Liouville Quantum Multifractality

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Everything is Complex

A conference in honor of Nikolaï Makarov

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Based on joint work with

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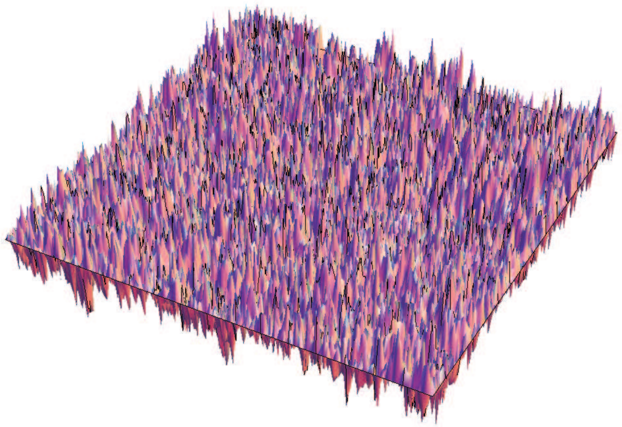
- Jason Miller (Cambridge), Scott Sheffield (MIT)

Gaussian Free Field (GFF)



[Courtesy of J. Miller]

Distribution h with *Gaussian weight* $\exp\left[-\frac{1}{2}(h,h)_\nabla\right]$, and **Dirichlet inner product** in domain D

$$(f_1, f_2)_\nabla := (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) dz = \mathrm{Cov}\big((h, f_1)_\nabla, (h, f_2)_\nabla\big)$$

Liouville QG
Random Measure

$$\mu = \text{“}e^{\gamma h} dz\text{”}$$

Quantum Measure

$$\mu_\gamma := \lim_{\varepsilon\to 0} \exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2} dz,$$

where $h_\varepsilon(z)$ *is the GFF average on a circle of radius* ε*, converges weakly for* $\gamma < 2$ *to a random measure, denoted by* $e^{\gamma h(z)} dz$.

[Høegh-Krohn '71; Kahane '85; D. & Sheffield '11]

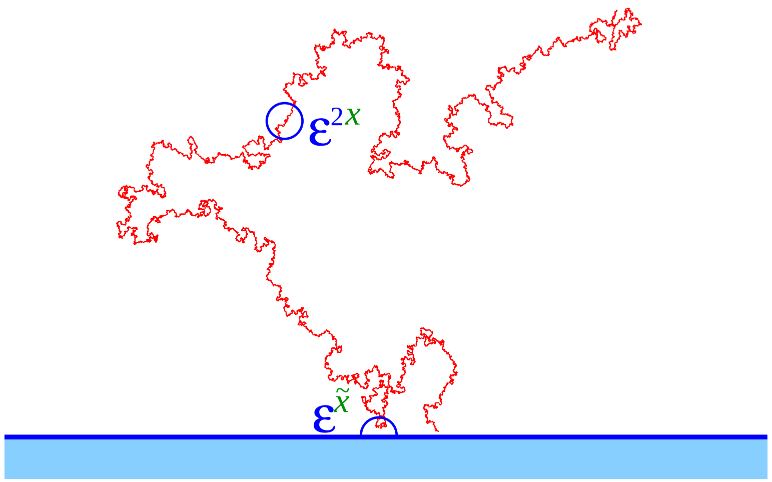
Quantum Boundary Measure

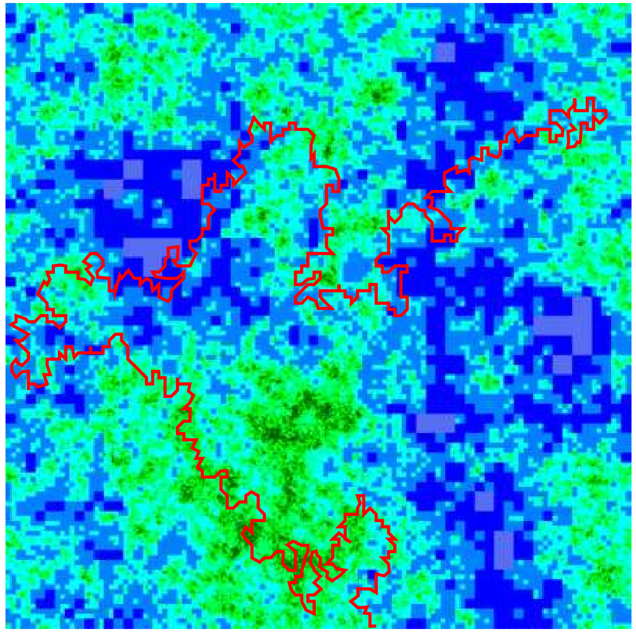
$$\nu_\gamma := \lim_{\varepsilon\to 0} \exp\left[\frac{\gamma}{2} \hat{h}_\varepsilon(z)\right] \varepsilon^{\gamma^2/4} dz,$$

converges weakly for $\gamma < 2$ *to a boundary random measure, denoted by* $e^{(\gamma/2)\, h(z)} dz$.

Scaling Exponents of (Random) Fractals

SAW in half plane - 1,000,000 steps





[Courtesy of T. Kennedy & J. Miller]

Probabilities & Hausdorff Dimensions (e.g., SLE_κ)

$\mathbb{P} \asymp \varepsilon^{2x}, \quad \widetilde{\mathbb{P}} \asymp \varepsilon^{\tilde{x}}, \qquad d = 2 - 2x \qquad (= 1 + \kappa/8)$

δ-*Quantum Ball:* $\quad \mathbf{P} \asymp \delta^{\Delta}, \quad \widetilde{\mathbf{P}} \asymp \delta^{\widetilde{\Delta}}$

KNIZHNIK, POLYAKOV, ZAMOLODCHIKOV ’88

x and Δ ($\tilde{x}$ and $\tilde{\Delta}$) are related by the **KPZ formula**

$$x = \left(1 - \frac{\gamma^2}{4}\right)\Delta + \frac{\gamma^2}{4}\Delta^2$$

Kazakov ’86; D. & Kostov ’88 [Random matrices]

David; Distler & Kawai ’88 [Liouville field theory]

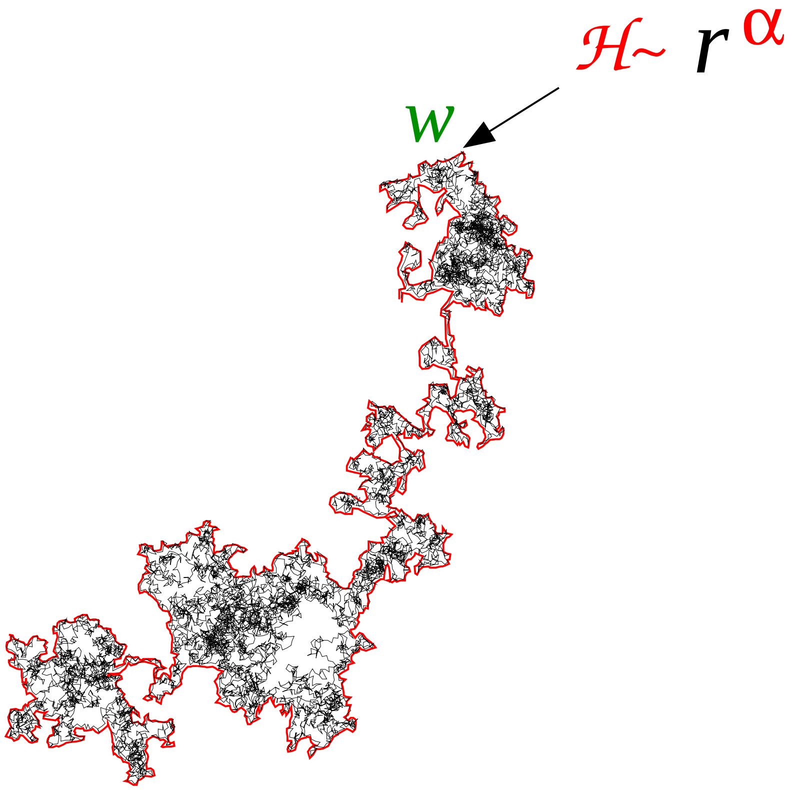
KPZ Theorem – *D. & Sheffield ’11*

Benjamini & Schramm ’09; Rhodes & Vargas ’11 [Hausdorff dimension]

David & Bauer ’09; Berestycki, Garban, Rhodes, Vargas ’14 [Heat kernel]

Multifractality

[Mandelbrot, '74; Frisch & Parisi '86; Halsey & al. '86]



$$w \in \mathcal{F}_\alpha : \quad \dim \mathcal{F}_\alpha = f(\alpha)$$

The standard multifractal formalism also relates $f(\alpha)$ to its symmetric Legendre transform $\tau(n)$ via

$$f(\alpha) + \tau(n) = \alpha n$$

$$n = \frac{\partial f(\alpha)}{\partial \alpha}, \alpha = \frac{\partial \tau(n)}{\partial n},$$

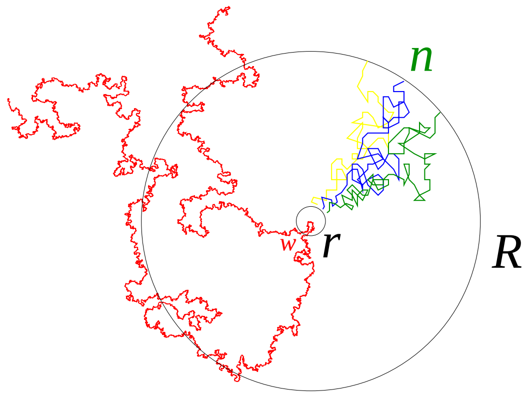
where n is the order of the multifractal moment of the harmonic measure. Then the *scaling dimension* $x(n)$ (which obeys KPZ) is canonically defined by

$$\tau(n) =: 2x(n) - 2,$$

and gives in particular the Hausdorff dimension of the fractal as $D = -\tau(0) = 2 - 2x(0)$.

Multifractal Exponents & QG

...or pretending to be a physysist [@ S. Treil]



$$x(n) \;=\; U_\kappa\left(\frac{1}{2}U_\kappa^{-1}(n)+\frac{1}{2}\right)$$

- Inspired by *Lawler and Werner's '98 cascade relations* for Brownian intersection exponents
- *Welding of quantum wedges [Sheffield '10; D., Miller & Sheffield '14; Astala, Jones, Kupiainen, Sacksman '10]*

KPZ & SLE

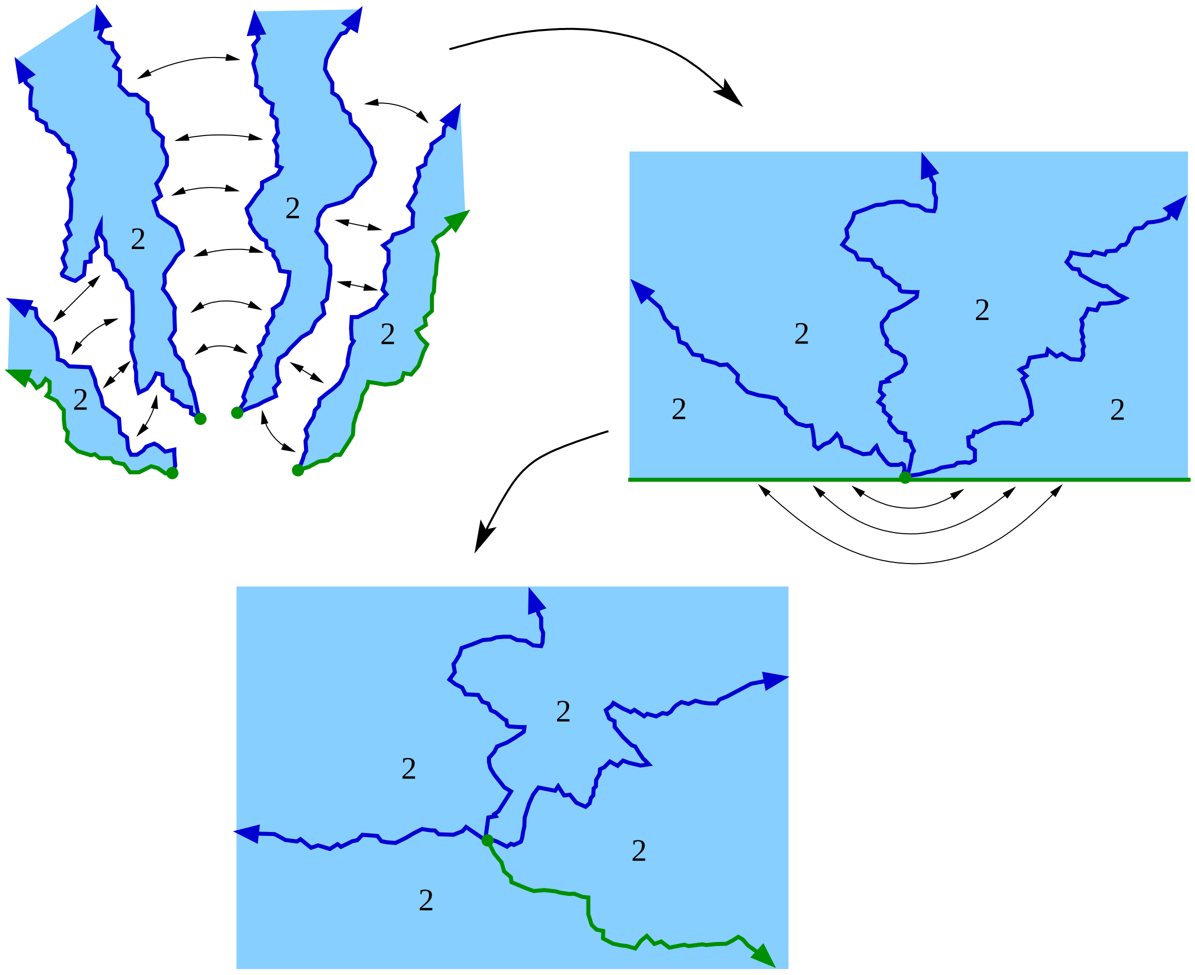
Conformal dimensions Δ in QG and x in $\mathbb{C}$

$$x = U_\kappa(\Delta) = \frac{1}{4}\Delta\left(\kappa\Delta + 4 - \kappa\right), \; \gamma^2 = \kappa = \frac{16}{\kappa'} \leqslant 4$$

Inverse KPZ map

$$\Delta = U_\kappa^{-1}(x) = \frac{1}{2\kappa}\left(\sqrt{16\kappa x + (\kappa - 4)^2} + \kappa - 4\right)$$

Quantum Wedge Welding



Tip Multifractal Exponents & QG

$$\hat{x}(n) = U_\kappa\left(\frac{1}{2}\hat{U}_\kappa^{-1}(n) + \frac{1}{2}\right)$$

- Modified **KPZ** relation *[D., Sheffield, Viklund '10]*
- Inverse modified KPZ map

$$\hat{U}_\kappa^{-1}(x) = \frac{1}{2\kappa}\left(\sqrt{16\kappa x + (\kappa-8)^2} + \kappa - 8\right)$$

Multifractal spectra

Bulk SLE harmonic measure spectrum

[D. ’00; Beliaev & Smirnov ’05; Rushkin et al. ’06; Gwynne, Miller & Sun ’14 (a.s.)]

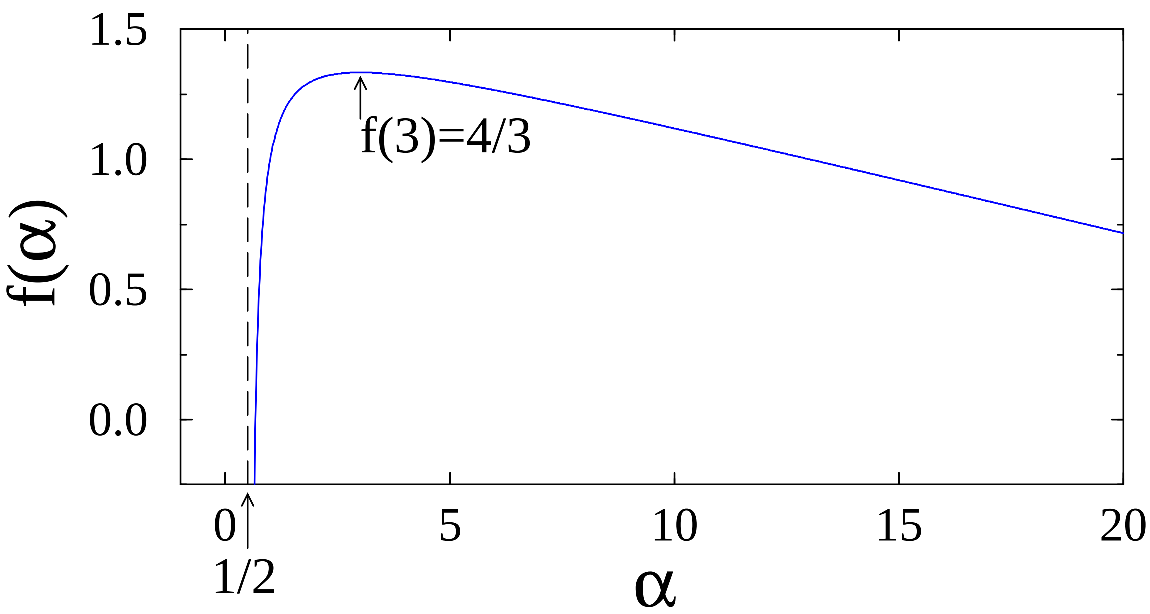
$$f_{\text{bulk}}(\alpha) = \alpha + \frac{(4+\kappa)^2}{8\kappa} - \frac{(4+\kappa)^2}{8\kappa}\frac{\alpha^2}{2\alpha-1}.$$

Tip spectrum *[Hastings ’02; Beliaev & Smirnov ’05; Johansson-Viklund & Lawler ’09 (a.s.)]*

$$f_{\text{tip}}(\alpha) = \alpha\left(1-\frac{4}{\kappa}\right) + \frac{(4+\kappa)^2}{8\kappa} - \frac{\kappa}{8}\frac{\alpha^2}{2\alpha-1}.$$

Boundary SLE collisions spectrum *[Alberts, Binder & Viklund ’16 (a.s.)]*

$SLE_{8/3}$ Multifractal Spectrum

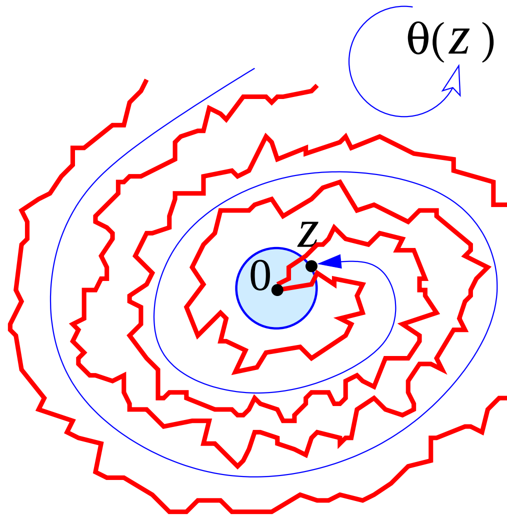


$$f(\alpha) = \alpha + b - \frac{b\alpha^2}{2\alpha - 1}, \quad b = \frac{25}{12}$$

$$f(\alpha = 1) = 1, \quad \text{Makarov}$$

Mixed Multifractal Spectrum

[Binder '96] Logarithmic spirals



$$w \in \mathcal{F}_{\alpha,\lambda} \iff \begin{Bmatrix} \mathcal{H}(w,r) \sim r^{\alpha} \\ \vartheta(w,r) \sim \lambda \ln r \end{Bmatrix}$$

$$\dim \mathcal{F}_{\alpha,\lambda} = f(\alpha,\lambda)$$

Scaling Law

$$f(\alpha) = \alpha + b - \frac{b\alpha^2}{2\alpha - 1}, \qquad b = \frac{1}{2\kappa}\left(2 + \frac{\kappa}{2}\right)^2$$

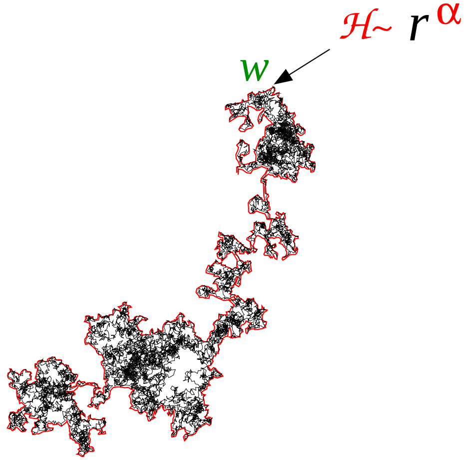
$$\begin{aligned} f(\alpha, \lambda) &= (1+\lambda^2) f\left(\frac{\alpha}{1+\lambda^2}\right) - b\lambda^2 \\ &= \alpha + b - \frac{b\alpha^2}{2\alpha - 1 - \lambda^2} \end{aligned}$$

Binder & D. ’02 (QG) ’08 (CG) ’16 (SLE)

Belikov, Gruzberg & Rushkin ’08 (CG)

Aru ’13 (SLE)

Liouville Quantum Multifractality



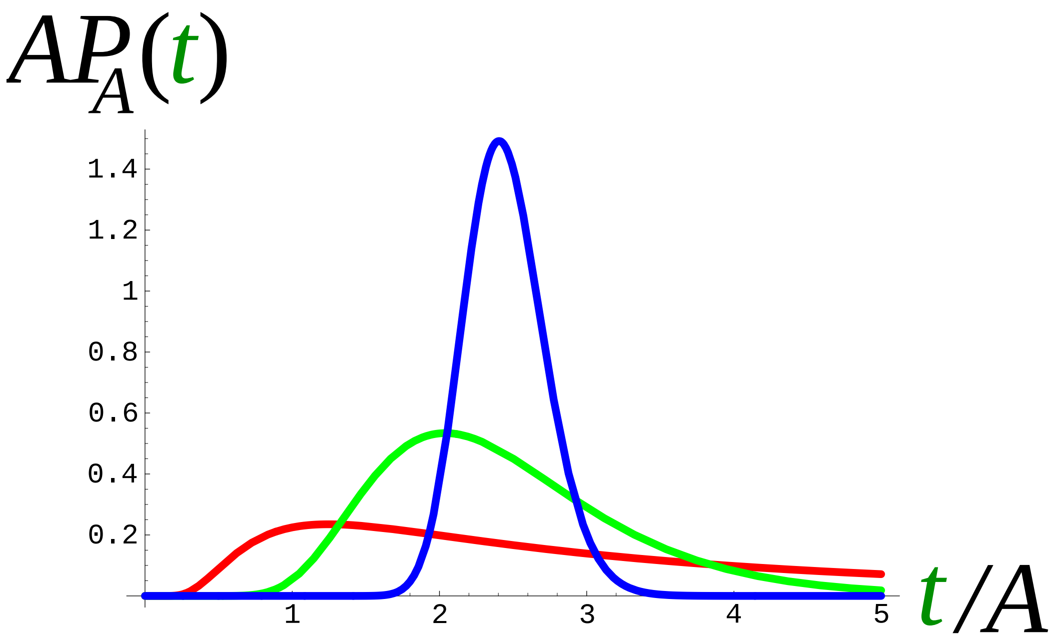
$$w \in \mathcal{F}_{\alpha}: \quad \mathcal{H} \sim r^{\alpha}, \; \dim \mathcal{F}_{\alpha} = f(\alpha)$$

Quantum ball: $\delta := \int_{B(w,r)} \mu_{\gamma}$

$$w \in \mathcal{F}^{G}_{\alpha_G}: \quad \mathcal{H} \sim \delta^{\alpha_G}, \; \dim \mathcal{F}^{G}_{\alpha_G} = f_G(\alpha_G)$$

$f_G(\alpha_G)$ is the *quantum multifractal spectrum* [dimension here up to a factor $D_{\mathcal{H}}$, the Hausdorff dimension of the LQG surface].

Probability Distribution $(\gamma = \sqrt{8/3})$ $[A = 2;\ 20;\ 200]$



$$t := -\log r,\ A := -\log\delta,\ \delta := \int_{B(w,r)} \mu_\gamma$$

$$P_A(t) = \frac{A}{\sqrt{2\pi t^3}} \exp\left[-\frac{1}{2t}(A - at)^2\right]$$

Large Deviations

Planar case: in a ball of radius r

$$\mathbb{P}_{\alpha}(t) \asymp e^{-t[2-f(\alpha)]}$$

$$t := -\log r, \ -\log \mathcal{H} = \alpha t,$$

Quantum case: *Conditioned on* A *and* α_G*, convolution*

$$\int_0^{\infty} \mathbb{P}_{\alpha}(t) P_A(t) dt \asymp e^{-A[1-f_G(\alpha_G)]}$$

$$A := -\log \delta, \ -\log \mathcal{H} = \alpha_G A, \ \ \frac{\alpha}{\alpha_G} = \frac{A}{t}.$$

$f_G(\alpha_G)$ is the quantum multifractal spectrum, i.e., the "dimension" of the set of points where the measure on a δ-quantum ball scales as δ^{α_G} [up to a factor $D_{\mathcal{H}}$, the Hausdorff dimension of the LQG surface].

Quantum Multifractal Spectrum

The *quantum* multifractal formalism relates $f_G(\alpha)$ to its symmetric Legendre transform $\tau_G(n) := \Delta(n) - 1$ via

$$f_G(\alpha) + \tau_G(n) = \alpha n$$

$$n = \frac{\partial f_G(\alpha)}{\partial \alpha}, \qquad \alpha = \frac{\partial \tau_G(n)}{\partial n},$$

where KPZ yields the *quantum* dimension

$$\Delta(n) = U_\kappa^{-1}(x(n))$$

from the scaling dimension $x(n)$ such that $\tau(n) := 2x(n) - 2$.

SLE Quantum Harmonic Spectrum

From the quantum gravity formulae above, we get

$$\tau(n) = \frac{n-1}{2} + \left(1+\frac{\kappa}{4}\right)\tau_G(n)$$

$$\alpha = \frac{1}{2} + \left(1+\frac{\kappa}{4}\right)\alpha_G$$

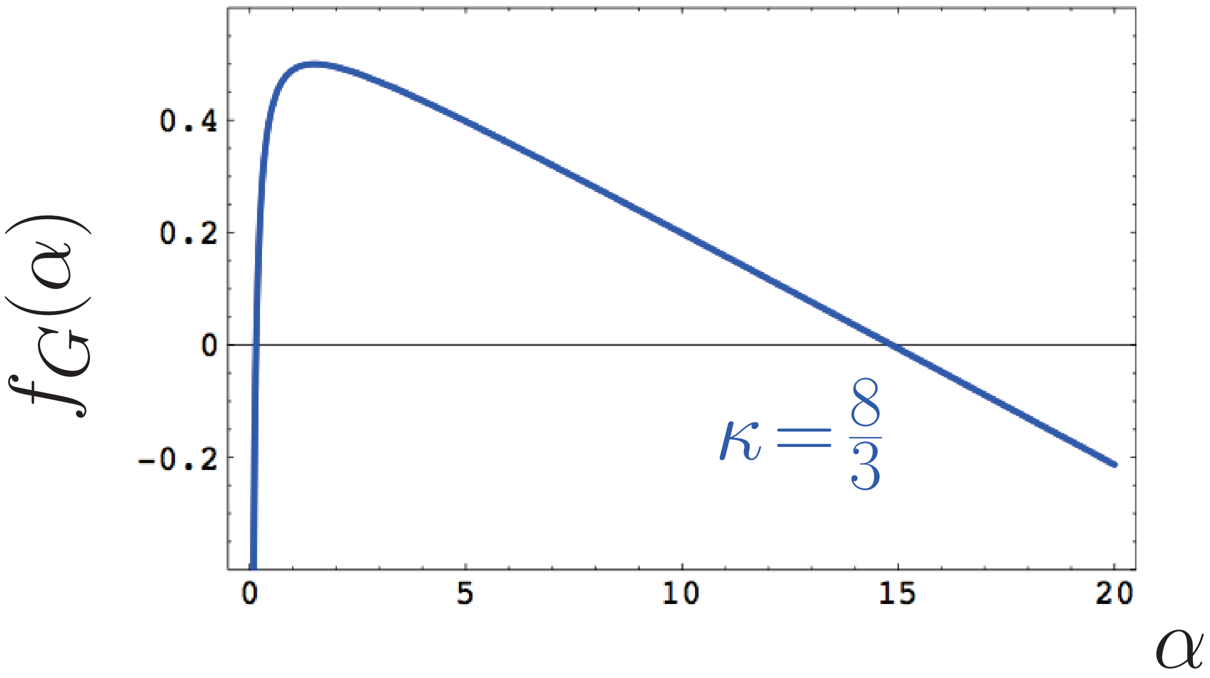
$$f(\alpha) = \frac{1}{2} + \left(1+\frac{\kappa}{4}\right)f_G(\alpha_G).$$

$$f_G(\alpha_G) = \frac{4+\kappa}{4\kappa} - \frac{1}{4\kappa}\frac{1}{\alpha_G} - \frac{(4-\kappa)^2}{16\kappa}\alpha_G, \quad \alpha_G > 0,$$

$$0 < \kappa \leqslant 4,\ \kappa = 16/\kappa',\ 4 \leqslant \kappa',$$

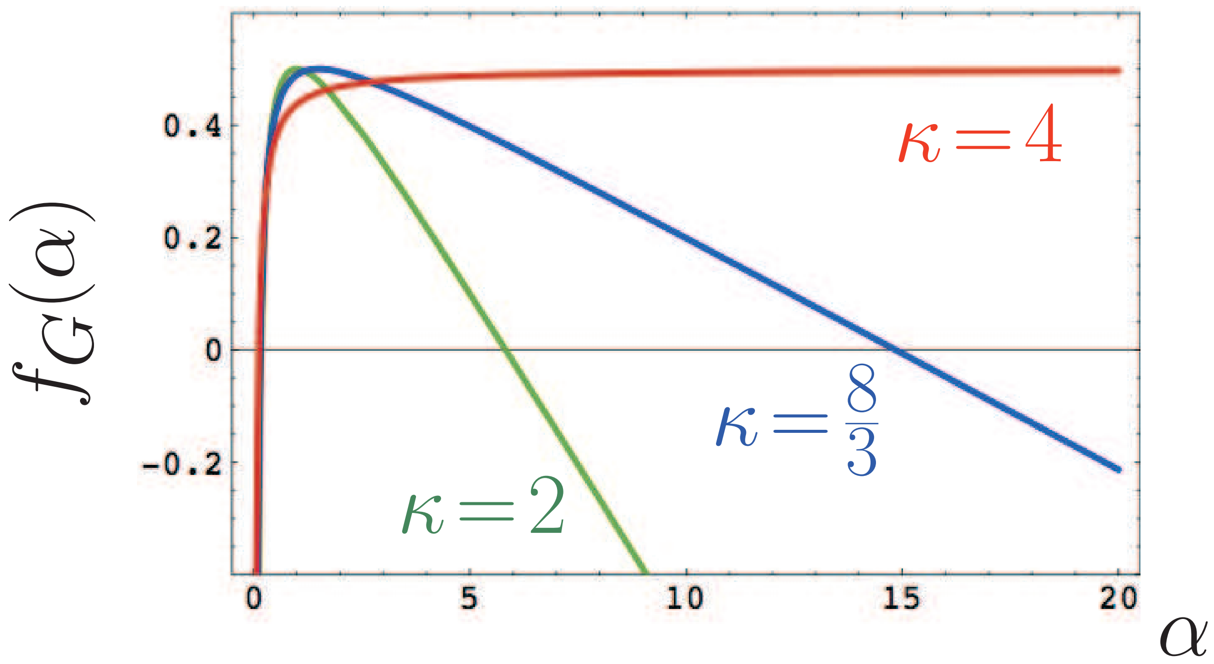
$$\max_{\alpha_G} f_G(\alpha_G) = 1/2, \forall\kappa.$$

$SLE_{8/3}$ Quantum Multifractal Spectrum



$$f_G(\alpha) = \frac{5}{8} - \frac{3}{32\alpha} - \frac{1}{24}\alpha, \quad \alpha > 0$$

SLE$_\kappa$ Quantum Multifractal Spectra



$$f_G(\alpha) = \frac{4+\kappa}{4\kappa} - \frac{1}{4\kappa}\frac{1}{\alpha} - \frac{(4-\kappa)^2}{16\kappa}\alpha, \quad \alpha > 0,$$

$$0 < \kappa \leqslant 4, \ \kappa = 16/\kappa', \ 4 \leqslant \kappa',$$

$$\max_{\alpha} f_G(\alpha) = 1/2, \forall \kappa.$$

SLE Quantum Tip Spectrum

We get

$$f_G^{\text{tip}}(\alpha_G) = \frac{8+\kappa}{4\kappa} - \frac{1}{4\kappa}\frac{1}{\alpha_G} - \frac{(8-\kappa)^2}{16\kappa}\alpha_G, \quad \kappa \leqslant 4, \quad \alpha_G > 0,$$

$$\max_{\alpha_G} f_G^{\text{tip}}(\alpha_G) = 1/2, \quad \forall \kappa \leqslant 4;$$

$$f_G^{\text{tip}}(\alpha_G) = \frac{8+\kappa'}{16} - \frac{\kappa'}{64}\frac{1}{\alpha_G} - \frac{(8-\kappa')^2}{16\kappa'}\alpha_G, \quad 4 \leqslant \kappa', \quad \alpha_G > 0,$$

$$\max_{\alpha_G} f_G^{\text{tip}}(\alpha_G) = \min\{\kappa'/8, 1\}, \quad \forall \kappa', \quad 4 \leqslant \kappa'.$$

SLE Quantum Mixed Spectrum

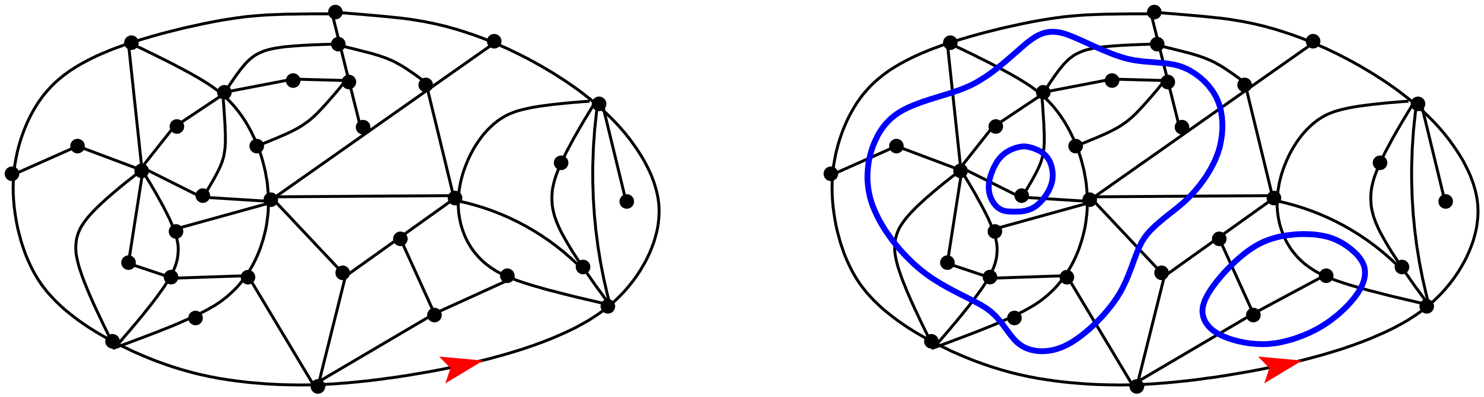
After some computations, we get the parametric form

$$
\begin{aligned}
f_G(\alpha_G, \lambda_G) &= \frac{4+\kappa}{2\kappa}\left[1 - \frac{1}{2\sqrt{1-\lambda^2}}\right] - \frac{1}{4\kappa}\frac{1+\lambda^2}{\alpha_G} - \frac{(4-\kappa)^2}{16\kappa}\alpha_G, \\
\lambda_G &= \frac{2\lambda}{\frac{1+\lambda^2}{\alpha_G} + \frac{4+\kappa}{2}\sqrt{1-\lambda^2}} \in [-\alpha_G, \alpha_G],\ \lambda \in (-1,1), \\
\alpha_G &\in [\lambda_G, +\infty), \\
& 0 < \kappa \leqslant 4,\ \kappa = 16/\kappa',\ 4 \leqslant \kappa'.
\end{aligned}
$$

$$
\max_{\alpha_G} f_G(\alpha_G, \lambda_G) = \frac{1}{2} + \frac{2}{\kappa}\left(1 - \sqrt{1 + 4\lambda_G^2}\right),
$$

$$
\max_{\lambda_G} f_G(\alpha_G, \lambda_G) = f_G(\alpha_G, 0).
$$

Random Planar Maps & Loop Models



[Courtesy of E. Guitter]

Clusters: Percolation, closed SAWs, Ising, $O(N)$ *and Potts models.*
Random quadrangulation weighted by the partition function of an $O(N)$ *loop model.*

Extreme nesting in CLE

Let $\mathcal{N}_z(\varepsilon)$ be the number of loops of a CLE_κ, $\kappa \in (8/3, 8)$ surrounding the ball $B(z, \varepsilon)$, and Φ_ν the set of points z where

$$\lim_{\varepsilon \to 0} \mathcal{N}_z(\varepsilon) / \log(1/\varepsilon) = \nu.$$

Theorem *[Miller, Watson & Wilson '14]*

$$\dim_{\mathcal{H}} \Phi_\nu = 2 - \gamma_\kappa(\nu)$$

$$\gamma_\kappa(\nu) = \nu \Lambda^*(1/\nu), \nu > 0; 1 - 2/\kappa - 3\kappa/32, \nu = 0$$

$$\Lambda^*(x) := \sup_{\lambda \in \mathbb{R}} (\lambda x - \Lambda(\lambda))$$

$$\Lambda(\lambda) = \log \left(\frac{-\cos(4\pi/\kappa)}{\cos\left(\pi \sqrt{\left(1 - \frac{4}{\kappa}\right)^2 + \frac{8\lambda}{\kappa}}\right)} \right)$$

Nesting of $O(N)$-loops on a random planar map

Large deviations in volume V *[Borot & Bouttier '15]*

$\mathbb{P}[\mathcal{N} = ad \ln V] \asymp_{V\to\infty} V^{-\Theta(d)}$

$$2\pi\Theta(d) = d\log\left(\frac{2d}{N\sqrt{1+d^2}}\right) + \operatorname{arc\,cot}(d) - \arccos\frac{N}{2}$$

$$a = \frac{1}{2\pi}\ \text{(dilute)};\quad a = \frac{1}{2\pi(1-b)}\ \text{(dense)}$$

$$N = 2\cos(\pi b),\ b \in [0, \frac{1}{2}],\ N \in [0,2]$$

$$2\pi\Theta(d) = \sup_{s\in[0,\frac{2}{N}]}\left\{d\log s + \arccos\frac{Ns}{2} - \arccos\frac{N}{2}\right\}$$

Q: Are these two results related? **A**: **LQG** with parameter γ

Tip: $b = \left|1 - \frac{4}{\kappa}\right|$; $\gamma = \min\{\sqrt{\kappa}, 4/\sqrt{\kappa}\}$.

Quantum Thermodynamics

$$P_A(t) \propto \exp\left[-\frac{1}{2t}\left(A - a_\gamma t\right)^2\right], \; a_\gamma := 2/\gamma - \gamma/2$$

$$t := -\log \varepsilon; \; A = -\gamma^{-1} \log \delta$$

In the plane, Legendre transform

$$\gamma_\kappa(\nu) = \lambda - \nu\Lambda(\lambda), \; \frac{1}{\nu} = \frac{\partial \Lambda(\lambda)}{\partial \lambda}.$$

In **Liouville Quantum Gravity**

$$\Theta(d) = \Delta(\lambda) - d\Lambda(\lambda), \; \frac{1}{d} = \frac{\partial \Lambda(\lambda)}{\partial \Delta(\lambda)},$$

where $\Delta(\lambda) := \sqrt{a_\gamma^2 + 2\lambda} - a_\gamma$ is **KPZ** for λ. □

Happy Birthday, Nikolaï!