Network models, quenched quantum gravity, and critical behavior at quantum Hall transitions

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General setting

- Critical points in 2D disordered systems, Anderson transitions
- Statistical treatment: all observables are random, need to find their distribution (or the mean and the moments)
- Average over disorder leads to conformal field theory with $c = 0$
- SLE connection: $\kappa = 8/3$ (SAW) or $\kappa = 6$ (percolation)
- Many more critical points
- Conformal restriction measures and $\mathrm{SLE}(8/3, \rho)$
- Recent development: coupling to quantum gravity

Integer quantum Hall effect

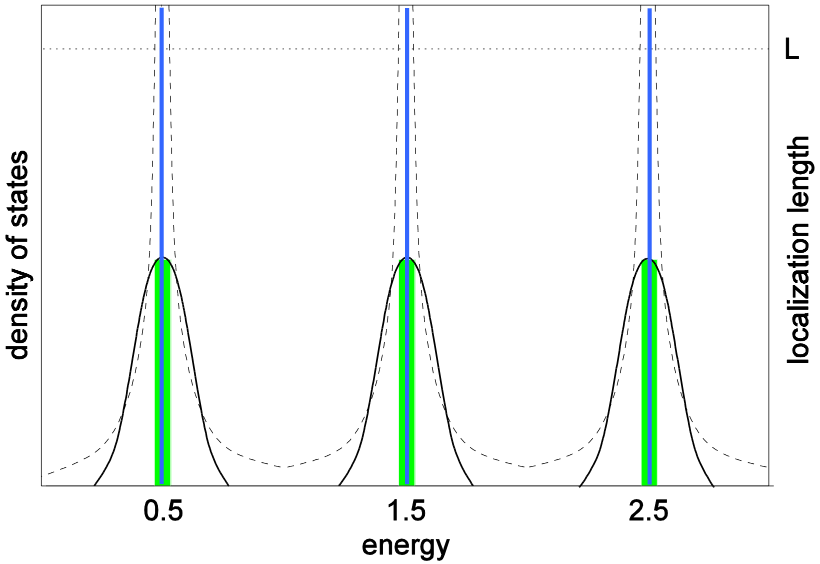
- Single electron in a magnetic field and a random potential

$$H = \frac{1}{2m}\Big(-i\hbar\nabla + \frac{e}{c}\mathbf{A}\Big)^2 + U(\mathbf{r})$$

- Most states are localized
- Extended states at E_n
- Localization length diverges

$$\xi(E) \propto |E - E_n|^{-\nu}$$

- Universal critical exponent ν

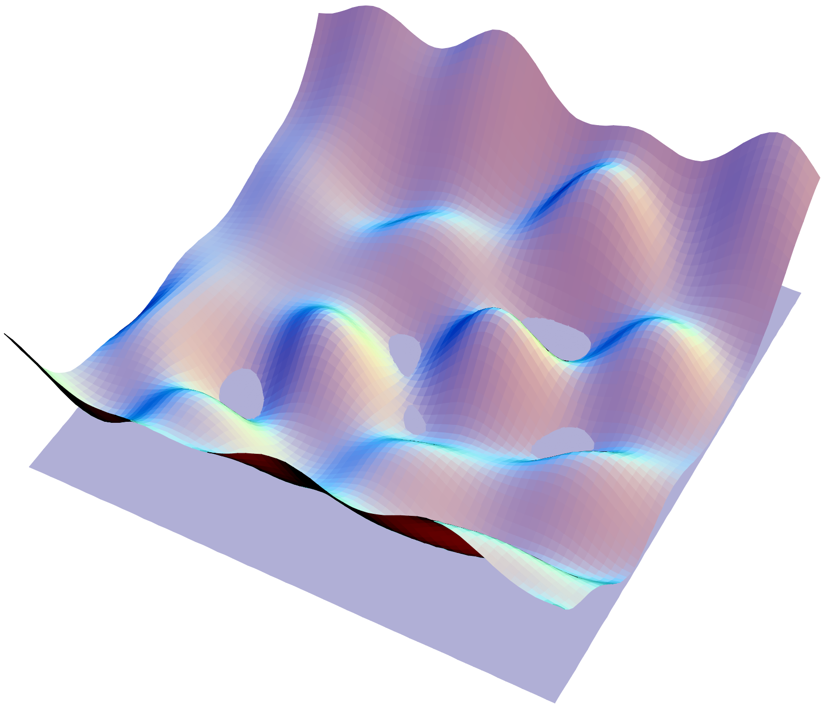


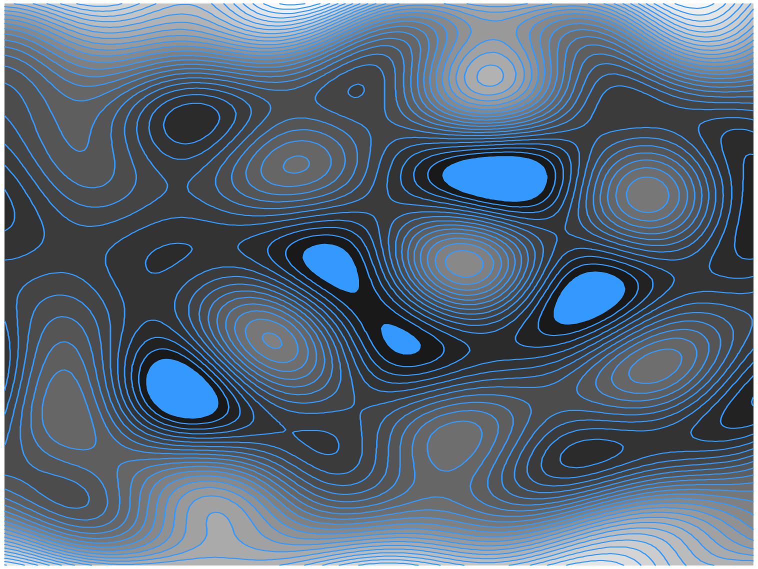
Theory of IQH plateau transition

- Goals for a theory of the transition:
  - Critical exponents and scaling functions
  - Correlation functions at the transition
- Expect conformal invariance at the transition (confirmed numerically)
- A lot of intuition comes from a network model, also numerics

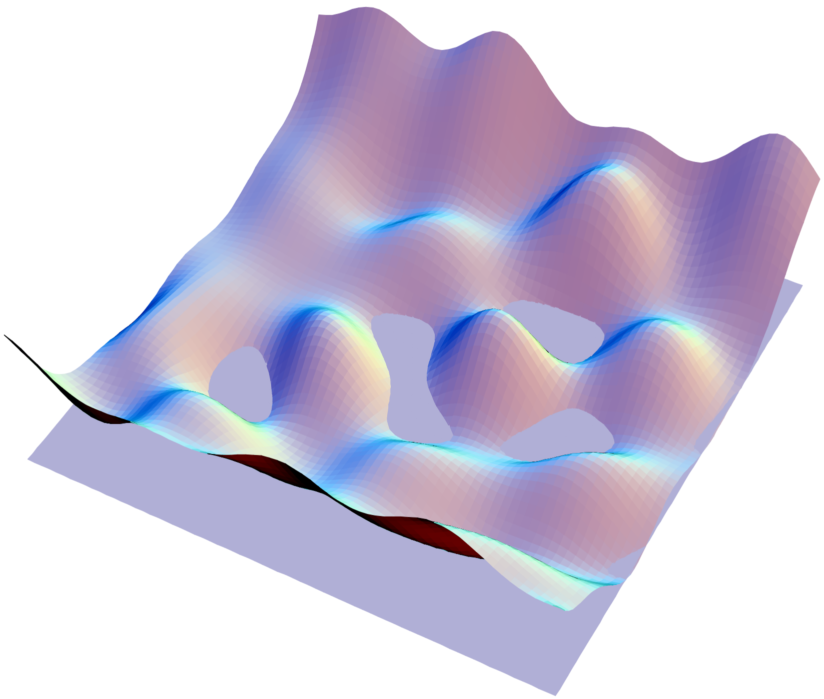
J. T. Chalker and P. D. Coddington `88

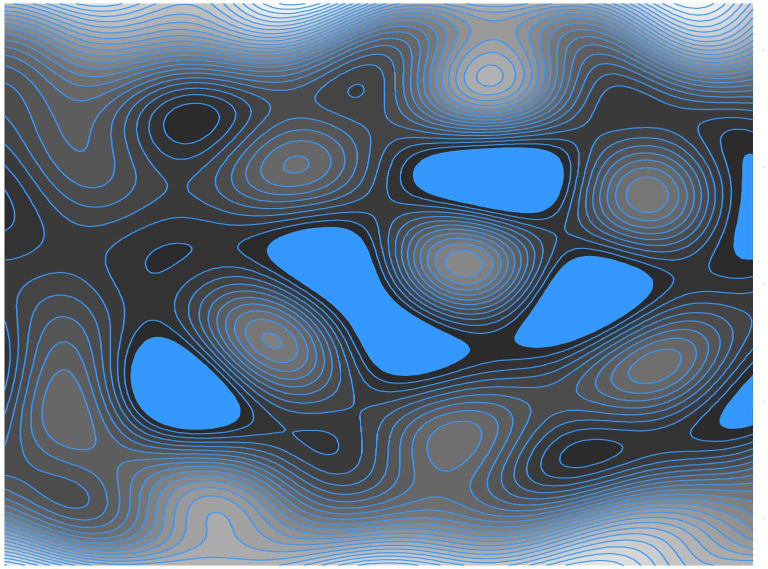
Motivation for the network model: Electrons in a smooth random potential



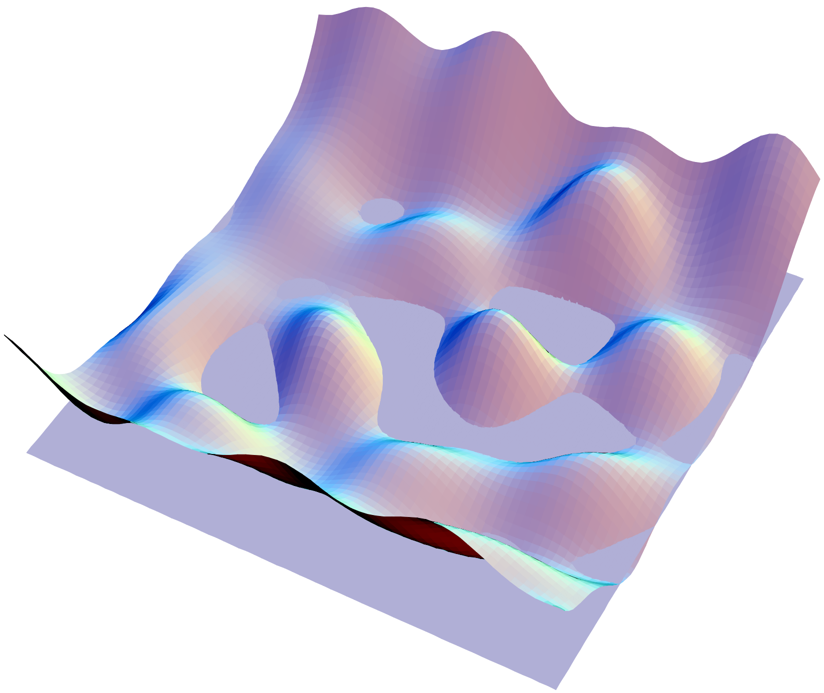


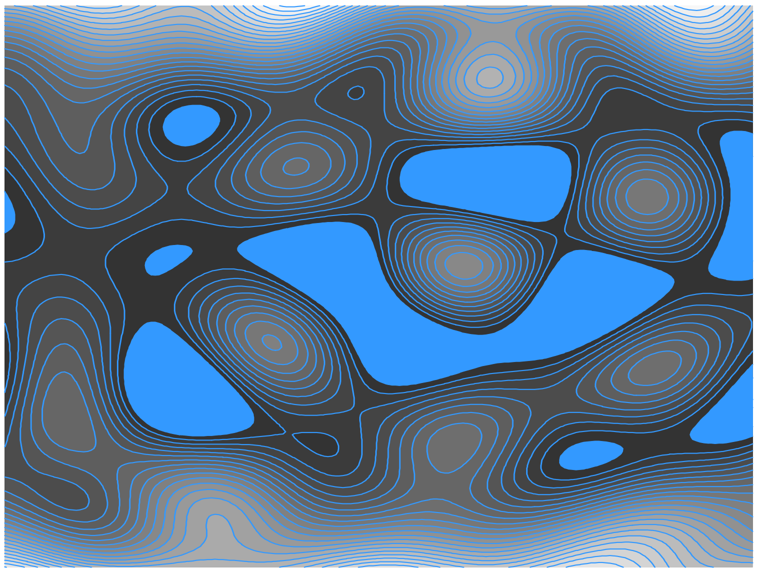
Electrons in smooth random potential



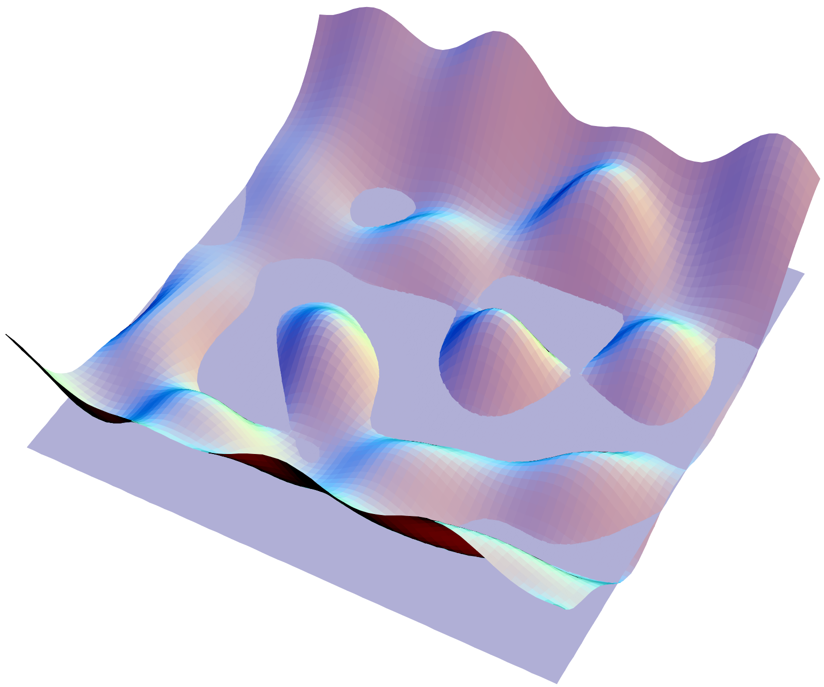


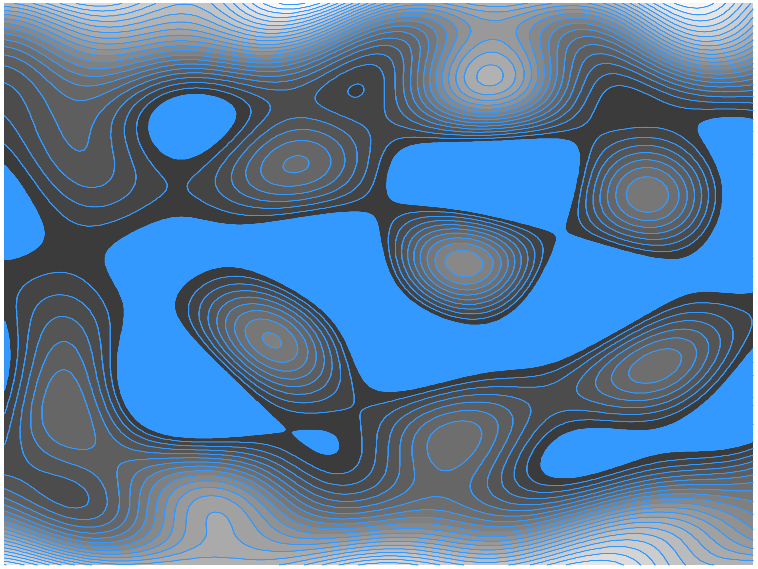
Electrons in smooth random potential



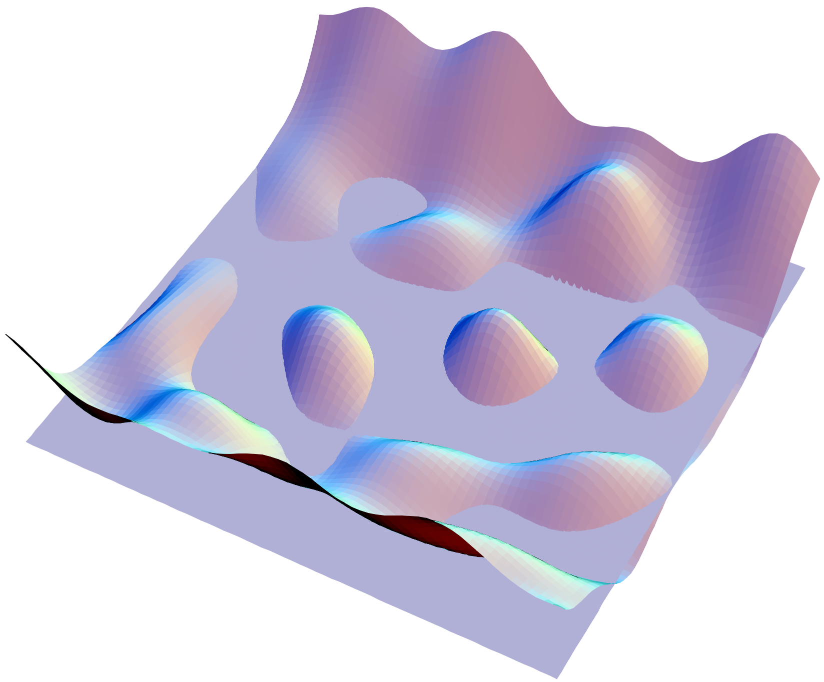


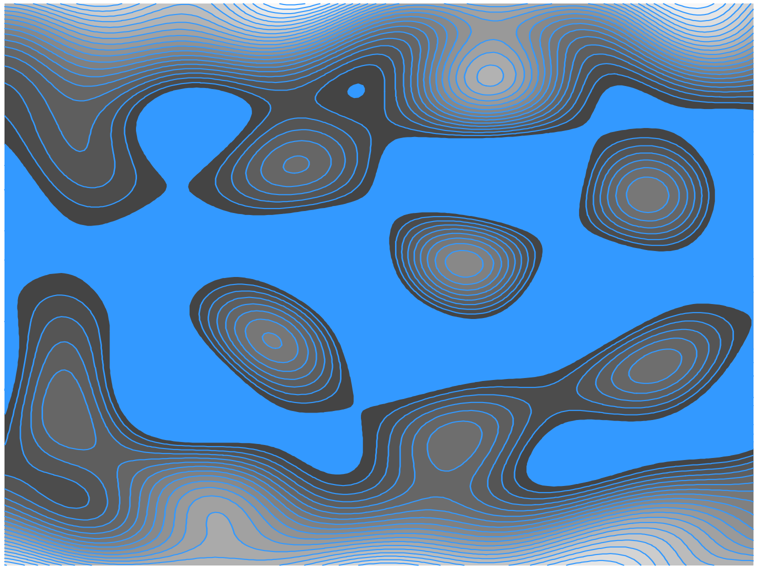
Electrons in smooth random potential



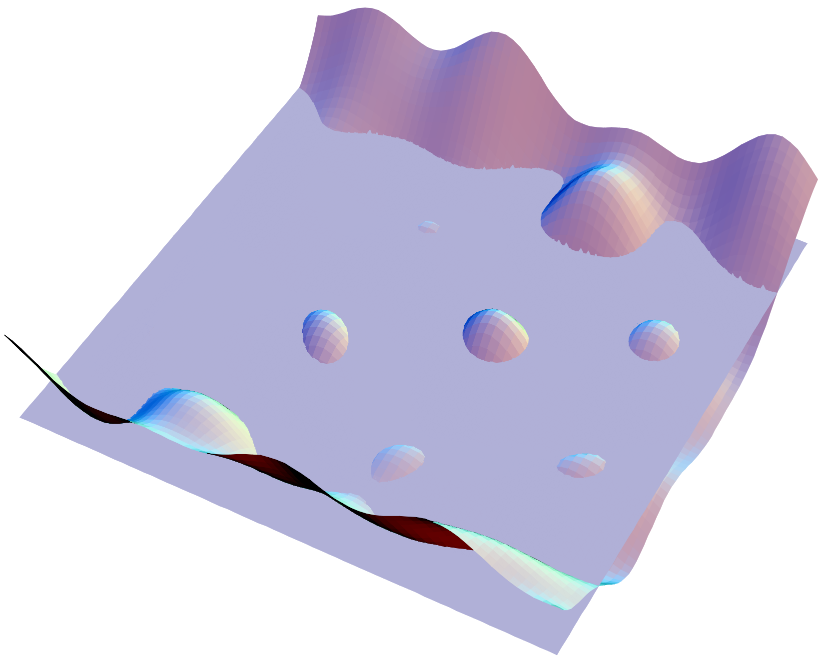


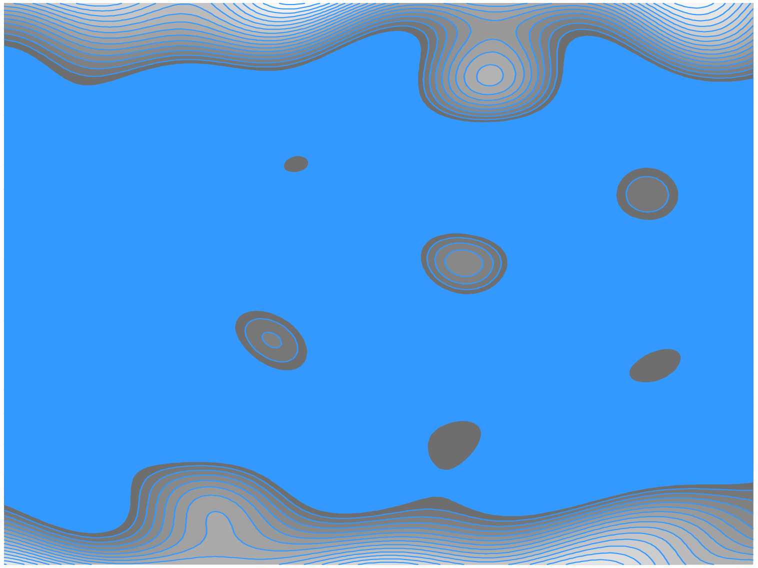
Electrons in smooth random potential





Electrons in smooth random potential



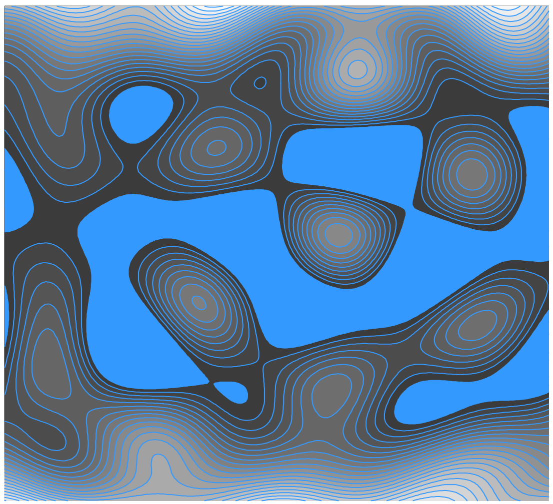


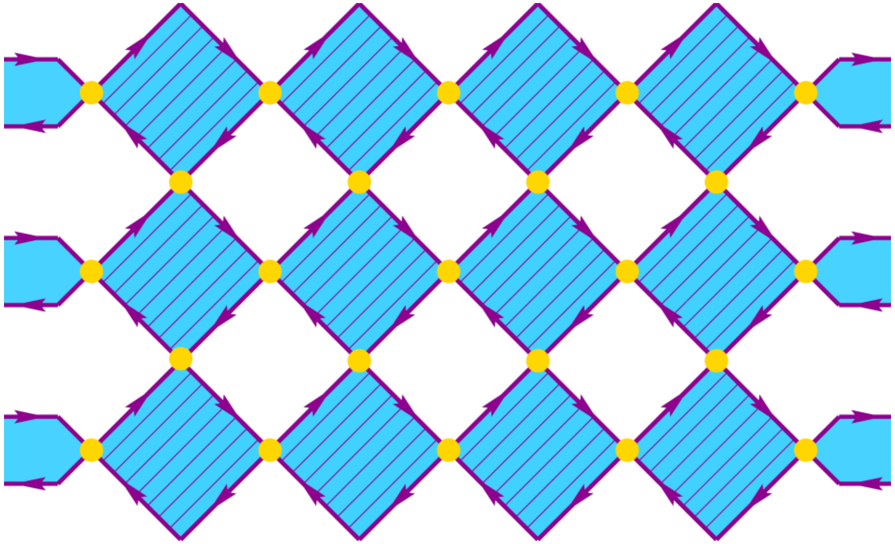
- The picture resembles classical percolation
- Essential difference:
  - tunneling across saddle points
  - quantum interference and random phases

Chalker-Coddington network model

J. T. Chalker, P. D. Coddington `88

- Obtained from semi-classical drifting orbits in smooth potential

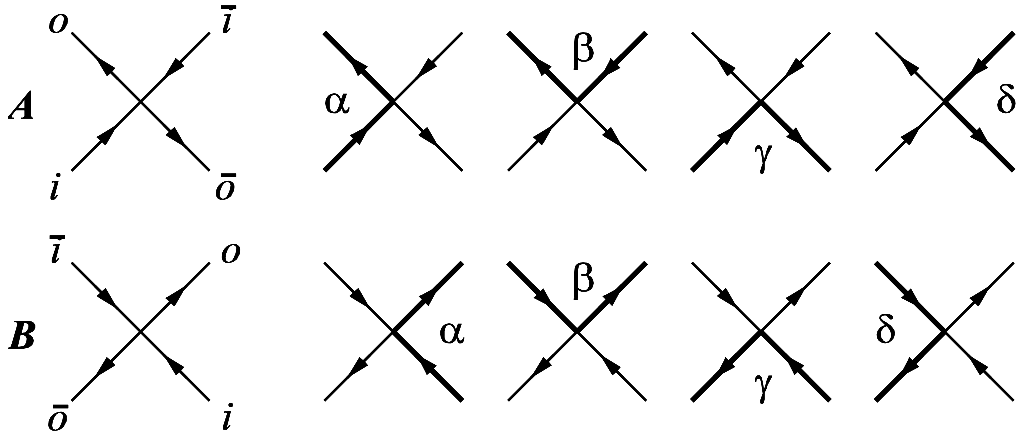




- Complex fluxes (currents) on links, scattering at nodes
- Regular lattice is convenient for numerical transfer matrix calculations

Chalker-Coddington network model

- States of the system specified by $Z \in \mathbb{C}^{N_l}$, N_l the number of links
- Evolution (discrete time) specified by a random $U \in \mathrm{U}(N_l)$

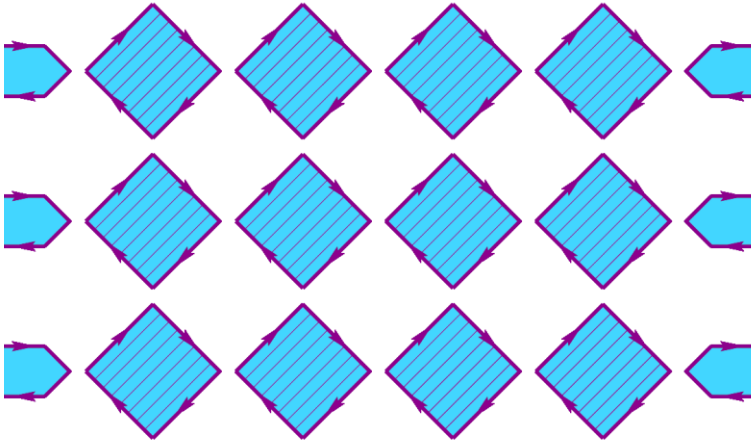


$$\begin{pmatrix} o \\ \bar{o} \end{pmatrix} = \mathcal{S} \begin{pmatrix} i \\ \bar{\imath} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} i \\ \bar{\imath} \end{pmatrix}$$

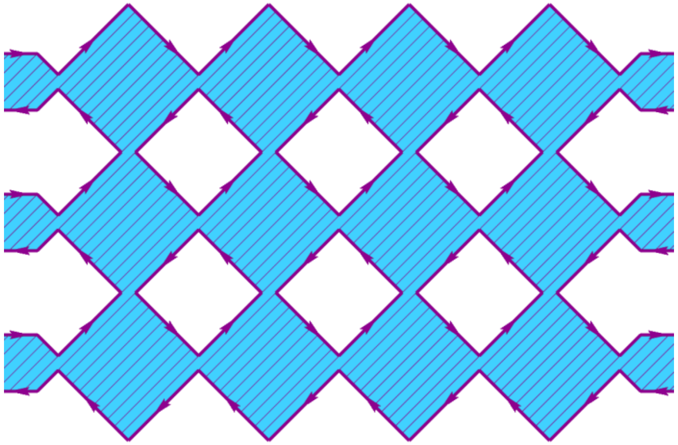
$$\mathcal{S}_S = \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \begin{pmatrix} \sqrt{1-t_S^2} & t_S \\ -t_S & \sqrt{1-t_S^2} \end{pmatrix} \begin{pmatrix} e^{i\phi_3} & 0 \\ 0 & e^{i\phi_4} \end{pmatrix}$$

Chalker-Coddington network model

- Extreme limits in the isotropic case (reminds percolation)



$t_A = 1$ Insulator



$t_A = 0$ Quantum Hall

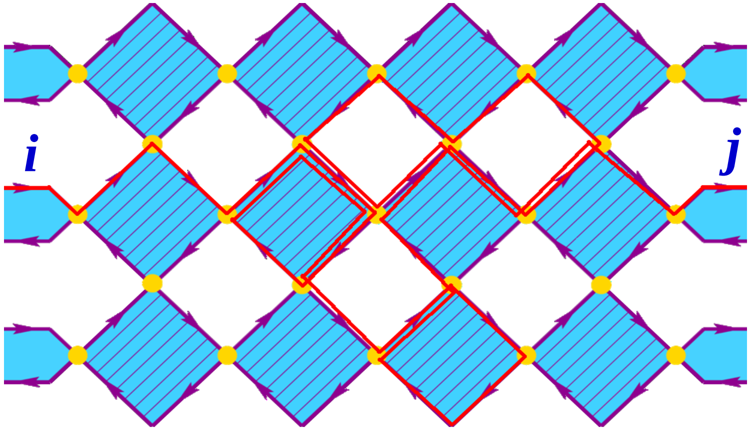
- Critical point at $t_A^2 = 1/2$

Observables in network model

- Propagator (resolvent matrix element) $G_{ij} = \langle i \big| (1 - e^{-\eta} U)^{-1} \big| j \rangle$
- Graphical representation in terms of a sum over (Feynman) paths

$$G_{ij} = \sum_{f: i \to j} A_{ij}(f)$$

$$A_{ij}(f) = \prod_{\mathrm{nodes} \in f} S_{ab}$$

$A_{ij}(f) =$ 

- Point contact conductance (PCC) $g_{ij} = |G_{ij}|^2$
- Its distribution (the mean and the moments)

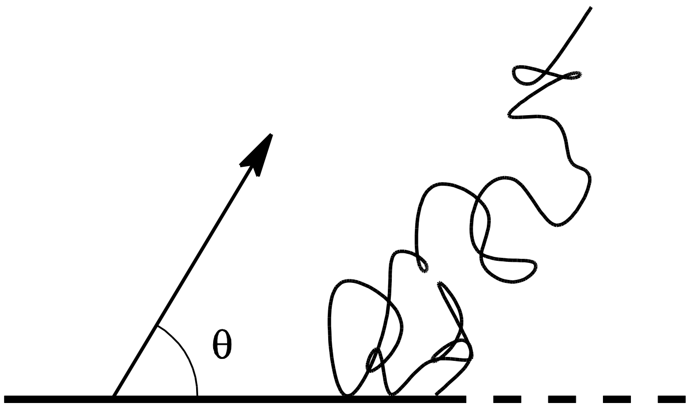
Our approach

E. Bettelheim, IAG, A. W. W. Ludwig, 2012

- We consider point contact conductances (PCC) within the Chalker-Coddington network model
- Map average PCC to a *classical* problem
- Establish a restriction property that in the continuum limit allows us to use the theory of conformal restriction
- Obtain PCC in systems with various boundary conditions

“Baby” version: classical CC network model

- Classical random walk on the CC network
- Right (left) turns with probability $p_R = t^2 \; (p_L = 1 - t^2)$
- This model is not random
- Continuum limit – reflected Brownian motion (diffusion in a magnetic field)



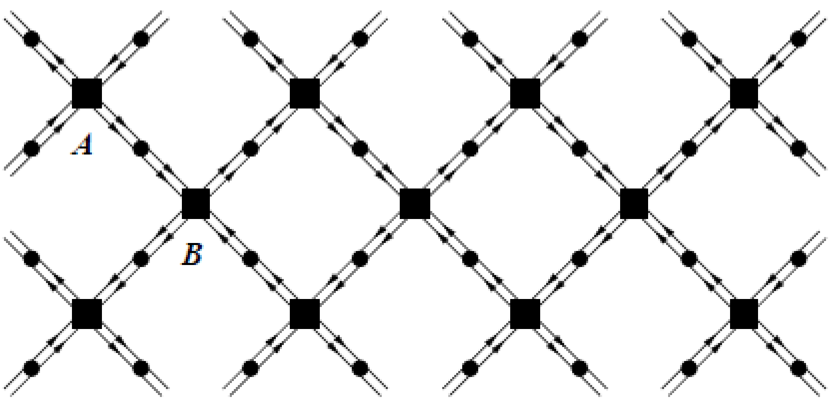
$$\tan\theta_R = \frac{p_L}{p_R}$$

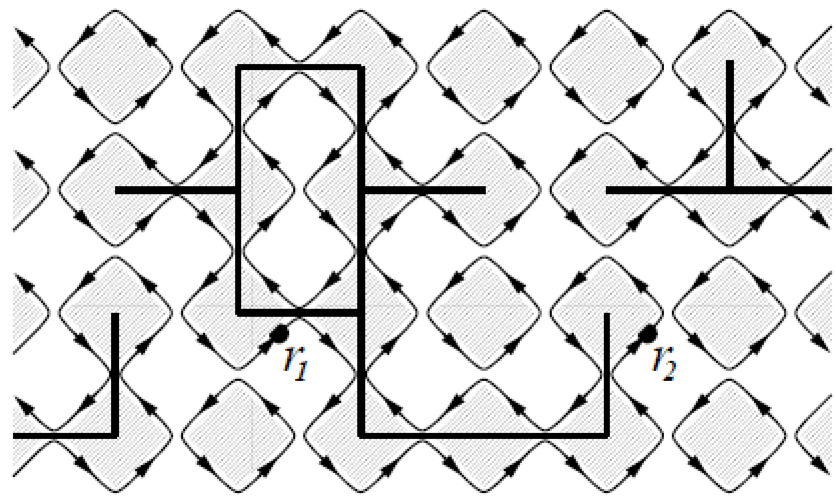
$$\tan\theta_L = -\frac{p_R}{p_L}$$

Another network model

IAG, A. W. W. Ludwig, N. Read, 1999

- Spin quantum Hall (SQH) transition
- Two channels on links, fluxes mixed by $\mathrm{SU}(2)$ matrices
- Average over $\mathrm{SU}(2)$: mapping to bond percolation on square lattice

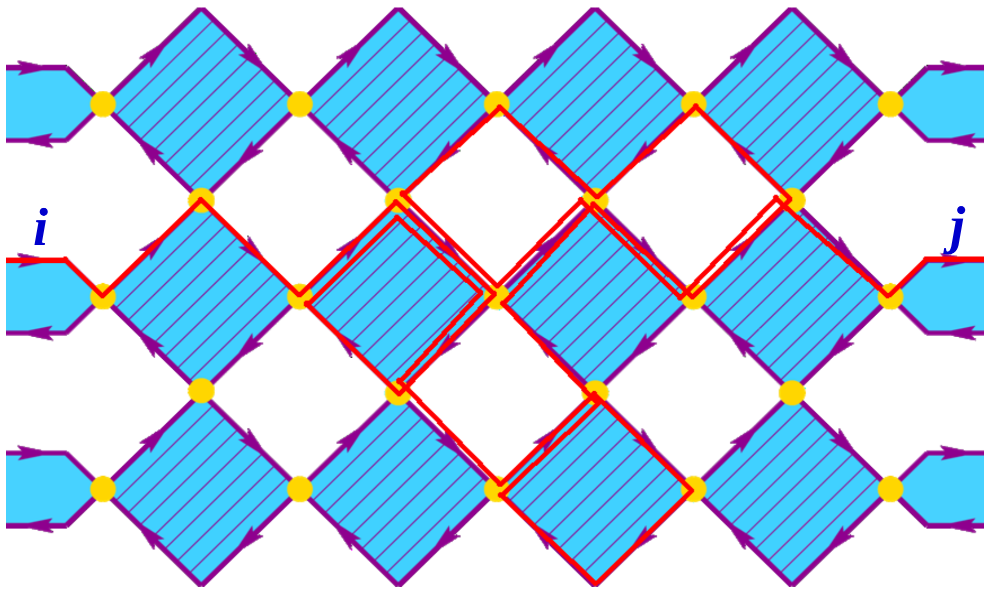




Quantum propagation in CC network model

- Graphical representation of propagator as a sum over (Feynman) paths

$$G_{ij} = \sum_{f:i\to j} A_{ij}(f), \qquad A_{ij}(f) = \prod_{\text{nodes}\in f} S_{ab}$$

$A_{ij}(f) =$ 

- Point contact conductance (PCC) $g_{ij} = |G_{ij}|^2$

Mapping to a classical problem

- Average point contact conductance over random phases on links

$$\langle g\rangle = \langle |A|^2\rangle = \sum_f \langle |A(f)|^2\rangle + \sum_{f_1\neq f_2} \langle A^*(f_1)A(f_2)\rangle$$

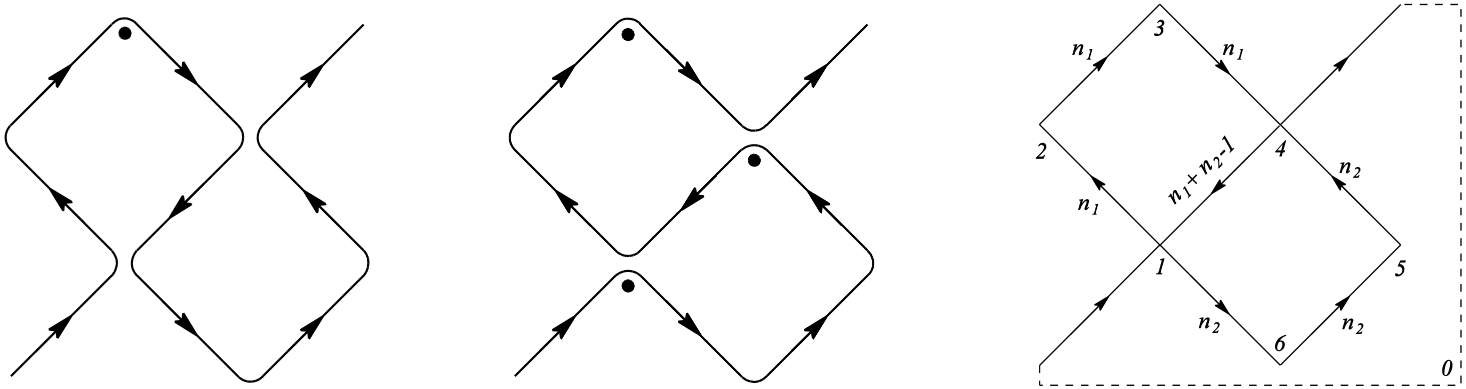
- Can write as a sum of positive terms $\langle g\rangle = \sum_p W(p)$

- $W(p)$ are intrinsic positive weights of "pictures" p

- This representation is valid at and away from the critical point as well as for anisotropic variants of the model

Pictures and paths

- Picture is obtained by “forgetting” the order in which links are traversed



$$W(p) = S^2(p), \qquad S(p) = 2^{-N(p)/2} \sum_{f \in F(p)} (-1)^{N_-(f)}$$

- Detailed analysis of the weights $W(p)$ may lead to a complete solution

- We try to go to continuum directly using restriction property

Restriction in the CC model

- Weights of pictures $W(p)$ are *intrinsic*
- Pictures satisfy restriction property with respect to *absorbing* boundaries
- Assume conformal invariance, then can use conformal restriction theory

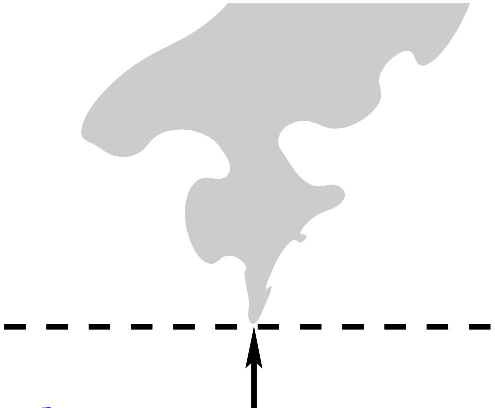
G. Lawler, O. Schramm, and W. Werner '03

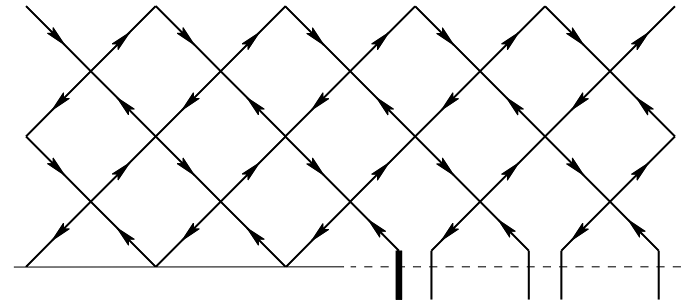
- Average point contact conductances are restriction partition functions

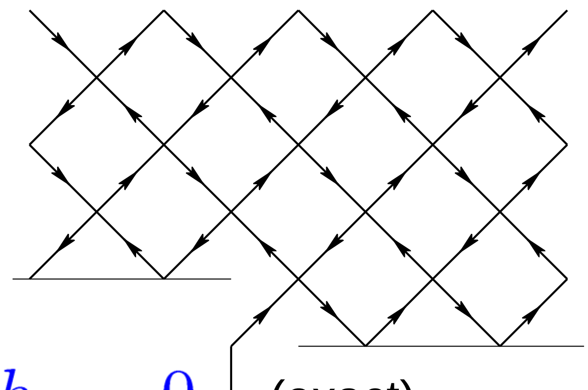
$$\langle g(a,b) \rangle = Z_D(a,b)$$

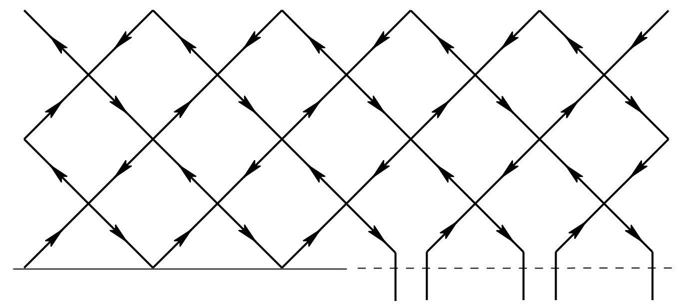
- Current insertions create pictures, and are *primary* CFT operators
- Important to know their scaling dimensions (restriction exponents)
- Explicit analytical results for average PCC with various boundaries

Boundary operators and dimensions



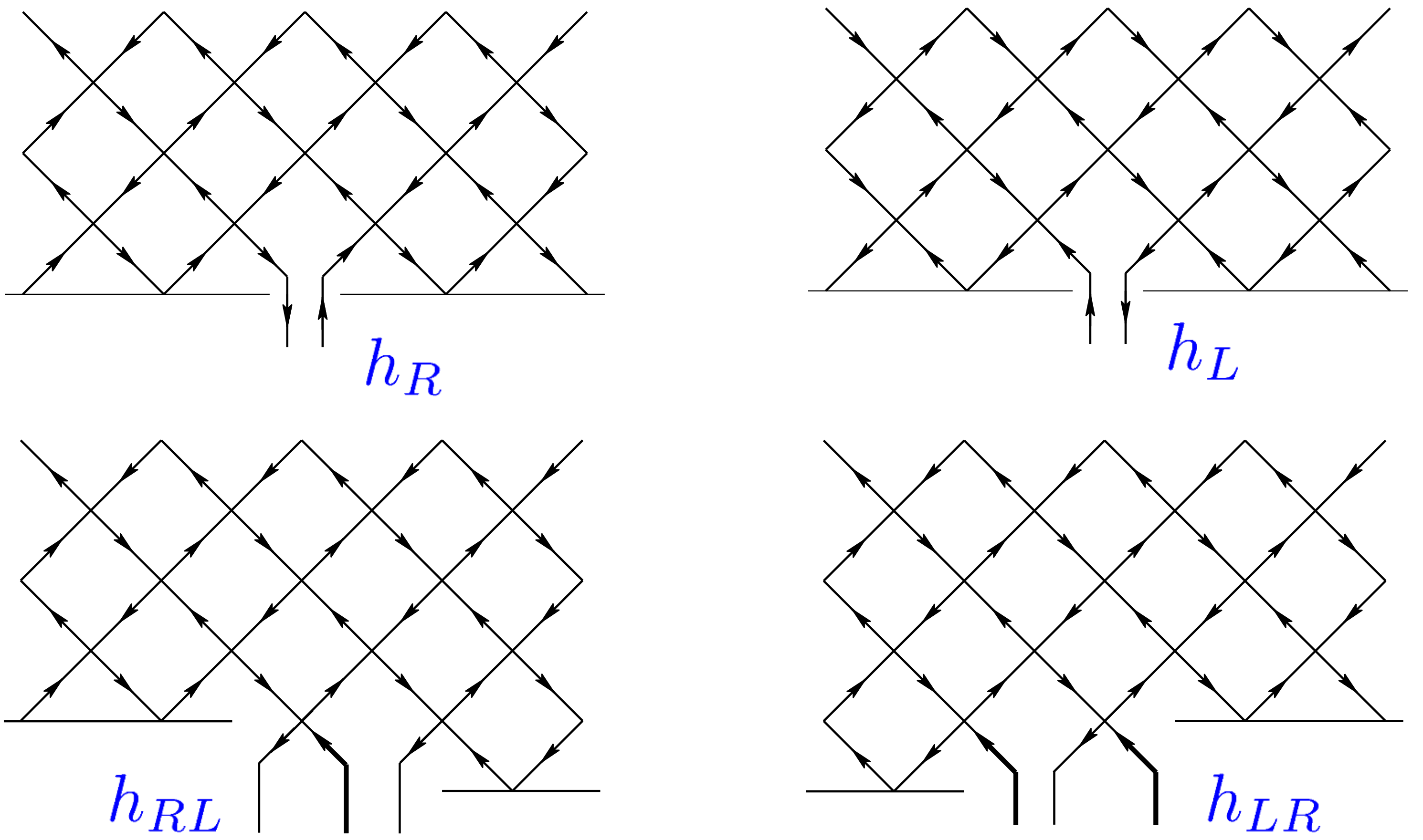






- Dimensions known numerically

Boundary operators and dimensions

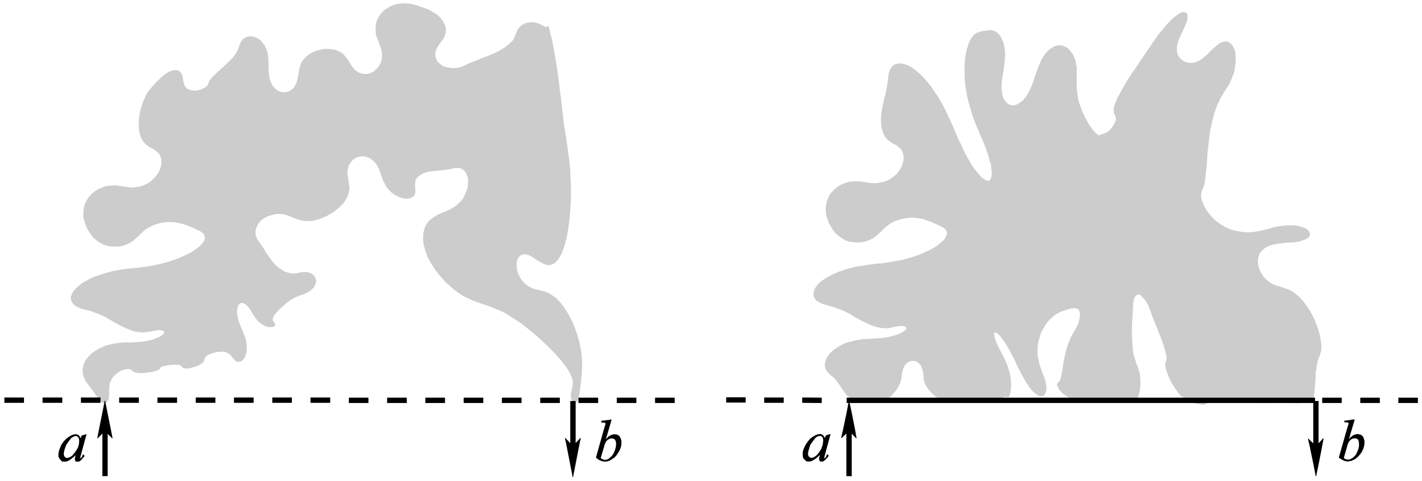


- Dimensions known numerically

Boundary operators and dimensions

	IQH transition	SQH transition	Classical CC model
h_A	1	$h_{1,4}(6) = 1$	1
h_T	2	$h_{1,5}(6) = 2$	2
h_l	1	$h_{1,4}(6) = 1$	1
h_{BC}	0	$h_{1,2}(6) = 0$	0
h_{RA}	0.8	$h_{1,4}(6) = 1$	$1/2 + \theta_R^H/\pi$
h_{LA}	0.32	$h_{1,3}(6) = 1/3$	$1/2 + \theta_L^H/\pi$
h_R	—	$h_{1,3}(6) = 1/3$	0
h_L	—	$h_{1,3}(6) = 1/3$	0
h_{RL}	—	$h_{1,4}(6) = 1$	1/2
h_{LR}	—	$h_{1,3}(6) = 1/3$	0
h_0	0	$h_{1,2}(6) = 0$	0

Exact results for PCC: two-point functions

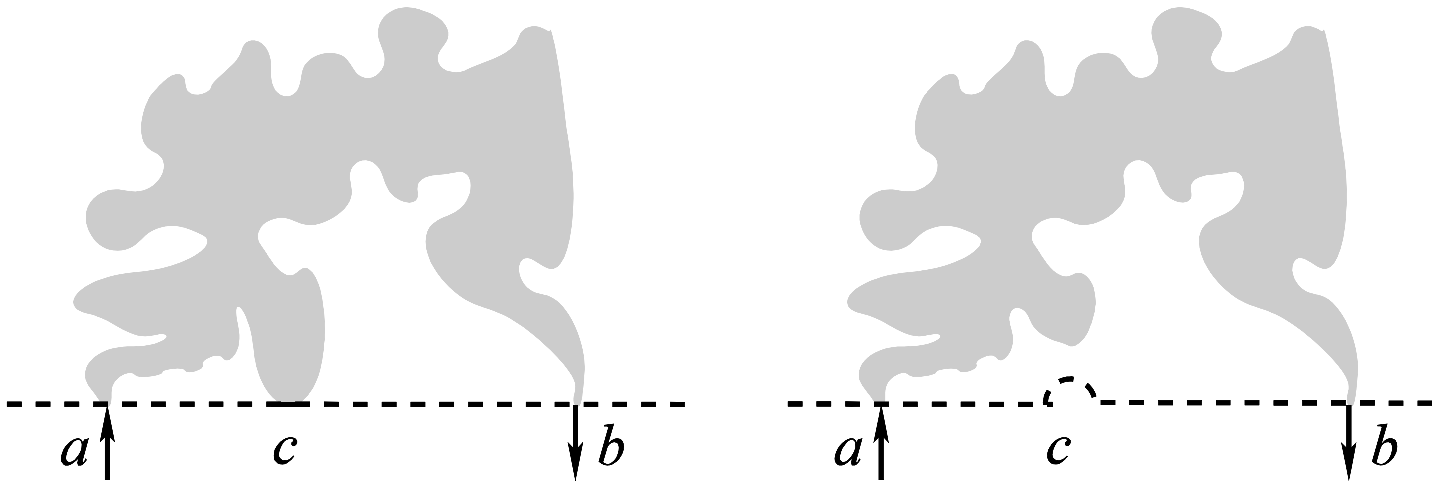


$h = h_A = 1$

$h = h_{LA}$ or h_{RA}

- Two-point PCCs $\langle g(a,b)\rangle = \frac{C}{|a-b|^{2h}}$

Exact results for PCC: three-point functions

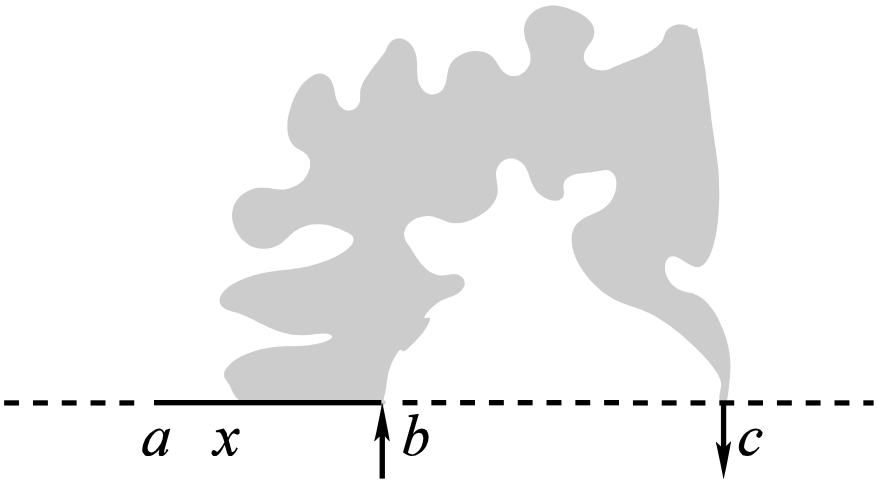


- Change in the PCC upon perturbing the boundary near c

$$\delta\langle g(a,b)\rangle = \frac{C\epsilon^{h_T}}{|a-b|^{2h_A-h_T}|a-c|^{h_T}|b-c|^{h_T}}$$

$$h_A = 1, \qquad h_T = 2$$

Exact results for PCC: three-point functions



- PCC with reflecting interval. “Lift-off” points.

$$\langle g(b,c;a)\rangle = \int_a^b dx\, \langle g(b,c;a,x)\rangle$$

$$\langle g(b,c;a,x)\rangle = \frac{C}{|x-b|^{h_l+h_b-h_A}|x-c|^{h_l+h_A-h_b}|b-c|^{h_b+h_A-h_l}}$$

$$h_l = h_A = 1$$

Conformal restriction gives restricted results

- The results obtained from conformal invariance alone are very generic
- No results for "interesting" dimensions and exponents h_{RA}, h_{LA}, ν that are specific to a particular critical point
- Work in progress: "interiors" of pictures, relation to trees, dimers, and CLEs

E. Bettelheim and IAG, 20??

- Especially interesting is the localization length exponent ν
- It has been measured experimentally and computed numerically in the CC network model

A puzzle

- All recent numerical results for ν agree with each other

$$\nu_{\mathrm{num}} \approx 2.6$$

- They all differ from the experimental value

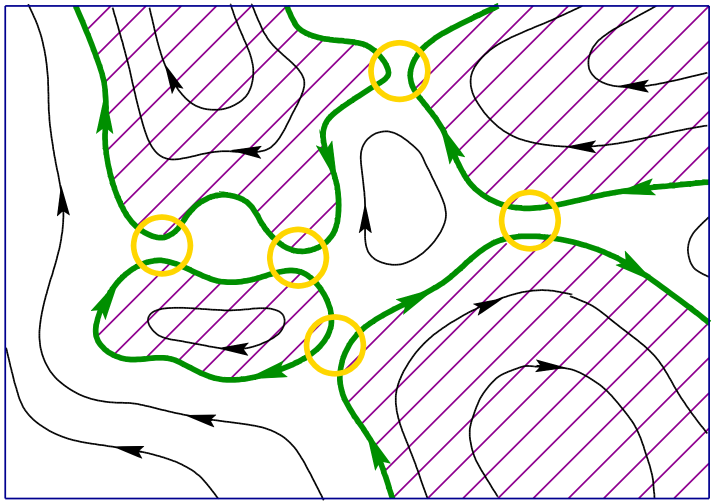
$$\nu_{\mathrm{exp}} \approx 2.38$$

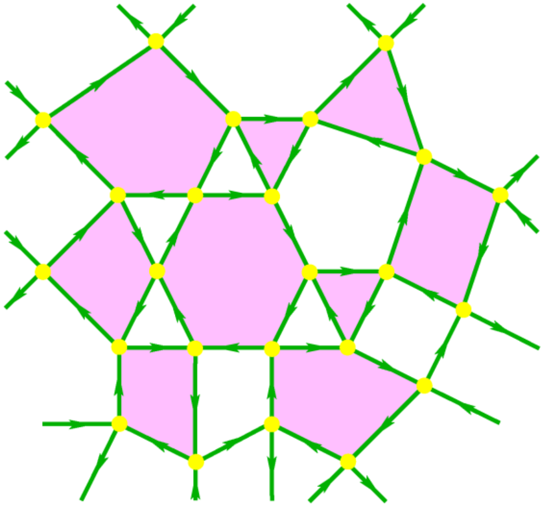
- We propose a possible explanation: Geometric disorder

A. Klümper, W. Nuding, IAG, A. Sedrakyan, 2016

Geometric disorder

- Arbitrary number of nodes around a puddle
- Network models on random graphs, disorder average includes summing over graphs

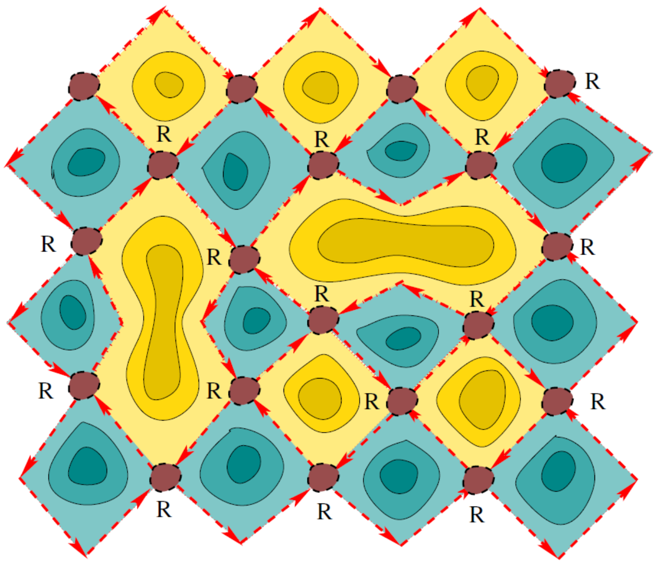




- Random graph similar to used in discrete models of quantum gravity
- Mapping to classical model still goes through
- Limit: restriction measures and SLE coupled to quantum gravity

Geometric disorder: numerical implementation

- Randomly choose a node and open it up



- Numerical result: $\nu = 2.37 \pm 0.02$

- Agrees with experiments, perhaps a coincidence

- Geometric disorder is relevant and needs to be understood

Continuum action for CC model: Dirac fermions

- Continuum limit of CC model

C. M. Ho, J. T. Chalker `96

$$S = \int d^2x\, \bar{\psi}[\sigma_\mu(i\overleftrightarrow{\partial}^\mu + A^\mu) + m\sigma_3 + V]\psi$$

- Random mass, gauge and scalar potentials $X = (A, m, V)$

- Quenched disorder, need supersymmetry (or replicas)

$$\overline{\langle O \rangle} = \int \mathcal{D}X \langle O \rangle = \int \mathcal{D}X \frac{\int \mathcal{D}\psi\, O(\psi)\, e^{-S[\psi,X]}}{\int \mathcal{D}\psi\, e^{-S[\psi,X]}}$$

$$= \int \mathcal{D}X \int \mathcal{D}\psi \mathcal{D}\phi\, O(\psi) e^{-S[\psi,X]-S[\phi,X]}$$

Continuum action for geometric fisorder: Dirac fermions coupled to quenched quantum gravity

- Continuum limit for the network with geometric disorder has additional coupling to quenched quantum gravity:

$$S = \int d^2x \, e \, \bar{\Psi}[\sigma_\mu e^\mu_\alpha (i \overleftrightarrow{\partial}^\alpha + A^\alpha) + m\sigma_3 + V]\Psi$$

- Disorder now includes geometric data (frames or metric)

$$X = (A, m, V, e)$$

- Needs further study: does KPZ apply?

- What is the meaning of ν in quantum gravity?

Summary

- PCCs in the CC network model in terms of a classical model
- Classical weights satisfy restriction property
- Conformal restriction gives some exact results for PCCs
- Geometric disorder in the network model changes critical behavior
- Critical exponent in a random network is close to experimental value
- Conformal restriction measures and SLE coupled to quantum gravity
- Field theory description with quenched quantum gravity
- Lots of open questions