On Makarov's principle in conformal mapping

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Dimensions of Quasicircles

Find $D(k)$, the maximal dimension of a k-quasicircle, the image of $\mathbb{S}^1$ under a k-quasiconformal mapping of the plane,

$$\text{homeomorphism}, \qquad \overline{\partial} w^{\mu}(z) = \mu(z) \cdot \partial w^{\mu}(z), \qquad \|\mu\|_{\infty} \leq k.$$

Theorem: (Becker-Pommerenke, 1987)

$$D(k) \leq 1 + 36\, k^2 + \mathcal{O}(k^3).$$

Astala's conjecture: (proved by Smirnov)

$$D(k) \leq 1 + k^2, \qquad \text{for } 0 < k < 1.$$

Bloch functions

Let b be a Bloch function on $\mathbb{D}$, i.e. a holomorphic function satisfying

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|b'(z)| < \infty.$$

Examples:

$$\log f', \qquad f : \mathbb{D} \to \mathbb{C} \ \text{ conformal}$$

$$P\mu = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\mu(w)}{(1-z\overline{w})^2}\,|dw|^2, \quad \mu \in L^\infty(\mathbb{D}).$$

Lacunary series:

$$z + z^2 + z^4 + z^8 + \dots$$

Asymptotic variance

For a Bloch function, define its asymptotic variance by

$$\sigma^2(b) = \limsup_{r \to 1^-} \frac{1}{2\pi |\log(1-r)|} \int_{|z|=r} |b(z)|^2 \, |dz|,$$

Set

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$$\text{(AIPP)} \quad 0.879 \le \Sigma^2 \le 1, \qquad \text{(Hedenmalm)} \quad \Sigma^2 < 1,$$

$$D(k) \;=\; 1 + k^2\,\Sigma^2 + \mathcal{O}(k^{8/3}),$$

$$\text{(Prause – Smirnov)} \quad D(k) < 1 + k^2 \quad \text{for all } 0 < k < 1.$$

McMullen's identity

Suppose μ is a **dynamical** Beltrami coefficient on the disk, either

- invariant under a co-compact Fuchsian group Γ,
- or eventually invariant under a Blaschke product $f(z)$.

Then,

$$\begin{aligned} 2\frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{M.dim}\, w^{t\mu}(\mathbb{S}^1) &= \sigma^2\left(\frac{d}{dt}\bigg|_{t=0} \log(w^{t\mu})'\right), \\ &= \sigma^2(P\mu), \\ &= \|\mu\|^2_{\mathrm{WP}}, \end{aligned}$$

where $\|\cdot\|^2_{\mathrm{WP}}$ is the Weil-Petersson metric.

Fractal approximation (AIPP)

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One argument to prove $\Sigma^2 \leq 1$:

$$\Sigma^2 = \sup_{|\mu| \leq \chi_{\mathbb{D}},\ \mu \in M_{\mathrm{I}}} \sigma^2(P\mu),$$

where

$$M_{\mathrm{I}} = \bigcup_{d \geq 2} M_{\mathrm{I}}(d), \qquad (z^d)^* \mu = \mu$$

in *some* neighbourhood of $\mathbb{S}^1$.

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Theorem: Fuchsian approximation does not work: $\Sigma_{\mathrm{F}}^2 < 2/3$.

Extremals are Gaussians

Theorem: Suppose μ is close to an extremal,

$$\frac{1}{2\pi|\log(1-r)|}\int_{|z|=r}|P\mu(z)|^2\,|dz| \;\geq\; \Sigma^2-\delta, \quad r\approx 1.$$

Then, as a random variable in $\theta\in[0,2\pi)$,

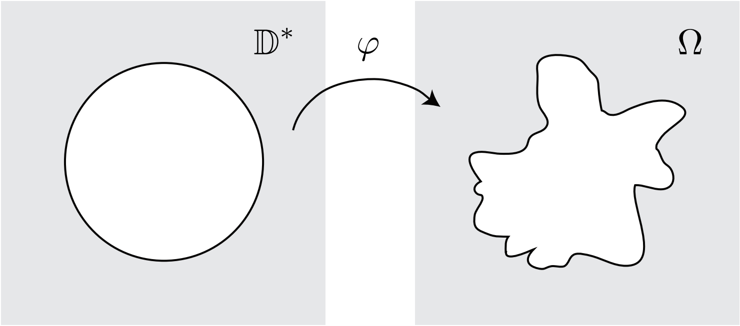
$$\frac{P\mu(re^{i\theta})}{\sqrt{\log\frac{1}{1-r}}} \approx \mathcal{N}_{\mathbb{C}}(0,\Sigma^2),$$

up to an additive error ε.

In other words, extremality invokes fractal structure.

Riemann Mapping Theorem

Let $\mathbb{D}^* = \{z : |z| > 1\}$ be the exterior unit disk.



"Complexity of the boundary $\partial\Omega$" is manifested in the "complexity of the Riemann map".

Makarov's theorem

In the 1980s, Makarov proved the following remarkable result:

Theorem: Suppose Ω is any simply connected domain, bounded by a Jordan curve. Then, the harmonic measure on $\partial\Omega$ has Hausdorff dimension 1.

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Makarov's principle: If $\partial\Omega$ is a regular fractal, then $\log|f'|$ behaves like a Gaussian random variable

$$N(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

Characteristics measuring σ^2

For $b = \log f'$, define its asymptotic variance by

$$\sigma^2(b) = \limsup_{r \to 1^-} \frac{1}{2\pi \cdot \log \frac{1}{1-r}} \int_{|z|=r} |b(z)|^2 \, |dz|,$$

and LIL constant $C^2_{\mathrm{LIL}}(b) = \operatorname{ess\,sup}_{\theta \in [0,2\pi)} C^2_{\mathrm{LIL}}(b,\theta)$ where

$$C_{\mathrm{LIL}}(b,\theta) = \limsup_{r \to 1^-} \frac{|b(re^{i\theta})|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}}.$$

Integral means spectra

For a conformal map $f : \mathbb{D} \to \Omega$, the integral means spectrum is given by

$$\beta_f(p) = \limsup_{r \to 1^-} \frac{\log \int_{|z|=r} |f'(z)^p| \, |dz|}{\log \frac{1}{1-r}}, \qquad p \in \mathbb{C}.$$

Problem: Find the **universal** integral means spectrum

$$B(p) := \sup_f \beta_f(p),$$

Kraetzer's conjecture. Is it $|p|^2/4$, for $|p| \leq 2$?

$$B(-2) = 1 \text{ ?} \quad B(1) = 1/4 \text{ ?} \qquad \text{Probably false.}$$

Equality of Characteristics

Przytycki, Urbański, Zdunik, Makarov, Binder, McMullen...

Dynamical setting: If $\partial\Omega$ is a regular fractal, e.g. a Julia set or a limit set of a quasi-Fuchsian group, then

$$2\frac{d^2}{dp^2}\bigg|_{p=0}\beta_f(p) \;=\; \sigma^2(\log f') \;=\; C^2_{\mathrm{LIL}}(\log f').$$

Set

$$h(t) = t\,\exp\left\{C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}\right\}, \qquad 0 < t < 10^{-7}.$$

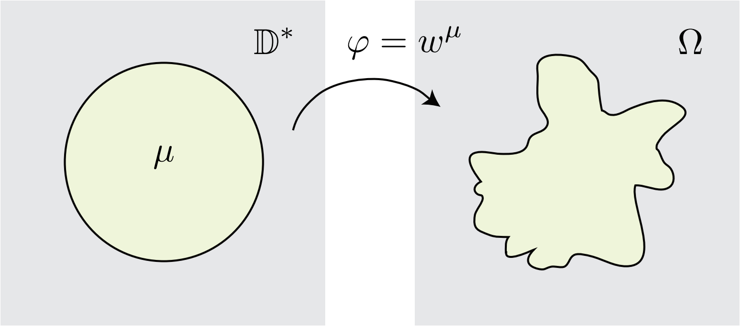
Then, $\omega \ll \Lambda_{h(t)}$ for $C \geq \sqrt{\sigma^2}$ and $\omega \perp \Lambda_{h(t)}$ for $C < \sqrt{\sigma^2}$.

Universal Teichmüller space

By definition,

$$\mathcal{T}(\mathbb{D}^*) := \bigcup_{0 \le k < 1} \mathbf{\Sigma}_k,$$

where $\mathbf{\Sigma}_k = \{\varphi :$ admit a k-quasiconformal extension to $\mathbb{C}\}$.



Equality of Characteristics

Theorem: (partly joint with I. Kayumov)

$$2\frac{d^2}{dp^2}\bigg|_{p=0} B_k(p) \;=\; \sup_{\varphi\in\boldsymbol{\Sigma}_k} \sigma^2(\log\varphi') = \sup_{\varphi\in\boldsymbol{\Sigma}_k} C^2_{\mathrm{LIL}}(\log\varphi'),$$

where $\Sigma^2(k)/k^2$ is a convex non-decreasing function of $k \in [0,1)$.

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Additionally, $\omega \ll \Lambda_{h(t)}$ for $C \geq \sqrt{\Sigma^2(k)}$. For any $C < \sqrt{\Sigma^2(k)}$, there exists a domain Ω such that $\omega \perp \Lambda_{h(t)}$.

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Theorem: (AIPP; Hedenmalm, Shimorin, Kayumov)

$$0.93 < \Sigma^2(1^-) < 1.24^2.$$

Bloch Martingales

Let b be a Bloch function on $\mathbb{D}$, satisfying

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|b'(z)| < \infty.$$

Identify $\mathbb{S}^1 \sim \mathbb{R}/\mathbb{Z}$ in the usual way. For a dyadic interval I, define

$$B_I = \lim_{r\to 1} \frac{1}{|I|} \int_I b(re^{i\theta})d\theta.$$

This is clearly a martingale, that is, if $I = I_1 \cup I_2$, then

$$B_I = \frac{B_{I_1} + B_{I_2}}{2}.$$

Bloch Martingales (cont.)

The local variance is defined as

$$\mathrm{Var}_I^n = \frac{1}{n \cdot 2^n} \sum_{j=1}^{2^n} |\Delta_j(x)|^2.$$

where $\Delta_j = B_{I_j}(x) - B_I(x)$ and $\{I_j\}$ ranges over generation n of children of I.

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Universal bounds: If $b = P\mu$, $|\mu| \le \chi_{\mathbb{D}}$, then $\fint \le \Sigma^2 + \mathcal{O}(1/n)$.

Dynamical coefficients: $\sigma^2 - \varepsilon \le \mathrm{Var}_I^n \le \sigma^2 + \varepsilon$ if n is large.

Thank you for your attention!