Remark. The preceding ideas are related to the work of Lyons, Pemantle and Peres (1995,1996) in a different setting.