Constructive Conformal Field Theory

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Scaling limits

Scaling limits of discrete 2d models: percolation, Ising, ...

- **SLE, CLE**: Statistics of boundary curves
- **Conformal Field Theory**: Correlation functions of local fields, operator algebras representing symmetries

While SLE is a probabilistic description the study of CFT has been mostly algebraic

Exceptions:

- Kang and Makarov on GFF
- Chelkak, Hongler, Izyurov, Kytölä, Smirnov, ... on Ising

Problem: direct continuum formulation of most CFT's missing

Liouville CFT:

- Explicite (formal) functional integral formulation
- One of the simplest and one of the most mysterious CFTs!

Our aim is to study its properties using probabilistic methods.

Conformal Field Theory

What does it mean to construct CFT?

- Expectation $\langle \dots \rangle$
- **Primary fields** (in general **not** distribution valued) $\Phi_\Delta(z)$, $z \in \mathbb{C}$ of **conformal weight** Δ
- Correlation functions $\langle \prod_i \Phi_{\Delta_i}(z_i) \rangle$ $z_i \neq z_j$
- **Global** conformal invariance: f Möbius

$$\langle \prod_i \Phi_{\Delta_i}(f(z_i)) \rangle = \prod_i |f'(z_i)|^{-2\Delta_i} \langle \prod_i \Phi_{\Delta_i}(z_i) \rangle$$

- **Local** conformal invariance: a **holomorphic** field $T(z)$ **"Energy-Momentum Tensor"**

Energy-Momentum tensor

Local conformal invariance:

$$\langle T(z)\prod_{i=1}^{n}\Phi_{\alpha_i}(z_i)\rangle \text{ and } \langle T(z)T(z')\prod_{i=1}^{n}\Phi_{\alpha_i}(z_i)\rangle$$

are analytic in $z \neq z' \in \mathbb{C} \setminus \{z_1, \dots, z_n\}$

Singularities given by **Conformal Ward identities**:

$$\langle T(z)\prod_i \Phi_{\Delta_i}(z_i)\rangle = \sum_j \left(\frac{\Delta_j}{(z-z_j)^2} + \frac{1}{z-z_k}\partial_{z_k}\right)\langle \prod_i \Phi_{\Delta_i}(z_l)\rangle$$

$$\langle T(z)T(z')\prod_i \Phi_{\Delta_i}(z_i)\rangle = \frac{c}{(z-z')^4}\langle \prod_i \Phi_{\Delta_i}(z_i)\rangle + \frac{2}{(z-z')^2}\langle T(z')\prod_i \Phi_{\Delta_i}(z_i)\rangle + \frac{1}{z-z'}\partial_{z'}\langle T(z')\prod_i \Phi_{\Delta_i}(z_i)\rangle + regular(z-z')$$

c **central charge**

Algebraic Structure

OS-positivity of $\langle \dots \rangle \implies$ Hilbert Space $\mathcal{H}$

Ward identities $\implies$ unitary representation of Virasoro Algebra on $\mathcal{H}$

How does this representation reduce?

Fusion rules for tensoring representations

Conformal bootstrap for determining correlations

Gaussian Free Field

$$\langle F \rangle = \int_{Map(\mathbb{C}\to\mathbb{R})} F(X) e^{-\frac{1}{4\pi}\int_{\mathbb{C}} |\partial_z X|^2 dz} DX$$

To define this pick a smooth metric $g(z)dz^2$ on $\hat{\mathbb{C}}$ and let X_g be Gaussian Free Field normalized with

$$\int_{\mathbb{C}} X_g \, g dz = 0$$

Then $X = X_g + c$, $c \in \mathbb{R}$ i.e.

$$\langle F \rangle = \int_{\mathbb{R}} [\mathbb{E} F(X_g + c)] dc := \int F(X) d\nu_{GFF}(X)$$

- $\nu_{GFF}(dX) = \mathbb{P}(dX_g)dc$ is **not** a probability measure
- $\langle \dots \rangle$ is independent of the metric since $X_g \overset{law}{=} X'_g + const$.
- Central charge $= 1$
- Primary fields $e^{i\alpha X}$ (renormalized), $\alpha \in \mathbb{R}$. $\Delta_\alpha = \frac{\alpha^2}{4}$.

Liouville theory

Perturbation of GFF

$$\nu_L = e^{-\frac{1}{4\pi}\int_{\mathbb{C}}(QR_g X + \mu e^{\gamma X})g dz}\nu_{GFF}$$

- $R_g = -\Delta \log g$ scalar curvature
- $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$, $\gamma > 0$, $\mu > 0$
- $e^{\gamma X} gdz := M_g(dz)$ is **Gaussian multiplicative chaos**

$$M_g(dz) = \lim_{\epsilon \to 0} e^{\gamma X_{g,\epsilon} - \frac{\gamma^2}{2}\mathbb{E}[X_{g,\epsilon}^2]} gdz$$

- M_g is a random **multifractal** measure on $\mathbb{C}$.
- $M_g \neq 0$ **iff** $\gamma < 2$ and $M_g(\mathbb{C}) < \infty$ a.s.

Knizhnik, Polyakov, and Zamolodchikov '88

Let Φ_Δ be a primary field of a CFT with $c = 25 - 6Q^2 < 1$.

Then the corresponding field on a **random surface** is

$$\Phi_\Delta e^{\gamma \Delta_q X}$$

with X the Liouville field and

$$\Delta = \Delta_q + \frac{\gamma^2}{4}\Delta_q(\Delta_q - 1)$$

Hence to understand CFT on a random surface need to understand correlations of **vertex operators**

$$V_\alpha(z) = e^{\alpha X(z)}$$

in Liouville theory.

These can be reduced to the study of Multiplicative Chaos.

Liouville correlations

$$\langle\prod_i V_{\alpha_i}(z_i)\rangle_L = \int \prod_i e^{\alpha_i X(z_i)} e^{-\frac{1}{4\pi}\int_{\mathbb{C}}(QR_g X+\mu e^{\gamma X})gdz} d\nu_{GFF}(X)$$

Since $X = X_g + c$ and by Gauss-Bonnet: $\int_{\mathbb{C}} R_g gdz = 8\pi$ we get

$$= \int \prod_i e^{\alpha_i X_g(z_i)} e^{(\sum_i \alpha_i - 2Q)c} e^{-\frac{1}{4\pi}\int_{\mathbb{C}}(QR_g X_g+\mu e^{\gamma c} e^{\gamma X_g})gdz} d\mathbb{P}(X_g)dc.$$

We can perfom the c integral to get

$$\langle\prod_i V_{\alpha_i}(z_i)\rangle_L = \frac{\Gamma(s)}{\mu^s\gamma}\mathbb{E}[\prod_i e^{\alpha_i X_g(z_i)} e^{-\frac{1}{4\pi}\int_{\mathbb{C}} QR_g X_g g} Z_0^{-s}]$$

provided

$$s := \frac{1}{\gamma}(\sum_i \alpha_i - 2Q) > 0.$$

and

$$Z_0 = \int_{\mathbb{C}} e^{\gamma X_g} gdz$$

Liouville correlations

Covariance of X_g

$$C(z,z') = \mathbb{E} X_g(z) X_g(z') = \log|z-z'|^{-1} + \dots$$

By Girsanov theorem we make a shift

$$X_g(z) \to X_g(z) + \sum \alpha_i C(z,z_i) + \int C(z,z') R_g g \, dz'$$

and Liouville correlation becomes

$$\langle \prod_i V_{\alpha_i}(z_i) \rangle_L = \frac{\Gamma(s)}{\mu^s \gamma} \prod_{i<j} |z_i - z_j|^{-\alpha_i \alpha_j} \mathbb{E}\, Z^{-s}$$

where Z is an integral over Multiplicative Chaos:

$$Z = \int_{\mathbb{C}} e^{\gamma X_g} \prod_i |z - z_i|^{-\gamma \alpha_i} g^{1 - \frac{\gamma}{4} \sum \alpha_i} dz$$

Liouville correlations

Modulus of continuity of Chaos: $M_g(B_r) \sim r^{\gamma Q} \implies |z - z_i|^{-\gamma\alpha_i}$ integrable iff

$$\alpha_i < Q$$

Corollary. Since $\sum \alpha_i > 2Q$ need at least **three** vertex operators to have finite correlators.

The vertex operator $e^{\alpha_i X(z_i}$ creates a **conical** singularity in the quantum metric $e^{\gamma X} g$.

Classical vs. Quantum

Extrema of classical Liouville action functional (**with** $Q = 2/\gamma$)

$$\int_{\mathbb{C}} (|\partial_z X|^2 + QR_g gX + \mu e^{\gamma X} g) dz$$

are **constant negative curvature metrics** $e^{\gamma X} g$ on $\hat{\mathbb{C}}$.

To have these classically need to have at least three conical singularities.

This holds for quantum Liouville too.

Classical Liouville action is Möbius invariant.

This holds for the quantum Liouville correlations too.

Global Conformal Invariance

Theorem

(a) **Conformal covariance**. Let ψ be Möbius. Then

$$\langle V_{\alpha_1}(\psi(z_1)) \dots V_{\alpha_n}(\psi(z_n))\rangle = \prod_i |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \langle V_{\alpha_1}(z_1) \dots V_{\alpha_n}(z_n)\rangle$$

where $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$.

(b) **Weyl invariance**. Let $g' = e^\phi g$. Then

$$\langle V_{\alpha_1}(z_1) \dots V_{\alpha_n}(z_n)\rangle' = e^{S(\phi,g)} \langle V_{\alpha_1}(z_1) \dots V_{\alpha_n}(z_n)\rangle$$

where $S(\phi, g) = \frac{c_L - 1}{96\pi}(\int |\partial\phi|^2 \, gdz + \int 2R_{\hat{g}}\phi \, gdz)$ and

$$c_L = 1 + 6Q^2$$

is the **central charge** of the Liouville theory.

Liouville measure

If $\alpha_i < Q$ and $\sum_i \alpha_i - 2Q > 0$ we can define probability

$$d\mathbb{P}_{\alpha_i,z_i}(X) := \frac{1}{Z_{\alpha_i,z_i}} \prod_{i=1}^{n} V_{\alpha_i}(z_i) d\mu_L(X)$$

Consider the law of the chaos measure $M_{\hat{g}}$ under $\mathbb{P}_{\mu,\gamma}$. Let

$$A = M_{\hat{g}}(\hat{\mathbb{C}})$$

be the "volume of the universe". Then
Theorem. Under $\mathbb{P}_{\alpha_i,z_i}$ the law of A is $\Gamma(s,\mu)$ ($s = \frac{\sum_i \alpha_i - 2Q}{\gamma}$) :

$$\mathbb{E}F(A) = \frac{\mu^s}{\Gamma(s)} \int_0^\infty F(y) y^s e^{-\mu y}\, dy.$$

Remark. With $n = 3$, $\alpha_i = \gamma$ this agrees with Planar Maps with an $O(n)$ loop model of $c = 25 - 6Q^2$ justifying KPZ.

Planar maps with matter

Let T be a triangulation of $\mathbb{S}^2$ with three marked points.

Define probability

$$\mathbb{P}(T) \propto e^{-\mu_0 |T|} Z_Q(T)$$

$Z_Q(T)$ partition function of loop model on T with $c = 25 - 6Q^2$.

Map T conformally to $\hat{\mathbb{C}} \setminus \{z_1, z_2, z_3\}$

Area measure on $T \to$ random measure ν_{μ_0} on $\hat{\mathbb{C}}$.

Take scaling limit as $\mu_0 \downarrow \mu_{crit}$. Then the total mass has $\Gamma(\frac{\sum_i \alpha_i - 2Q}{\gamma}, \mu)$ law ($n = 3$, $\alpha_i = \gamma$).

Conformal Ward identities

Define the **Energy-momentum tensor**

$$T(z) = Q\partial_z^2 X(z) - ((\partial_z X(z))^2 - \mathbb{E}(\partial_z X(z))^2)$$

Theorem

$$\mathbb{E}\Big(T(z)\prod_{i=1}^n V_{\alpha_i}(z_i)\Big) \text{ and } \mathbb{E}\Big(T(z)T(z')\prod_{i=1}^n V_{\alpha_i}(z_i)\Big)$$

are analytic in $z, z' \in \mathbb{C} \setminus \{z_1, \dots, z_n\}$ and satisfy the conformal Ward identities:

$$\begin{aligned} T(z)T(z') &= \frac{c}{(z-z')^4} + \frac{2}{(z-z')^2}T(z') + \frac{1}{z-z'}\partial T(z') + \mathcal{O}(1) \\ T(z)V(z_i) &= \frac{\Delta_i}{(z-z_i)^2}V(z_i) + \frac{1}{z-z_i}\partial V(z_i) + \mathcal{O}(1) \end{aligned}$$

Conformal Ward identities

Proof. Use integration by parts, e.g.

$$\langle \partial_z^2 X(z) \prod e^{\alpha_i X(z_i)} \rangle_L = \frac{1}{2} \sum_i \frac{\alpha_i}{(z - z_i)^2} \langle \prod e^{\alpha_i X(z_i)} \rangle_L - \frac{\mu\gamma}{2} \int \frac{1}{(z-u)^2} \langle e^{\gamma X(u)} \prod e^{\alpha_i X(z_i)} \rangle_L du$$

Need to control **Beurling transforms** of Liouville correlations.

These are singular as $z_i - z_j \to 0$ i.e. need to understand Liouville **operator product expansion**

$$V_\alpha(u) V_\beta(v) \sim |u - v|^{-\delta} V_\gamma(v)$$

This relates to the **freezing** phenomenon in Chaos theory.

3-point function

In CFT 3-point function believed to determine whole theory.
Möbius invariance $\implies$ suffices to consider

$$
\begin{aligned}
C(\alpha_1, \alpha_2, \alpha_3) &= \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle \\
&= \mu^{-s} \Gamma(s) \, \mathbb{E} \Big(\int e^{\gamma X(z)} \frac{1}{|z|^{\gamma\alpha_1} |z-1|^{\gamma\alpha_2}} \rho dz \Big)^{-s}
\end{aligned}
$$

This is finite only if $s = \frac{1}{\gamma}(\sum_i \alpha_i - 2Q) > 0$.

For $s = -k$ $k \in \mathbb{N}$ it may be evaluated formally in terms of diverging Selberg-integrals.

These have a finite "analytic continuation", the explicit **DOZZ formula** due to Dorn, Otto, Zamolodchikov and Zamolodchikov

How to prove this?

BPZ Equations

CFT's have **degenerate fields** leading to PDE's for correlations.

In Liouville $V_{-\frac{\gamma}{2}}$ is degenerate: we prove

Theorem. Let $F(z, z_1, \ldots, z_N) = \langle V_{-\frac{\gamma}{2}}(z) \prod_l V_{\alpha_l}(z_l) \rangle$ then

$$\frac{4}{\gamma^2}\partial_z^2 F + \sum_k \frac{\Delta_{\alpha_k}}{(z - z_k)^2} F + \sum_k \frac{1}{z - z_k}\partial_{z_k} F = 0.$$

For $N = 3$ this equation has a unique solution expressed in terms of hypergeometric functions.

As a corollary we prove a recursion for $C(\alpha_1, \alpha_2, \alpha_3)$ whose unique solution in analytic functions is the DOZZ conjecture.

A major problem is to prove analyticity.

Hilbert space

ν_{GFF} and μ_L are **reflection positive**:

$$(F,G) := \int \overline{F(X)}(\Theta G)(X)d\mu_L(X) \geq 0 \quad \forall F,G \in \mathcal{F}_{\mathbb{D}}$$

- $\mathcal{F}_{\mathbb{D}} = \{F(X) \text{ supported on } X|_{\mathbb{D}}\}$.
- $(\Theta F)(X) := F(X(1/\bar{z}))$.

Define the **Physical Hilbert space**:

$$\mathcal{H} := \overline{\mathcal{F}_{\mathbb{D}}/\{F : (F,F) = 0\}}$$

Hilbert space

Splitting of GFF to independents:

$$X_g = c + X_{\mathbb{D}} + X_{\mathbb{D}^c} + P\phi$$

- $X_{\mathbb{D}}$ and $X_{\mathbb{D}^c}$ Dirichlet GFF
- ϕ GFF on $\partial\mathbb{D}$ ("1/f noise")
- $P\phi$ harmonic extension of ϕ on $\partial\mathbb{D}$.

Then

$$U_{GFF}F = \mathbb{E}_{\mathbb{D}}F(c + X_{\mathbb{D}} + P\phi)$$

is a unitary map $U_{GFF} : \mathcal{H}_{GFF} \to L^2(dc) \times L^2(\mathbb{P}(d\phi))$.

For Liouville

$$UF = e^{-Qc}\mathbb{E}_{\mathbb{D}}(e^{-\mu \int_{\mathbb{D}} e^{\gamma X} dz} F(c + X_{\mathbb{D}} + P\phi))$$

is a unitary map $U : \mathcal{H}_L \to L^2(dc) \times L^2(\mathbb{P}(d\phi))$.

Dilation semigroup

Dilation $z \to \lambda z$ acts on $\mathcal{F}_{\mathbb{D}} \implies$ contraction semigroup

$$\lambda^{L_0}\bar{\lambda}^{\bar{L}_0} : \mathcal{H} \to \mathcal{H}$$

$H := L_0 + \bar{L}_0 \geq 0$ **Hamiltonian** operator of the CFT.

For GFF get

$$H_{GFF} = -\tfrac{1}{2}\frac{d^2}{dc^2} + \sum_{n>0} n(a_n^* a_n + b_n^* b_n)$$

and for Liouville formally

$$H_L = \tfrac{1}{2}(-\frac{d^2}{dc^2} + Q^2) + \sum_{n>0} n(a_n^* a_n + b_n^* b_n) + (\int_{\mathbb{T}} e^{\gamma\phi})_{renormalized}$$

Virasoro algebra

Ward identities $\implies$ unitary representation of **Virasoro algebra** on $L^2(dc) \times L^2(\mathbb{P}(d\phi))$:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}\delta_{m,-n}$$

with

$$L_n = U \oint_{|z|=r<1} z^{n+1} T(z) U^{-1}$$

and $c = 1$ for GFF, $c = 1 + 6Q^2$ for Liouville.

For Liouville the domain of L_n poses some challenges!

Spectrum

Reduction of the Virasoro representation on $\mathcal{H}$?

- Highest weight module $M_\alpha = \mathrm{span}\{L_n\psi_\alpha, n \leq 0\}$, $L_0\psi_\alpha = \Delta_\alpha\psi_\alpha$, $L_n\psi_\alpha = 0$, $n > 0$.

Conjectures:

- Each $\alpha = Q + iP$ occurs with multiplicity one:

$$\mathcal{H} = \int_{\mathbb{R}}^{\oplus} \mathcal{H}_P \, dP$$

- All correlation functions $\langle V_{\alpha_1} \dots V_{\alpha_n} \rangle$ determined by the three point functions $\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} \rangle$ (**"conformal boostrap"**).

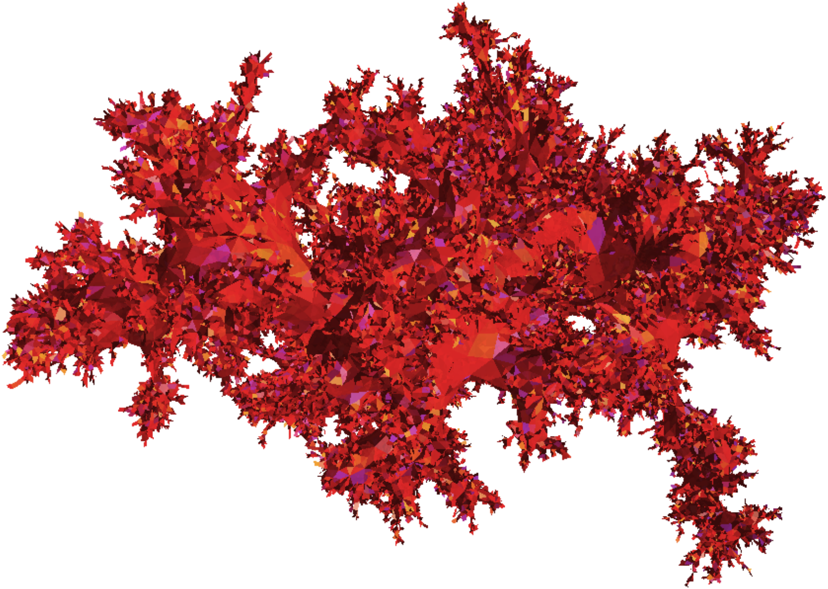
Constructive CFT

• Liouville CFT has a straightforward probabilistic definition.

• Probabilistic methods allow to prove some of its basic properties: Seiberg bounds, KPZ scaling and Ward identities.

• It remains to be seen if they can bridge the gap to the axiomatic-algebraic approach by justifying its assumptions on the spectrum and correlation functions.

$\gamma = \sqrt{2}$, ($c = -2$) Quantum Sphere



Punctures

We are interested in $\mathbb{S}^2$ with **3 marked points** z_1, z_2, z_3.

To have a constant negative curvature metric on $\mathbb{S}^2$ one needs to include **conical singularities**. Let us do this at the points z_1, z_2, z_3. Such metrics are extrema of the functional

$$S(X, g, \alpha_i, z_i) = S(X, g) - \sum_{i=1}^{3} \alpha_i X(z_i)$$

suitably renormalized. The angle of the cone at z_i is

$$\Omega_i = 2\pi(1 - \alpha_i/Q_{class}), \quad Q_{class} = \gamma/2$$

and one needs

$$\sum_i (2\pi - \Omega_i) > 4\pi \quad \text{i.e.} \quad \sum \alpha_i > 2Q_{class}.$$

hence since $\Omega_i > 0$ need at least three conical singularities.

Vertex operators

Do the same in the random case. The density

$$e^{-S(X,g,\alpha_i,z_i)} = \prod_{i=1}^{3} e^{\alpha_i X(z_i)} e^{-S(X,g)}$$

is defined in terms of the **vertex operators**

$$V_\alpha(z) := \lim_{\epsilon \to 0} e^{\gamma X_{g,\epsilon} - \frac{\gamma^2}{2}\mathbb{E}[X_{g,\epsilon}^2]}$$

Define the probability for punctured sphere

$$d\mathbb{P}_{\alpha_i,z_i}(X) := \frac{1}{Z_{\alpha_i,z_i}} \prod_{i=1}^{3} V_{\alpha_i}(z_i) d\mu_L(X)$$

with normalization

$$Z_{\alpha_i,z_i} = \int_{\mathbb{R}} e^{-2Qc} \Big(\mathbb{E}_{\hat{g}} \prod_{i=1}^{n} V_{\alpha_i}(z_i) e^{-\mu e^{\gamma c} M_{\hat{g}}(\mathbb{S}^2)} \Big) dc$$

Seiberg bounds

Now the c-integral becomes

$$\int_{\mathbb{R}} e^{(\sum \alpha_i - 2Q)c - \mu e^{\gamma c} M_{\hat{g}}(\mathbb{S}^2)}\, dc$$

This is finite if and only if $\sum \alpha_i > 2Q$.

Theorem Let $\sum \alpha_i > 2Q$. Then $Z < \infty$ and $Z > 0$ if and only if $\alpha_i < Q$ for all i.

Proof Girsanov theorem: punctures give rise to volume form

$$\prod_i |z - z_i|^{-\gamma \alpha_i} e^{\gamma X(z)} g(z) dz$$

Modulus of continuity of Chaos: integrable iff $\alpha_i < Q$.

Corollary. Indeed, need at least **three** vertex operators as in the classical case.

Planar maps with matter

Let $\mathcal{T}_N$ = triangulations of $\mathbb{S}^2$, N faces **with 3 marked faces**

- $T \in \mathcal{T}_N$ is a graph with topology of $\mathbb{S}^2$ and faces triangles
- For $\gamma \in [\sqrt{2}, 2]$ $\exists$ **critical lattice model** on the graph T.
  - $\gamma = \sqrt{2}$ spanning trees, $\gamma = \sqrt{8/3}$ percolation, $\gamma = \sqrt{3}$, Ising model, $\gamma = 2$ GFF
- $Z_\gamma(T)$ **partition function** of the model
  - $Z_{\sqrt{2}}(T) = \det \Delta_T$, $Z_{\sqrt{8/3}}(T) = 1$, $Z_2(T) = \det^{-\frac{1}{2}} \Delta_T$

For $\mu_0 > 0$ consider the probability $\mathbb{P}_{\mu_0,\gamma}$ on $\mathcal{T} = \cup_N \mathcal{T}_N$:

$$\mathbb{E}_{\mu_0,\gamma} F := \frac{1}{Z_{\mu_0,\gamma}} \sum_N e^{-\mu_0 N} \sum_{T \in \mathcal{T}_N} Z_\gamma(T) F(T)$$

Planar maps with matter

It is conjectured

$$\sum_{T \in \mathcal{T}_N} Z_\gamma(T) = N^{1-\frac{4}{\gamma^2}} e^{\bar{\mu} N}(1 + o(1)).$$

Hence

$$\lim_{\mu_0 \downarrow \bar{\mu}} \mathbb{E}_{\mu_0,\gamma} |T| \to \sum_N N^{2-\frac{4}{\gamma^2}} = \infty$$

provided $\gamma \in [\sqrt{2}, 2]$.

As $\mu_0 \downarrow \bar{\mu}$ $\mathbb{P}_{\mu_0,\gamma}$ concentrated on large triangulations.

Random measure on $\mathbb{S}^2$

Conformal structure on T: triangles equilateral area 1.

Map T conformally to $\mathbb{S}^2$ s.t. marked faces map to z_1, z_2, z_3.

Image of volume on T is a measure $\nu_T(dz)$ on $\mathbb{S}^2$.

Under $\mathbb{P}_{\mu_0,\gamma}$, ν_T becomes a **random measure** $\nu_{\mu_0,\gamma}$ on $\mathbb{S}^2$.

Scaling limit

As $\mu_0 \downarrow \bar{\mu}$ typical size of triangulation diverges.

Let $\mu > 0$ and define

$$\rho^{(\epsilon)}_{\mu,\gamma} := \epsilon \nu_{\bar{\mu}+\epsilon\mu,\gamma}$$

Conjecture. $\rho^{(\epsilon)}_{\mu,\gamma}$ converges in law as $\epsilon \to 0$ to a random multifractal measure $\rho_{\mu,\gamma}$ on $\mathbb{S}^2$.

Since $\epsilon \nu_T(\mathbb{S}^2) = \epsilon N$ the law of total volume $\rho^{(\epsilon)}_{\mu,\gamma}(\mathbb{S}^2)$ is:

$$\mathbb{E}[F(\rho^{(\epsilon)}_{\mu,\gamma}(\mathbb{S}^2))] = \frac{1}{Z_\epsilon} \sum_N e^{-\mu\epsilon N} N^{1-\frac{4}{\gamma^2}} F(\epsilon N).$$

It converges to $\Gamma(2 - \frac{4}{\gamma^2}, \mu)$ as $\epsilon \to 0$.

We will construct a measure $\rho_{\mu,\gamma}$ on $\mathbb{S}^2$ with this law for its total mass.

2d Gravity a la Polyakov

Polyakov (81), Kniznik, Polyakov, Zamolochikov (88):

$$\rho_{\mu,\gamma}(dz) = e^{\gamma X(z)} dz$$

$X(z)$ is a random field on $\mathbb{S}^2$, **Liouville CFT**

The law $\mathbb{P}_{\gamma,\mu}$ of X is formally given by functional integral

$$\mathbb{E}_{\gamma,\mu} f(X) = Z^{-1} \int f(X) \, e^{-S(X,g)} DX \tag{1}$$

where S is the **Liouville** action functional:

$$S(X,g) := \int_{\mathbb{S}^2} \left(|\nabla^g X|^2 + Q R_g X + \mu e^{\gamma X} \right) g dz$$

- $g = g(z) dz^2$ is any smooth conformal metric on $\mathbb{S}^2$
- $R_g = -\Delta \log g$ scalar curvature
- $Q = 2/\gamma + \gamma/2$

Classical Liouville theory

If we take

$$Q = 2/\gamma$$

extrema of $S(X, g)$ are solutions of **Liouville equation**

$$R_{e^{\gamma X} g} = -\frac{1}{2}\mu\gamma^2.$$

Solutions define metrics $e^{\gamma X} g$ with constant negative curvature and lead to the uniformisation theorem of Riemann surfaces.

For the quantum (random) case we need to take

$$Q = 2/\gamma + \gamma/2.$$

The resulting quantum geometry has interesting parallels with the classical one.

Quantum Liouville theory

We want to give meaning to the integral

$$\int F(X)e^{-\int_{\mathbb{S}^2}(|\nabla^g X|^2+QRX+\mu e^{\gamma X})gdz}DX$$

by viewing it as a perturbation of

$$e^{-\frac{1}{4\pi}\int_{\mathbb{S}^2}(|\nabla^g X|^2dz}DX \tag{2}$$

which looks like a Gaussian measure.

Indeed, one could interpret (2) as the **Gaussian Free Field** on $\mathbb{S}^2$ i.e. the Gaussian measure with covariance $(-\Delta_g)^{-1}$.

Constant mode

However, GFF is defined only **up to an additive constant** since Δ_g annihilates constants.

- We want to include constant fields to DX: the "gaussian" measure will **not** be a probability measure
- Quadratic part of action is independent on metric:

$$\int_{\mathbb{S}^2} |\nabla^g X|^2 \, g dz = \int_{\mathbb{S}^2} |\nabla X|^2 \, dz.$$

so look for a measure having this property.

Gaussian Free Field (GFF)

Let X_g be Gaussian Free Field with zero average in metric g:

$$\langle X_g \rangle_g := \frac{1}{vol_g(\mathbb{S}^2)} \int_{\mathbb{S}^2} X_g(z) g(z) dz = 0$$

As g varies the fields differ by random additive constants:

$$X_g - \langle X_g \rangle_{g'} \overset{law}{=} X_{g'}.$$

Define

$$X = X_g + c$$

where c is uniform on $\mathbb{R}$.

"Law" of X is independent of g