Natural Parametrization of the Schramm-Loewner Evolution and the Loop-Erased Random Walk

Everything is Complex (Makarov Fest)
Saas-Fee, Switzerland

Gregory F. Lawler

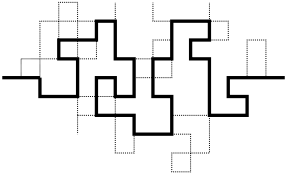
Department of Mathematics
University of Chicago
5734 S. University Ave.
Chicago, IL 60637

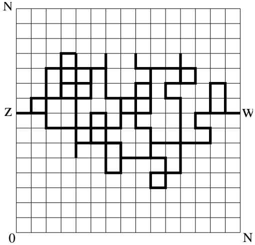
lawler@math.uchicago.edu

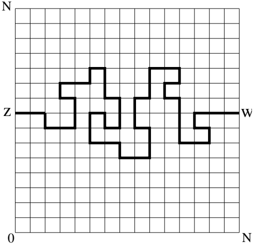
March 10, 2016

LOOP-ERASED RANDOM WALK (LERW)

- Start with simple random walks and erase loops in chronological order to get a path with no self-intersections.







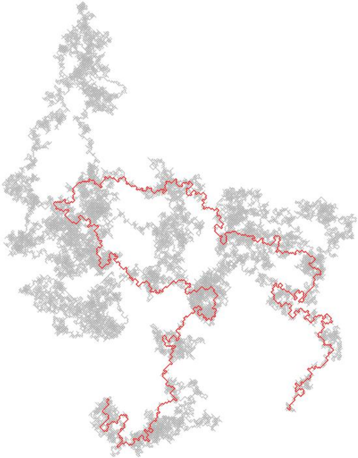


Figure : Realization of LERW

- Can also be described as Laplacian random walk.
- LERW is Laplacian growth at the tip.
- Diffusion limited aggregation (DLA) is Laplacian growth on the entire cluster.
- In two dimensions, study of LERW led to construction of Schramm-Loewner evolution (SLE_2) of parameter κ.
- Can also consider growth by harmonic measure to a power b.
- There is a corresponding continuous process called SLE_κ with

$$b = \frac{6-\kappa}{2\kappa}.$$

- Unknown (even nonrigorously) if there is convergence for $b \neq 1$. The only discrete model we will consider today is LERW ($b = 1$) but we will consider SLE_κ for various values of κ.

Scaling Limit

- Given bounded simply connected domain D containing the origin with analytic boundary and two boundary points $a', b' \in \partial D$.
- For each N let A_N be the lattice approximation of ND.
- Choose $a_N, b_N \in \partial A_N$ close to Na', Nb'. Let $a = a_N/N, b = b_N/b$. Consider a probability measure P_N on paths from a_N to b_N in A_N.
- Critical exponent fractal dimension d. We assume that a typical path under P_N has of order N^d points.

- Given P_N, d we have a measure on scaled paths where

$$\eta = [\eta_0, \eta_1, \ldots, \eta_k]$$

is scaled to be the continuous curve $\tilde{\eta}$ of time duration $T = k/N^d$ given by

$$\tilde{\eta}\left(\frac{k}{N^d}\right) = \frac{\eta_k}{N}.$$

- $\tilde{\eta}$ becomes a continuous curve by linear interpolation.
- A scaling limit is a limit for the induced probability measure on scaled curves.
- We have given the discrete curves the (normalized) natural or length parametrization.

The fractal dimension for LERW

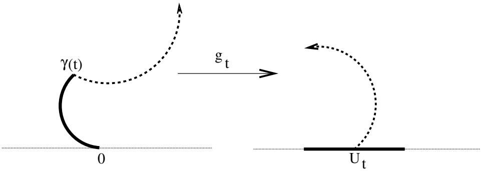
- Given conformal invariance of planar Brownian motion, good reason to believe conformal invariance of limit.
- Kenyon (2000) used relationship with dimer models to rigorously show that $d = \frac{5}{4}$.
- Although Kenyon's result was exact on the value of the dimension he did not give sufficiently sharp asymptotics that our needed for our proof.
- While his result did not suffice, an important idea (zippers) from his proof was used in later work.

Find the limit measure

- Schramm (2000) considered possible scaling limits for the LERW assuming conformal invariance and another property, domain Markov property.
- He showed there is a one parameter family of processes satisfying conformal invariance and domain Markov property — these are now called the Schramm-Loewner evolution (SLE_κ).
- Using another property of LERW he was able to determine that $\kappa = 2$ would have to be the limit of LERW assuming the scaling limit existed and was conformally invariant.
- This measure was really on curves modulo reparametrization.

(Half plane) Loewner equation

- Let $\gamma : (0, \infty) \to \mathbb{H}$ be a simple curve with $\gamma(0+) = 0$ and $\gamma(t) \to \infty$ as $t \to \infty$.
- $g_t : \mathbb{H} \setminus \gamma(0, t] \to \mathbb{H}$



- Choose g_t and reparametrize so that

$$g_t(z) = z + \frac{2t}{z} + \cdots , \qquad z \to \infty$$

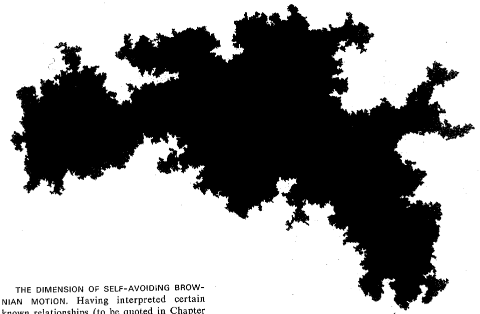
Then g_t satisfies the (half plane) Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Moreover, $U_t = g_t(\gamma(t))$ is continuous in t.

- (Schramm) The only possible conformally invariant limit for LERW is the solution to the Loewner equation with $U_t = \sqrt{2}\, B_t$, where B_t is a standard Brownian motion. More generally, $U_t = \sqrt{\kappa}\, B_t$ gives (chordal) SLE_κ.
- $SLE_\kappa, \kappa < 8$ is a measure on simple curves of fractal dimension $d = 1 + \frac{\kappa}{8}$. We will consider only $\kappa < 8$ in this talk.
- The paths are simple for $\kappa \leq 4$. (Rohde-S)
- We have also chosen a particular parametrization called the capacity parametrization.
- The measure is defined in other simply connected domains by conformal invariance.
- (L-Schramm-Werner) The scaling limit of LERW in the capacity parametrization is SLE_2.

Example: outer boundary of Brownian motion



THE DIMENSION OF SELF-AVOIDING BROWNIAN MOTION. Having interpreted certain known relationships (to be quoted in Chapter 36) as implying that a self-avoiding random walk is of dimension 4/3, I conjecture that the same is true of self-avoiding Brownian motion.

An empirical test of this conjecture provides an excellent opportunity to test also the length-area relation of Chapter 12. The plate is covered by increasingly tight square lattices, and we count the numbers of squares of side G intersected by a) the hull, standing for G-area, and b) its boundary, standing for G-length. Graphs relating G-length to G-area, using doubly logarithmic coordinates, were found to be remarkably straight, with a slope indistinguishable from $D/2=(4/3)/2=2/3$.

The resemblance between the curves in Plates 243 and 231, and their dimensions, is worth stressing.

NOTE. In Plate 243, the maximal open domains that B(t) does not visit are seen in gray. They can be viewed as tremas bounded by fractals, hence the loop is a net in the sense of Chapter 14.

◅ The question arises, of whether the loop is a gasket or a carpet from the viewpoint of the order of ramification. I conjectured that the latter is the case, meaning that Brown nets satisfy the Whyburn property, as described on p. 133. This conjecture has been confirmed in Kakutani & Tongling (unpublished). It follows that the Brown trail is a universal curve in the sense defined on page 144. ► ■

- Mandelbrot conjectured that outer boundary has dimension $\frac{4}{3}$.
- Burdzy - L: outer boundary is Jordan curve.
- Makarov: Parametrization by harmonic measure is carried on a set of dimension 1.
- LSW: Outer boundary is locally the same as $SLE_{8/3}$ and hence has dimension $\frac{4}{3}$.
- The capacity parametrization from $SLE_{8/3}$ is carried on a set of dimension strictly between 1 and $4/3$.
- Under this parametrization the curve is weakly-Hölder continuous of order $\frac{10-4\sqrt{6}}{11-4\sqrt{6}} = .168\cdots$.
- The "natural parametrization" should be carried on a set of dimension $\frac{4}{3}$ and should be weakly-Hölder continuous of order $\frac{3}{4}$.

- Some results for random walk boundaries, e.g.,
  L (1993). A Discrete Analogue of a Theorem of Makarov, Combinatorics, Probability and Computing, 2, pp 181-199.

- The capacity parametrization for $SLE_\kappa, \kappa < 8$ is carried on a set dimension (roughly speaking) $1 + \frac{\kappa}{\kappa+8} \quad (< 1 + \frac{\kappa}{8})$.
- This is one case of a general result about the tip multifractal spectrum of the curve. Another special case is the Hausdorff dimension of the path (originally proved by Beffara).
- The curve is weakly-Hölder continuous of order

$$1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8+\kappa}}.$$

(One direction first proved by Lind.)

Parametrization by harmonic measure is a special case of full boundary (two-sided) multifractal spectrum (in the case of outer boundary of Brownian motion it is Brownian intersection exponents).

Natural parametrization of $SLE_\kappa, \kappa < 8$

- We want to parametrize SLE path by the normalized limit of the length (number of steps) of the discrete walk.
- Natural parametrization of a smooth curve is another term for parametrization by arclength.
- For a d-dimensional curve, we would like to parametrize by "d-dimensional length".
- Although the Hausdorff dimension of SLE_κ is $d = 1 + \frac{\kappa}{8}$, the Hausdorff d-measure is zero (Rezaei).
- Finding a nontrivial Hausdorff gauge function is an unsolved problem and seems very difficult.

Minkowski content

- The d-dimensional Minkowski content of a compact $V \subset \mathbb{C} = \mathbb{R}^2$ is given by (provided that the limit exists)

$$\mathrm{Cont}(V) = \mathrm{Cont}_d(V) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2}\, \mathrm{Area}\{z : \mathrm{dist}(z, V) < \epsilon\}.$$

- Interested in this for $V = \gamma$, an SLE_κ curve from w to w' in a domain D.
- One-point estimate: find d and function $G(z) = G_D(z; w, w')$ such that

$$\mathbb{P}\{\mathrm{dist}(z, \gamma) < \epsilon\} \sim G(z)\, \epsilon^{2-d}, \quad \epsilon \downarrow 0.$$

- If $f : D \to f(D)$ is conformal,

$$G_D(z; w, w') = |f'(z)|^{2-d}\, G_{f(D)}(f(z); f(w), f(w')).$$

- (Rohde-Schramm) If such a relation holds then $d = 1 + \frac{\kappa}{8}$ and

$$G_{\mathbb{H}}(z; 0, \infty) = c\,[\mathrm{Im} z]^{d-2}\,[\sin \arg z]^{\beta}, \quad \beta = \frac{8}{\kappa} - 1.$$

- (L, -Werness, -Rezaei) There exists c, α such that

$$\mathbb{P}\{\mathrm{dist}(z, \gamma) < \epsilon\} = G(z)\,\epsilon^{2-d}\,[1 + O(\epsilon^{\alpha})], \quad \epsilon \downarrow 0.$$

Also there exists a two-point function $G(z, z')$,

$$\mathbb{P}\{\mathrm{dist}(z, \gamma) < \epsilon, \mathrm{dist}(z', \gamma) < \epsilon\} \sim G(z, z')\,\epsilon^{2(2-d)}, \quad \epsilon \downarrow 0.$$

$$\mathrm{dist}(z, z') \asymp G(z)\,|z - z'|^{d-2}, \quad z' \to z.$$

Minkowski content

Theorem (L-Rezaei)

If γ is an $SLE_\kappa(\kappa < 8)$ curve from w to w' in D, then

$$\Theta_t := \mathrm{Cont}\left(\gamma[0,t]\right)$$

exists, is Hölder continuous, and strictly increasing in t.

- If D is bounded with analytic boundary, then

$$\mathbb{E}\left[\mathrm{Cont}(\gamma)\right] = \mathbb{E}\left[\Theta_\infty\right] = \int_D G_D(z)\, dA(z) < \infty,$$

$$\mathbb{E}\left[\mathrm{Cont}(\gamma)^2\right] = \mathbb{E}\left[\Theta_\infty^2\right] = \int_{D\times D} G_D(z,z')\, dA(z)\, dA(z') < \infty$$

Natural parametrization of SLE

- $\gamma(t)$ is SLE_κ with the natural parametrization (or natural length) if it is the time change of Schramm's SLE with

$$\text{Cont}\,(\gamma_t) = t, \quad \gamma_t = \gamma[0,t].$$

- Note that

$$\mathbb{E}\left[\text{Cont}(\gamma_\infty) \mid \gamma_t\right] = \text{Cont}(\gamma_t) + \Psi_t,$$

$$\text{where} \quad \Psi_t = \int_{D\setminus\gamma_t} G_{D\setminus\gamma_t}(z;\gamma(t),w')\,dA(z).$$

- The first definition (L-Sheffield) of natural parametrization was as the increasing process Θ_t such that $\Theta_t + \Psi_t$ is a martingale (Doob-Meyer decomposition). This characterization is important in comparing to LERW.

- Conjecture (but not proven): SLE_κ under the natural parametrization is weakly Hölder-continuous of order $1/d$.
- Werness: There is a parametrization at least for $\kappa \leq 4$ that is weakly Hölder-continuous of order $1/d$.
- No way to reparametrize to get a higher order.

Outline to prove Minkowski content exists

- For $z \in D$ let
$$\sigma_\epsilon = \sigma_{\epsilon,z} = \inf\{t : |\gamma(t) - z| = \epsilon\}.$$
- Show that
$$\mathbb{P}\{\sigma_\epsilon < \infty\} \sim G(z)\, \epsilon^{2-d}.$$
- Let
$$\Upsilon_\epsilon(z) = \mathbb{P}\{\sigma_\epsilon < \infty \mid \gamma[0, \sigma_{2\epsilon}]\}.$$
Show that as $\epsilon \downarrow 0$, $\Upsilon_\epsilon(z)$ (given $\sigma_{2\epsilon} < \infty$) has a limit distribution independent of ϵ, z.
- Show that if z, z' are separated, then for ϵ small $\Upsilon_\epsilon(z)$ and $\Upsilon_\epsilon(z')$ are almost independent.
- Need double points of path to have strictly smaller dimension than the path.

Return to Loop-Erased Random Walk

- A, finite simply connected subset of $\mathbb{Z}^2$ containing the origin.
- $a, b \in \partial A$, distinct.
- Let $\mathbb{P}_{A,a,b}$ denote the probability measure of LERW from a to b in A. This is a measure on self-avoiding paths

$$\eta = [\eta_0, \eta_1, \dots, \eta_k]$$

from a to b in A. We write

$$\eta^n = [\eta_0, \dots, \eta_n],$$

- Let A_n denote the connected component of $A \setminus \eta^n$ that contains b on its boundary.

- Domain Markov property: the distribution of the remainder of η given η^n is $\mathbb{P}_{A_n,\eta_n,b}$.
- Let T denote the total number of steps of the walk.

$$\mathbb{E}_{A,a,b}\left[T\right] = \sum_{\zeta \in A} \mathbb{P}_{A,a,b}\{\zeta \in \eta\}.$$

- $$\mathbb{E}_{A,a,b}\left[T \mid \eta^n\right] = n + \Phi_n,$$

  where

$$\Phi_n = \mathbb{E}_{A_n,\eta_n,b}[T] = \sum_{\zeta \in A} \mathbb{P}_{A_n,\eta_n,b}\{\zeta \in \eta\}.$$

- We need to estimate $\mathbb{P}_{A,a,b}\{\zeta \in \eta\}$ precisely even for A with "rough boundary".

- Let D_A be the domain obtained from A by replacing each point z with a square of side length one centered at z

.

Theorem (Beneš-L-Viklund)

There exists $c_0, u > 0$ such that

$$\begin{aligned} \mathbb{P}_{A,a,b}\{\zeta \in \eta\} &= (c_0\,\hat{c})\,r^{-\frac{3}{4}}\,\left[S^3 + O(r^{-u})\right] \\ &= c_0\,G_{D_A,a,b}(\zeta) + O(r^{-\frac{3}{4}-u}), \end{aligned}$$

where $r = r_{D_A}(\zeta), S = S_{D_A}(\zeta; a, b)$ and G denotes the SLE_2 Green's function.

- The proof makes no use of SLE.
- The estimates are uniform over all finite simply connected A and hence they apply to $\mathbb{P}_{A_n,a_n,b}$.
- The proof uses the relationship between LERW and the random walk loop measure.

- This relationship is algebraic and does not require the bond weights on the random walks to be positive.
- Using the zipper idea of Kenyon, we assign weights of bonds crossing the zipper to be negative.
- A determinantal relationship (Fomin's identity) relates the probability we need with a certain measure of paths under the signed measure.
- We finally need to estimate the usual loop measure of loops in A that have odd winding number about ζ: There exists β such that

$$\frac{1}{8}\log r_A(\zeta) + \beta + o(1).$$

The exponent $-3/4$ in the theorem comes from the constant $1/8$ in this formula.

$$-\frac{3}{4} = -1 + 2\,\frac{1}{8}.$$

- The calculation is first done for a conformally invariant object, the Brownian loop measure (L.-Werner).
- The result for the random walk measure uses a very close coupling of the two (L-Trujillo Ferreras) which allows for sharp estimates of the error terms.
- There are also careful estimates for the random walk Poisson kernel in the domains D_A in terms of the Brownian kernel. This uses work in Kozdron-L (and is closed related to results in L-Schramm-Werner for random walk in slit domains).
- The exponent 3 can be calculated "in the limit" for Brownian motion but must be verified for the random walk.

Theorem (L- Viklund)

For any simple connected bounded domain D with analytic boundary and $a', b' \in \partial D$, the measures on loop-erased walks scaled with $d = \frac{5}{4}$ converge to the Schramm-Loewner evolution with parameter $\kappa = 2$ (SLE_2), parametrized by (a constant multiple times the) $\frac{5}{4}$-dimensional Minkowski content (natural parametrization).

- The key new part of this this theorem is that the convergence is with respect to curves parametrized by Minkowski content (natural length).
- The proof requires detailed estimates for both the LERW and the SLE.

Basic strategy of proof of main theorem

- Fix a lattice spacing $1/N$.
- Using the observable "probability that LERW goes through z", we can couple an SLE γ and a scaled LERW η so that their paths are close (similar to idea in LSW paper).
- Let

$$\tilde{\eta}_t = \tilde{\eta}_{t,N} = N^{-1}\,\eta\left(ctN^{5/4}\right)$$

be the scaled version of η of time duration

$$\tilde{T} = [cN^{5/4}]^{-1}T.$$

- Consider the martingales (actually considered at discrete times)

$$M_t := \mathbb{E}\left[\mathrm{Cont}(\gamma_\infty) \mid \gamma_t\right] = \mathrm{Cont}(\gamma_t) + \Psi_t,$$
$$\tilde{M}_t = \mathbb{E}\left[\tilde{T} \mid \tilde{\eta}_t\right] = \tilde{T}_t + \tilde{\Phi}_t,$$

- We have a martingale N_t

$$N_t = M_t - \tilde{M}_t = \mathrm{Cont}(\gamma_t) - \tilde{T}_t + \Psi_t - \tilde{\Phi}_t,$$

$$N_t = B_t + Y_t,$$

where B_t is bounded variation with $B_0 = 0$.

- Using the coupling of the SLE and LERW along with Green's function estimates for both SLE and LERW we see that $\mathbb{E}[{Y_t}^2]$ is very close to zero. (This requires the precise asymptotics of LERW Green's function.)
- We conclude that B_t is close to zero.
- Although the basic program of proof is not too hard to state, there are a lot of details to verify, both for SLE and LERW, to make this work.

HAPPY BIRTHDAY , NICK!