The six vertex model and randomly growing interfaces in (1+1) dimensions

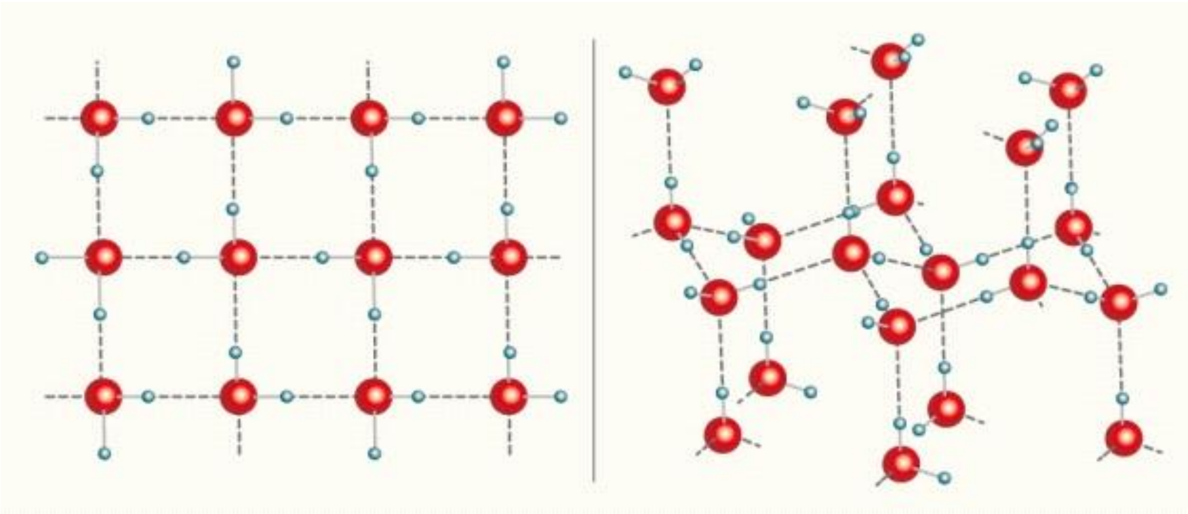
Alexei Borodin

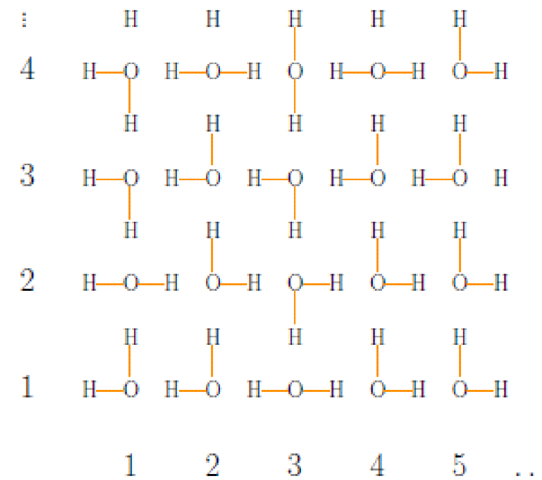
Through the last two decades, the large time asymptotics of a number of out-of-equilibrium random growth models in (1+1)d have been analyzed (Kardar-Parisi-Zhang universality class, Tracy-Widom distributions).

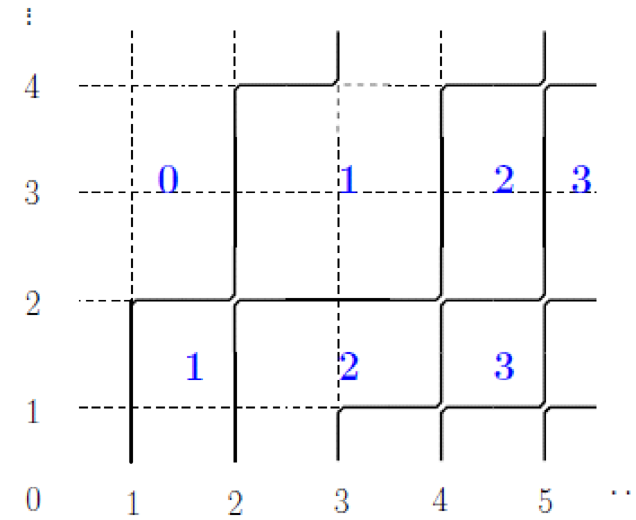
It turns out that the solvability of all the non-free-fermion ones can be traced to the Yang-Baxter integrability of the six vertex model.

Unraveling the basic structure that underlies the solvability leads to more powerful systems that go down to new analyzable physical (local) models and new phenomena.

The six vertex model (Pauling, 1935)

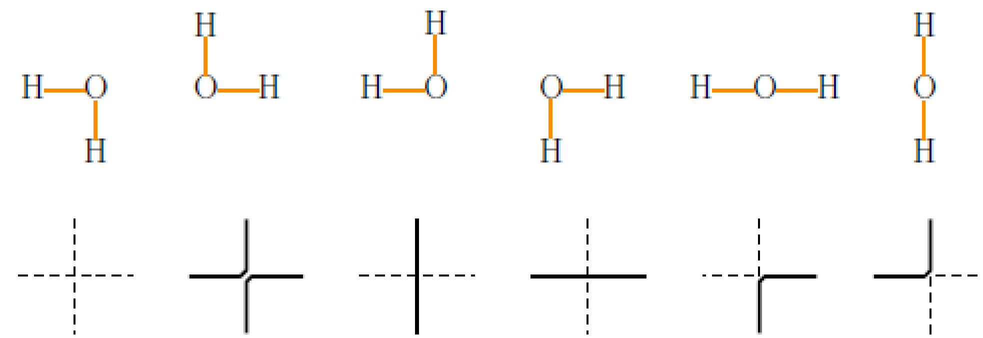






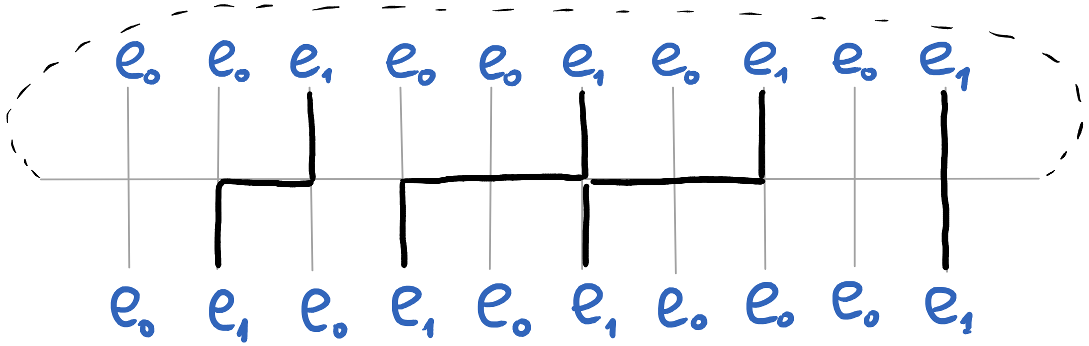
In 'square ice', which has been seen between graphene sheets, water molecules lock flat in a right-angled formation. The structure is strikingly different from familiar hexagonal ice (right).

From <http://www.nature.com/news/graphene-sandwich-makes-new-form-of-ice-1.17175>



Lieb in 1967 computed the partition function of the square ice on a large torus - an estimate for the residual entropy of real ice.

The six vertex model and the XXZ quantum spin chain



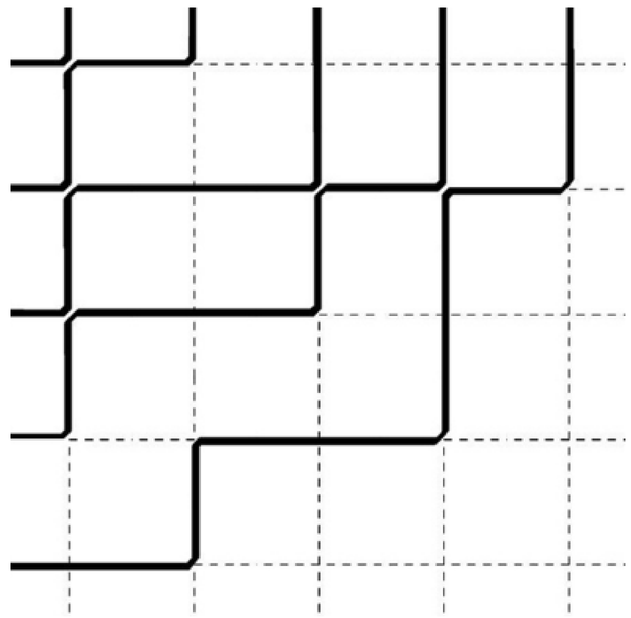
Encode rows of vertical edges as vectors in $(\mathbb{C}^2 = \langle e_0, e_1 \rangle)^{\otimes N}$.

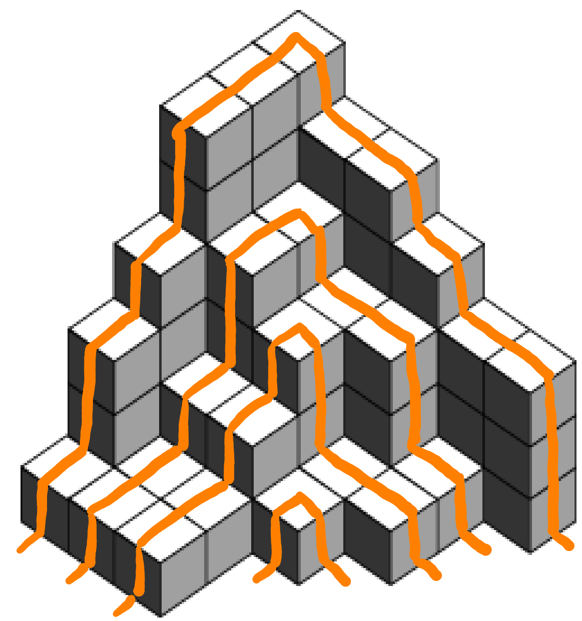
View products of weights of verticles in a horizontal row as matrix elements of an operator $(\mathbb{C}^2)^{\otimes N} \to (\mathbb{C}^2)^{\otimes N}$. For a certain choice of the six weights, the log derivative of this operator is

$$\sum_{j=1}^{N} J_x \sigma_j^x \sigma_{j+1}^x + J_x \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z , \qquad \sigma_j^{\left\{\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right\}} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \overset{(j)}{\left\{ \begin{matrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{matrix} \right\}} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} .$$

This is the Hamiltonian of a quantum mechanical (Heisenberg) model of ferromagentism known as the XXZ model.

The six vertex model vs. dimers



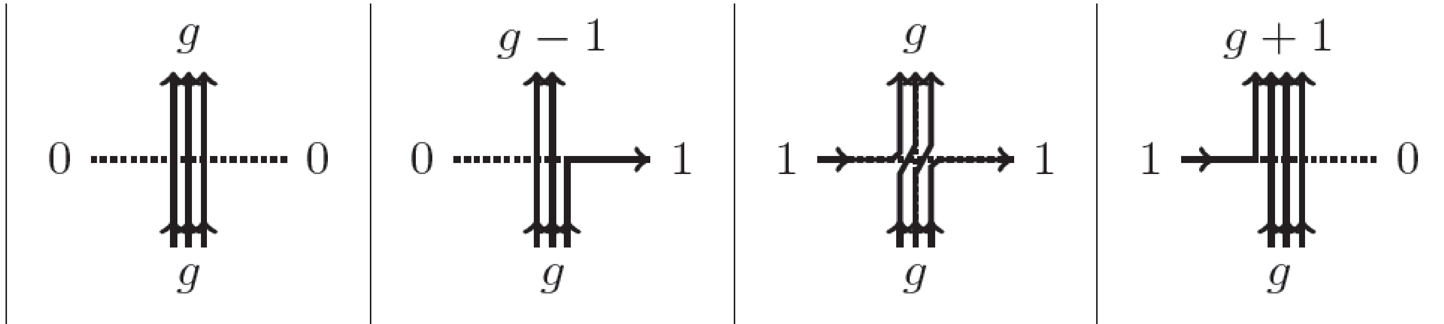


Partition function is the Izergin-Korepin det.

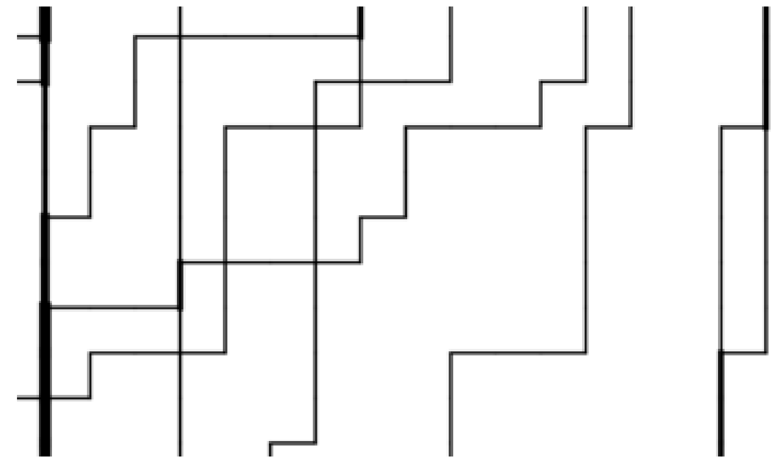
Partition function is a product, e.g. $\prod_{n \geq 1} \frac{1}{(1-q^n)^n}$.

Are there instances of the six vertex model with the partition function that looks like a product, not determinant?

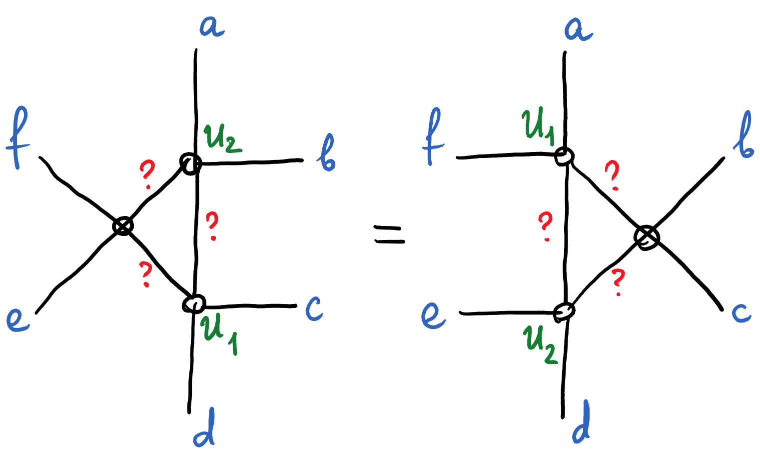
The higher spin six vertex model [Kulish-Reshtikhin-Sklyanin '81]

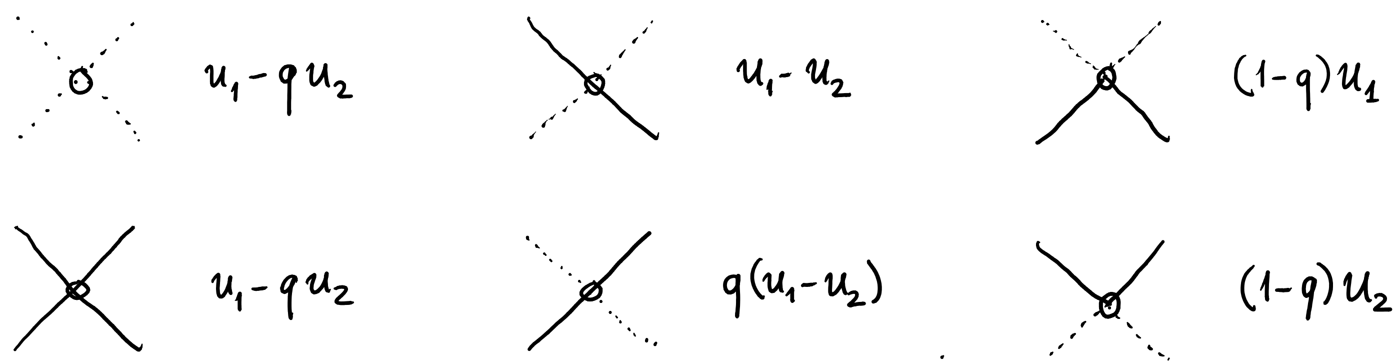


$w_{u,\mathsf{s}}$	$\dfrac{1-\mathsf{s}q^{g}u}{1-\mathsf{s}u}$	$\dfrac{(1-\mathsf{s}^2q^{g-1})u}{1-\mathsf{s}u}$	$\dfrac{u-\mathsf{s}q^{g}}{1-\mathsf{s}u}$	$\dfrac{1-q^{g+1}}{1-\mathsf{s}u}$

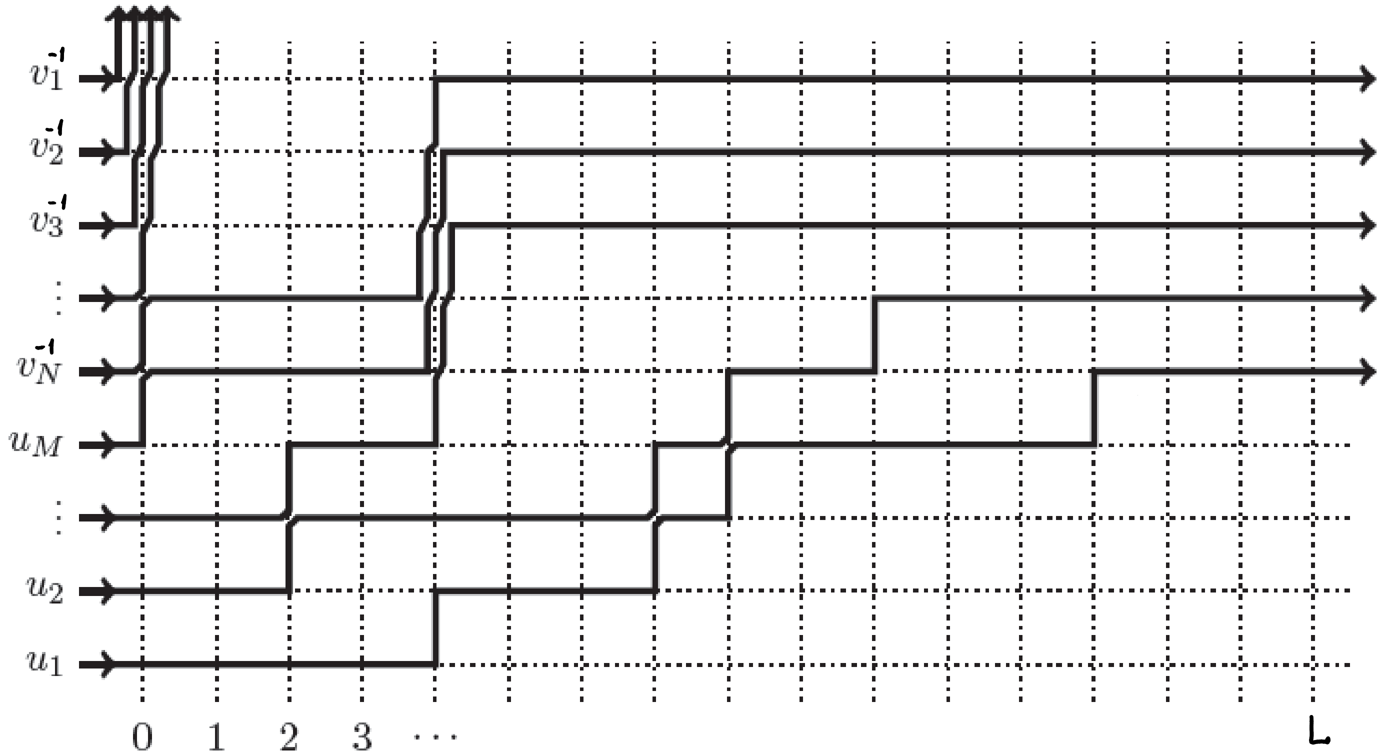


The Yang-Baxter (star-triangle) equation:





A product partition function

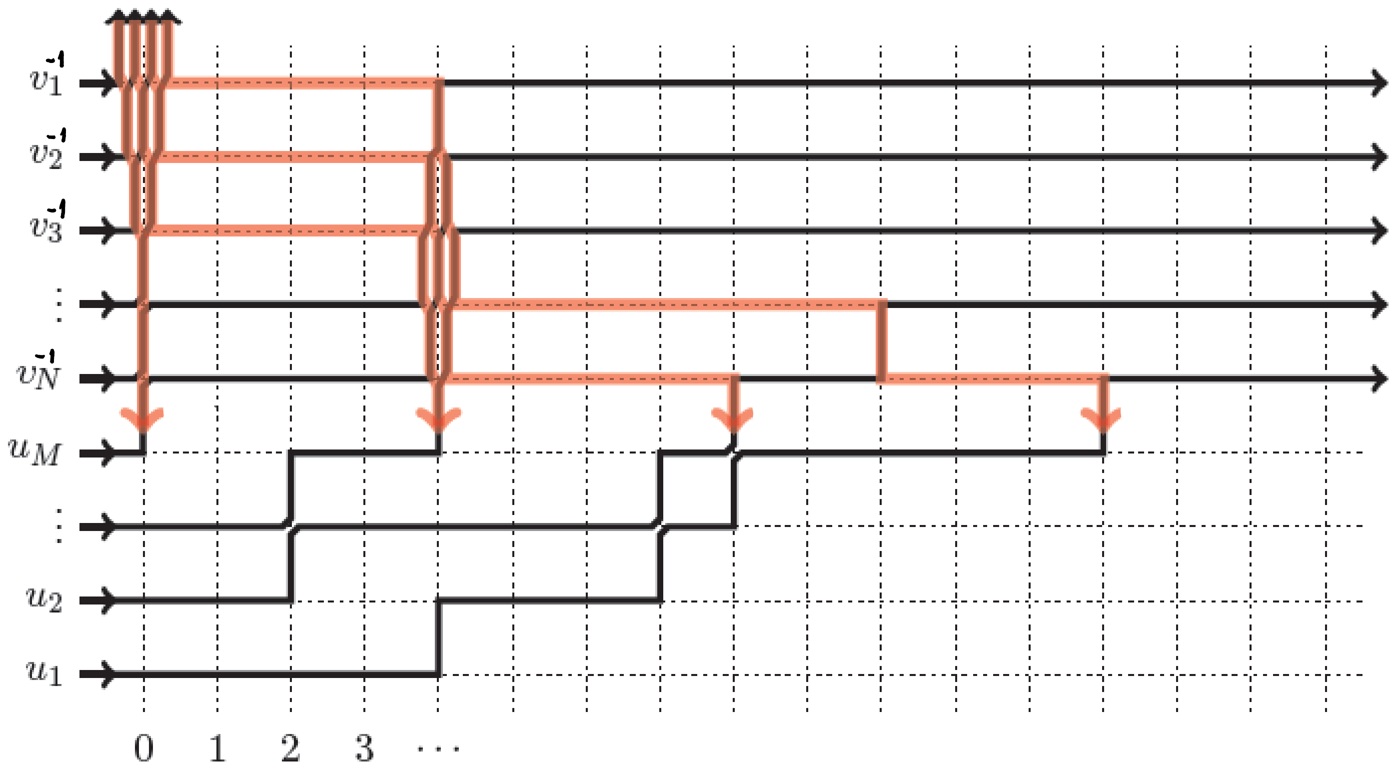


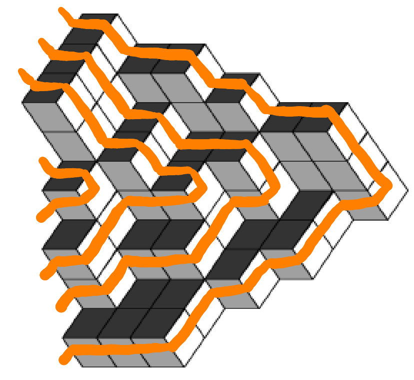
Theorem [B.'14] The partition function normalized by $\left(\prod_{j=1}^{N} \mathrm{weight}_{v_j^{-1}}(\to\!\!\cdot\!\!\to)\right)^{L}$ equals

$$\prod_{i=1}^{M} \frac{1-q^i}{1-su_i} \cdot \prod_{\substack{i=1,\dots,M \\ j=1,\dots,N}} \frac{1-qu_iv_j}{1-u_iv_j}.$$

Convergence: $\left| \frac{u_i - s}{1-su_i} \cdot \frac{v_j - s}{1-sv_j} \right| < 1 \quad \forall\, i,j.$

A product partition function

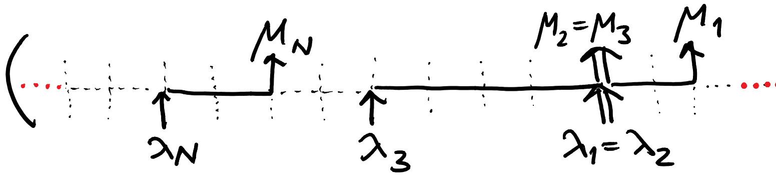


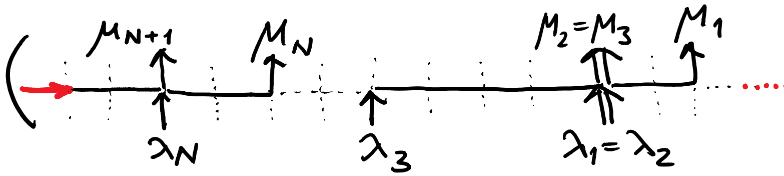


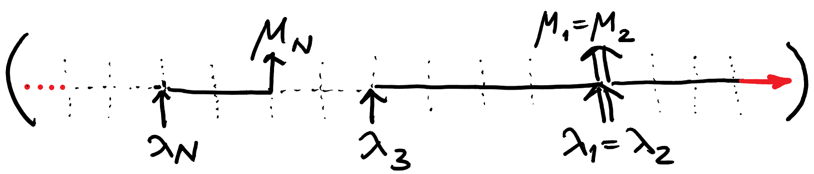
In the top part, one can replace occupied (black) horizontal edges and uoccupied ones. Then one has to normalize vertex weights in the top part rather than the partition function.

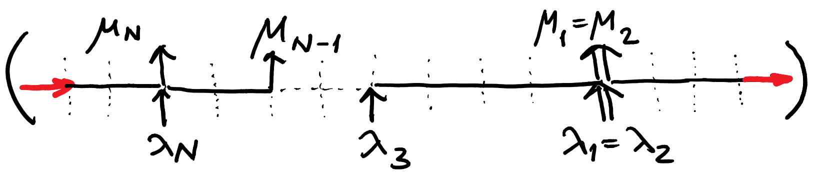
Proof – operator approach

In $\mathrm{Span}\{e_\lambda : \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0)\}$ define

$A(u)\, e_\lambda = \sum_\mu \mathrm{weight}_u\Big($  $\Big)\, e_\mu$

$B(u)\, e_\lambda = \sum_\mu \mathrm{weight}_u\Big($  $\Big)\, e_\mu$

$C(u)\, e_\lambda = \sum_\mu \mathrm{weight}_u\Big($  $\Big)\, e_\mu$

$\mathcal{D}(u)\, e_\lambda = \sum_\mu \mathrm{weight}_u\Big($  $\Big)\, e_\mu$

In infinite volume, C and $\mathcal{D}$ need to be normalized:

$$\overline{\mathcal{D}}(u) := \lim_{L\to\infty} \frac{\mathcal{D}^{(L)}(u)}{\big(\mathrm{weight}_u(\rightarrow\!\bullet\!\rightarrow)\big)^L},$$

L is the length of the strip

Proof – operator approach

The Yang–Baxter equation is equivalent to certain quadratic commutation relations between these operators. For example,

$$B(u_1)\,\mathcal{D}(u_2) = \frac{u_1 - u_2}{q u_1 - u_2}\,\mathcal{D}(u_2)\,B(u_1) + \frac{(1-q)u_2}{u_2 - q u_1}\,B(u_2)\,\mathcal{D}(u_1).$$

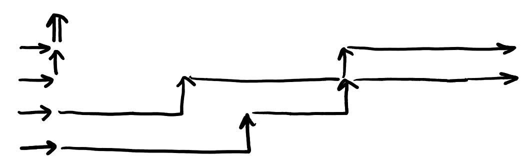
Assuming $|\mathrm{weight}_{u_1}(\to\bullet\to)| < |\mathrm{weight}_{u_2}(\to\bullet\to)|$, in infinite volume one gets

$$B(u_1)\,\overline{\mathcal{D}}(u_2) = \frac{u_1 - u_2}{q u_1 - u_2}\,\overline{\mathcal{D}}(u_2)\,B(u_1)$$

The result now follows from

$$\left(\overline{\mathcal{D}}(v_N^{-1})\cdots\overline{\mathcal{D}}(v_1^{-1})\,B(u_M)\cdots B(u_1)\,e_\varnothing,\ e_{0^M}\right) =$$

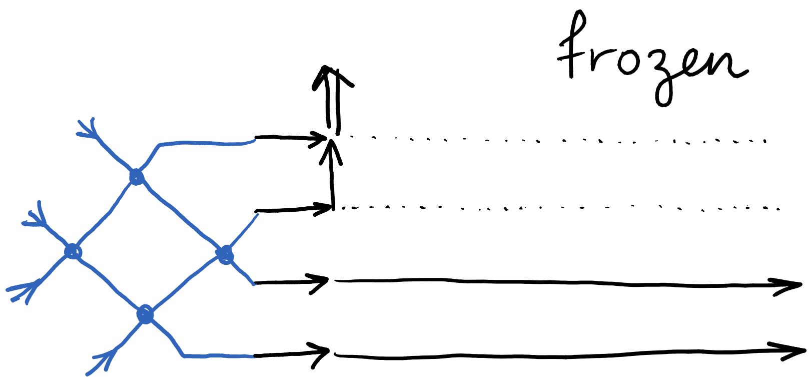
$$= \prod_{\substack{i=1,\dots,M\\ j=1,\dots,N}} \frac{1 - q u_i v_j}{1 - u_i v_j} \cdot \left(B(u_M)\cdots B(u_1)\,\overline{\mathcal{D}}(v_N^{-1})\cdots\overline{\mathcal{D}}(v_1^{-1})\,e_\varnothing, e_{0^M}\right)$$

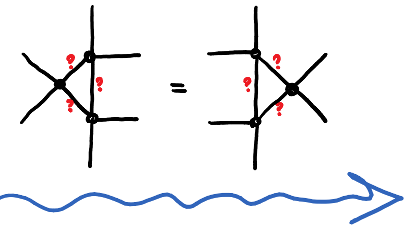


$\mathcal{D}\,\mathcal{D}\,\mathcal{B}\,\mathcal{B}\,e_\varnothing \rightsquigarrow e_{(0^2)}$

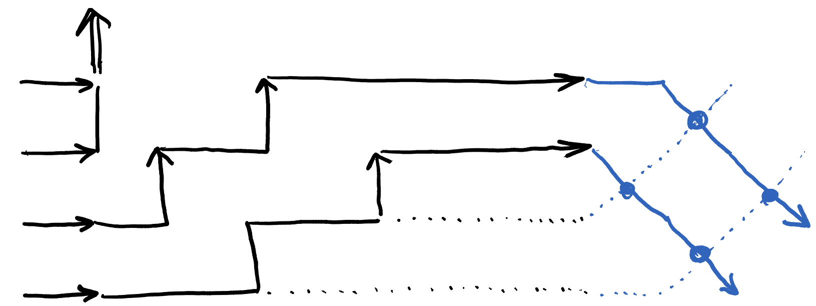
$\mathcal{B}\,\mathcal{B}\,\mathcal{D}\,\mathcal{D}\,e_\varnothing \rightsquigarrow e_{(0^2)}$

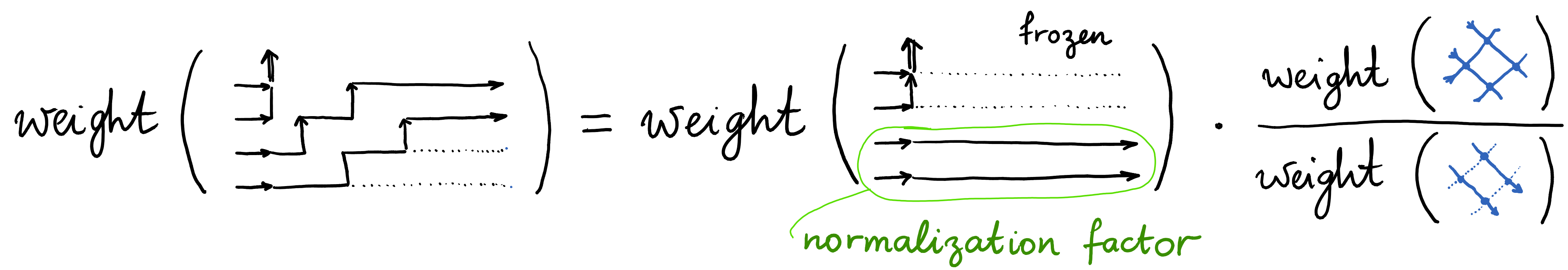
Proof – pictorial approach

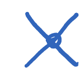




move  all the way to the right via YB





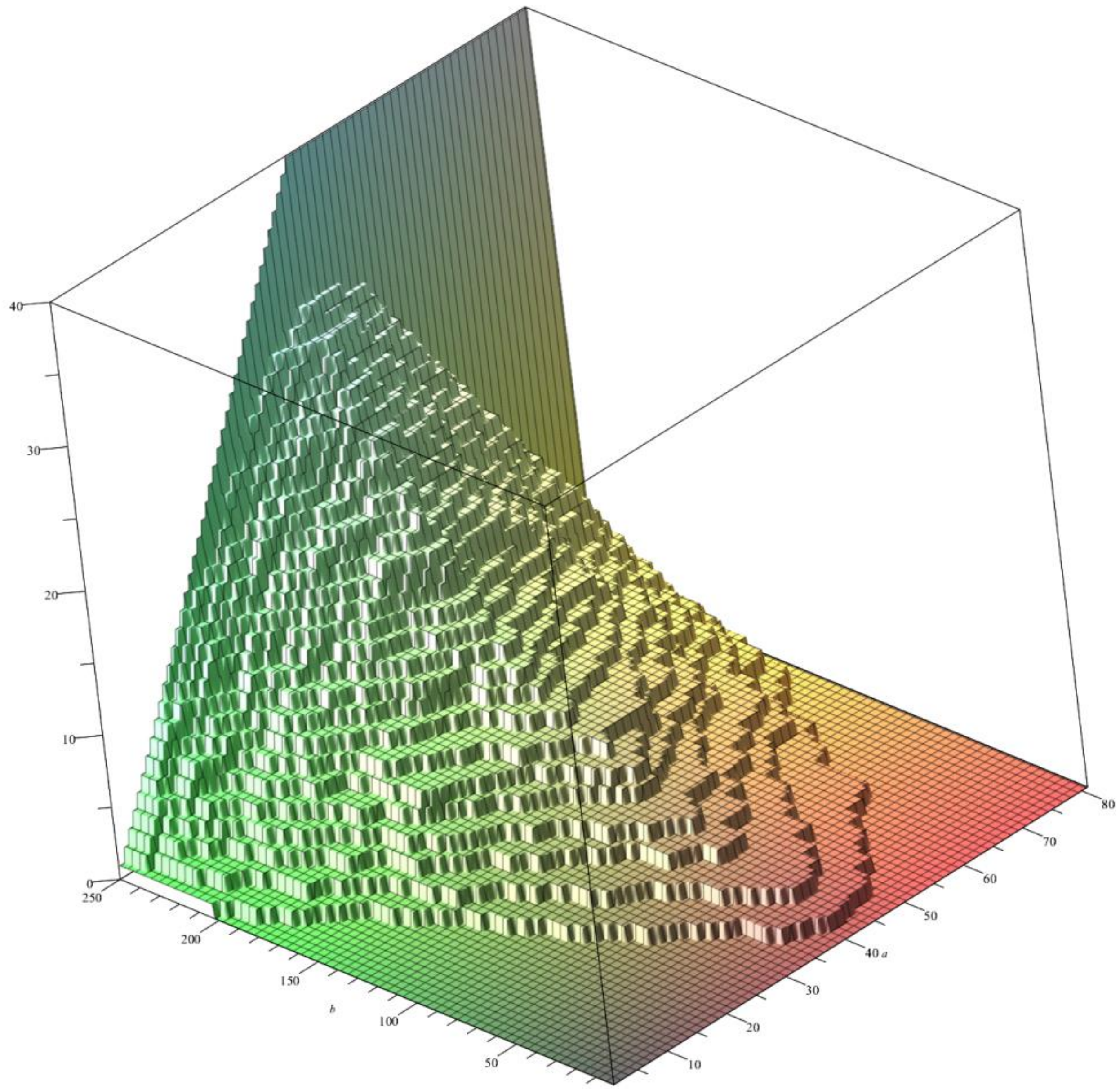
 $v_j^{-1} - q u_i$

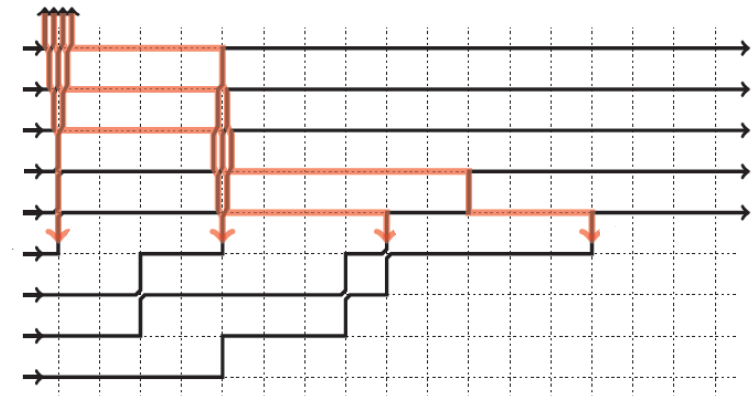
 $v_j^{-1} - u_i$

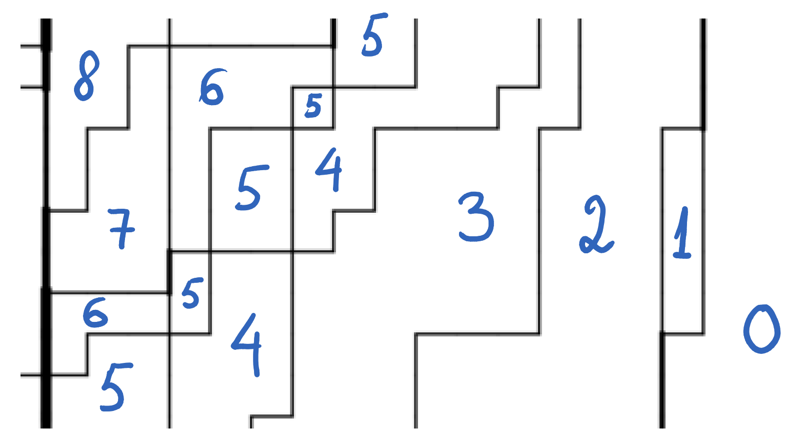
$$\prod_{i=1}^{M} \frac{1-q^i}{1-su_i} \cdot \prod_{\substack{i=1,\dots,M \\ j=1,\dots,N}} \frac{1-qu_iv_j}{1-u_iv_j}$$

A. Sportiello: This is also an exact sampling algorithm!

Sampling

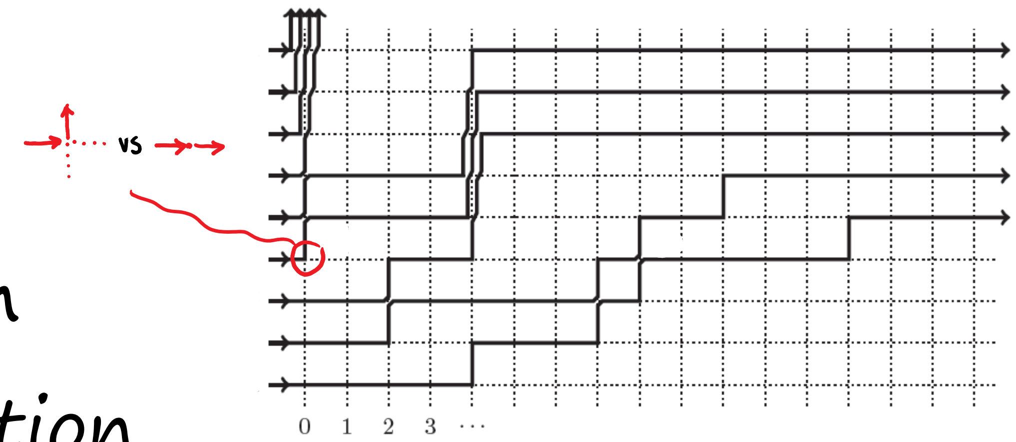






Height function

Removing the first column



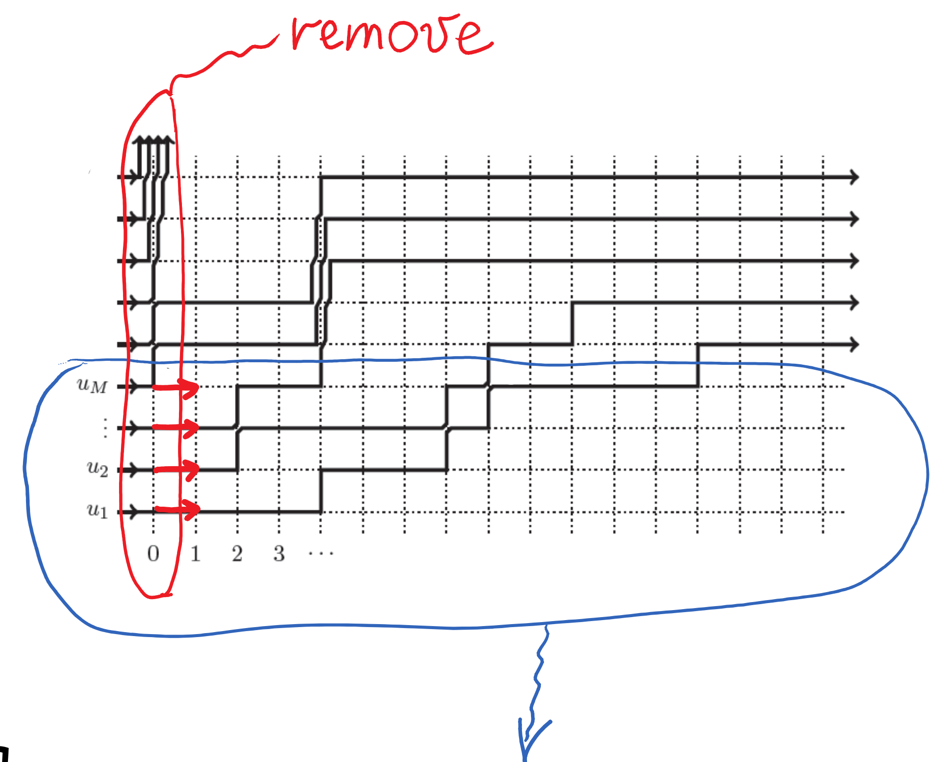
Making the 0th column deterministic turns it into a boundary condition.

If all 0th column vertices in the bottom half look like $\rightarrow\!\bullet\!\rightarrow$, the partition function has the factor $\prod_i w_{u_i}(\rightarrow\!\bullet\!\rightarrow) = \prod_i \frac{u_i - s}{1 - s u_i}$, thus must vanish at $u_i = s$.

$$\prod_j \frac{1 - q u v_j}{1 - u v_j}\Bigg|_{\substack{(v_1,\dots,v_N) = \\ = (v, qv, \dots, q^{N-1} v)}} = \frac{1 - q^N u v}{1 - u v}\Bigg|_{q^N = (sv)^{-1}} = \frac{1 - u/s}{1 - u v}\Bigg|_{v=0} = 1 - \frac{u}{s}$$

vanishes at $u = s$

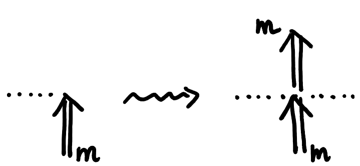
A stochastic model

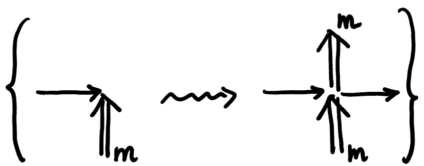


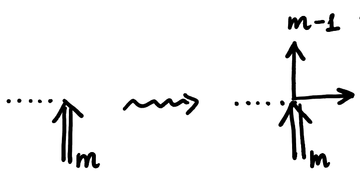
Specialize $\{v_j\}$ as

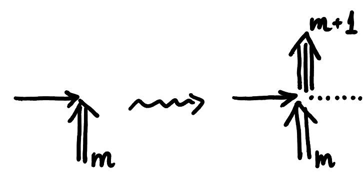
$$\prod_{i,j}\frac{1-q\,u_iv_j}{1-u_iv_j}\longrightarrow\prod_i\left(1-\frac{u_i}{s}\right)$$

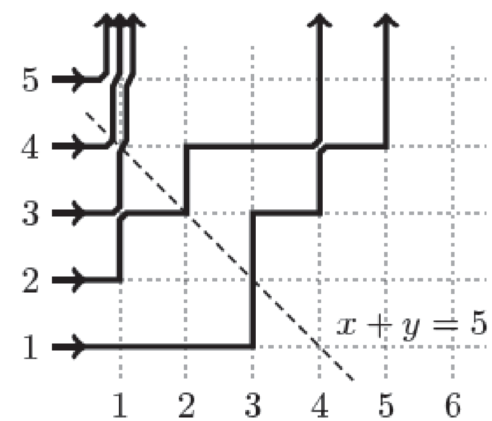
Theorem [B. '14, Corwin-Petrov '15]
The resulting random paths in the bottom part of the picture can be constructed recursively via

$\mathbb{P}\{$  $\} = \dfrac{1-sq^m u_y}{1-su_y}$

$\mathbb{P}\{$  $\} = \dfrac{s^2q^m-su_y}{1-su_y}$

$\mathbb{P}\{$  $\} = \dfrac{(q^m-1)\,su_y}{1-su_y}$

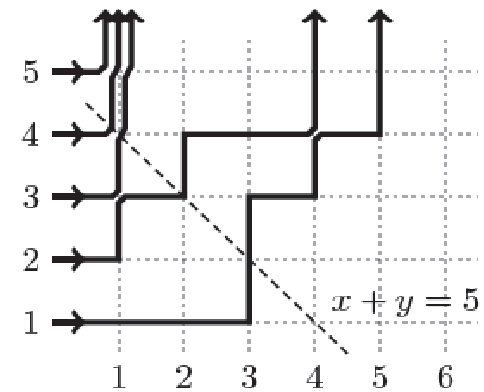
$\mathbb{P}\{$  $\} = \dfrac{1-s^2q^m}{1-su_y}$



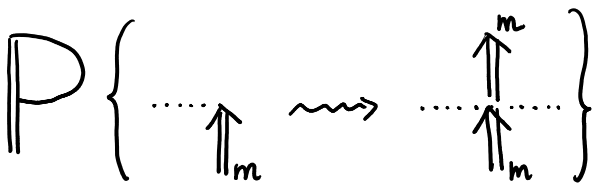
An inhomogeneous stochastic model

Specialize $\{v_j\}$ as

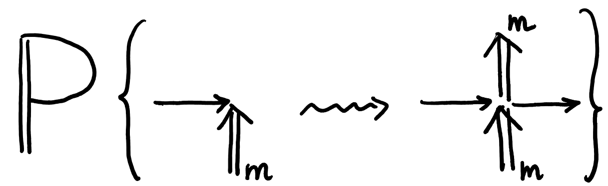
$$\prod_{i,j} \frac{1 - q u_i v_j}{1 - u_i v_j} \longrightarrow \prod_{i} \left(1 - \frac{u_i}{s}\right)$$



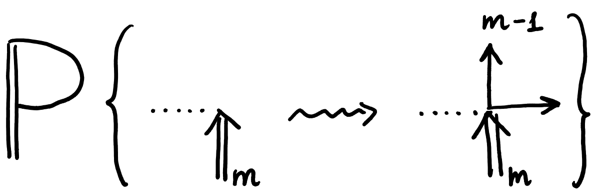
<u>Theorem</u> [B.-Petrov '16]
The resulting random paths in the bottom part of the picture can be constructed recursively via



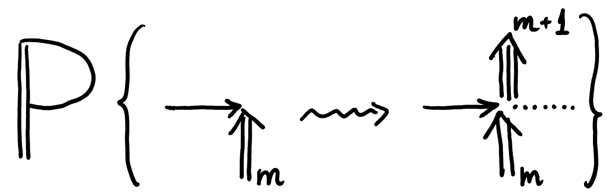
$$= \frac{1 - s_x q^m u_y \xi_x}{1 - s_x u_y \xi_x}$$



$$= \frac{s_x^2 q^m - s_x u_y \xi_x}{1 - s_x u_y \xi_x}$$



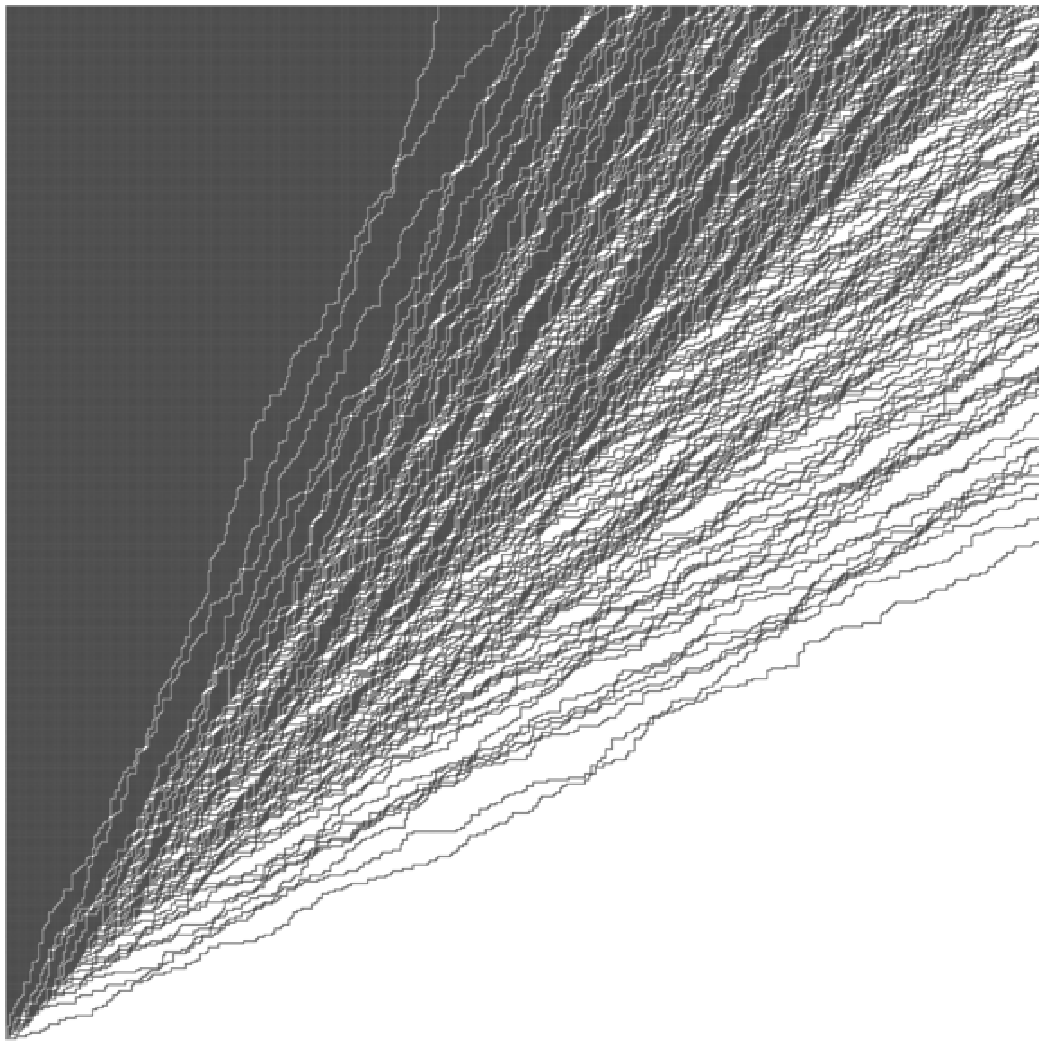
$$= \frac{(q^m - 1) s_x u_y \xi_x}{1 - s_x u_y \xi_x}$$

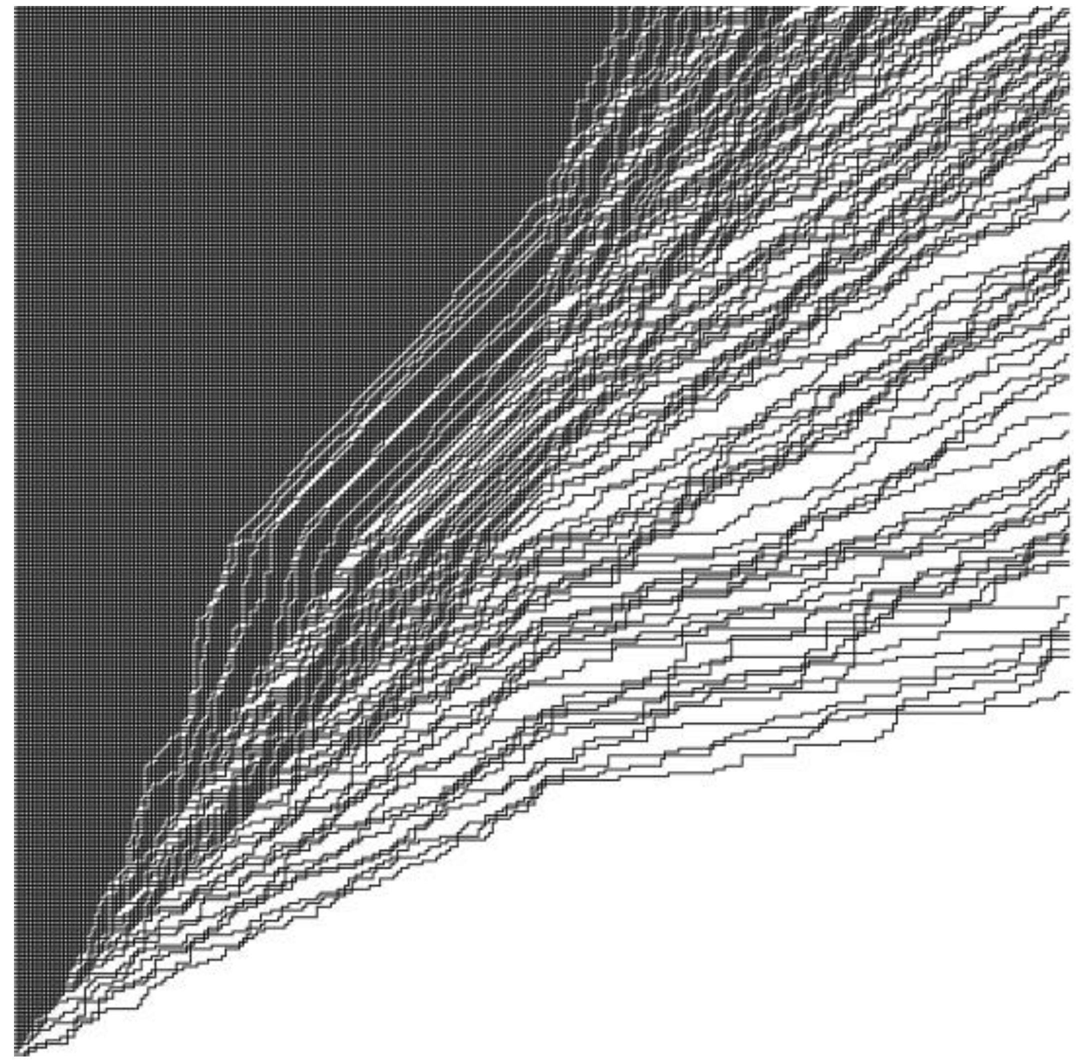


$$= \frac{1 - s_x^2 q^m}{1 - s_x u_y \xi_x}$$

with additional sets of parameters $\{s_x\}_{x \geq 1}$, $\{\xi_x\}_{x \geq 1}$.

Sampling (the six vertex case)



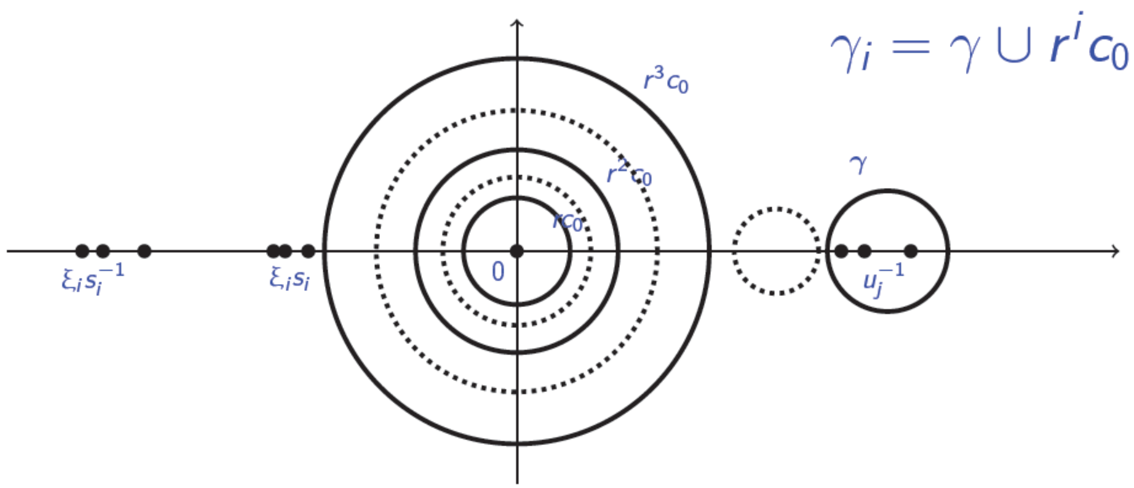


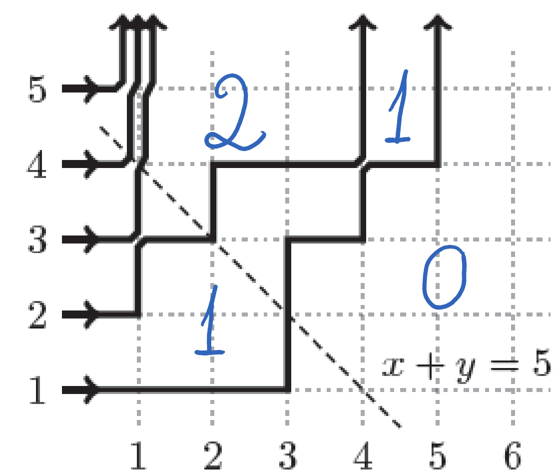
Courtesy of Leo Petrov

q-Moments of the height function

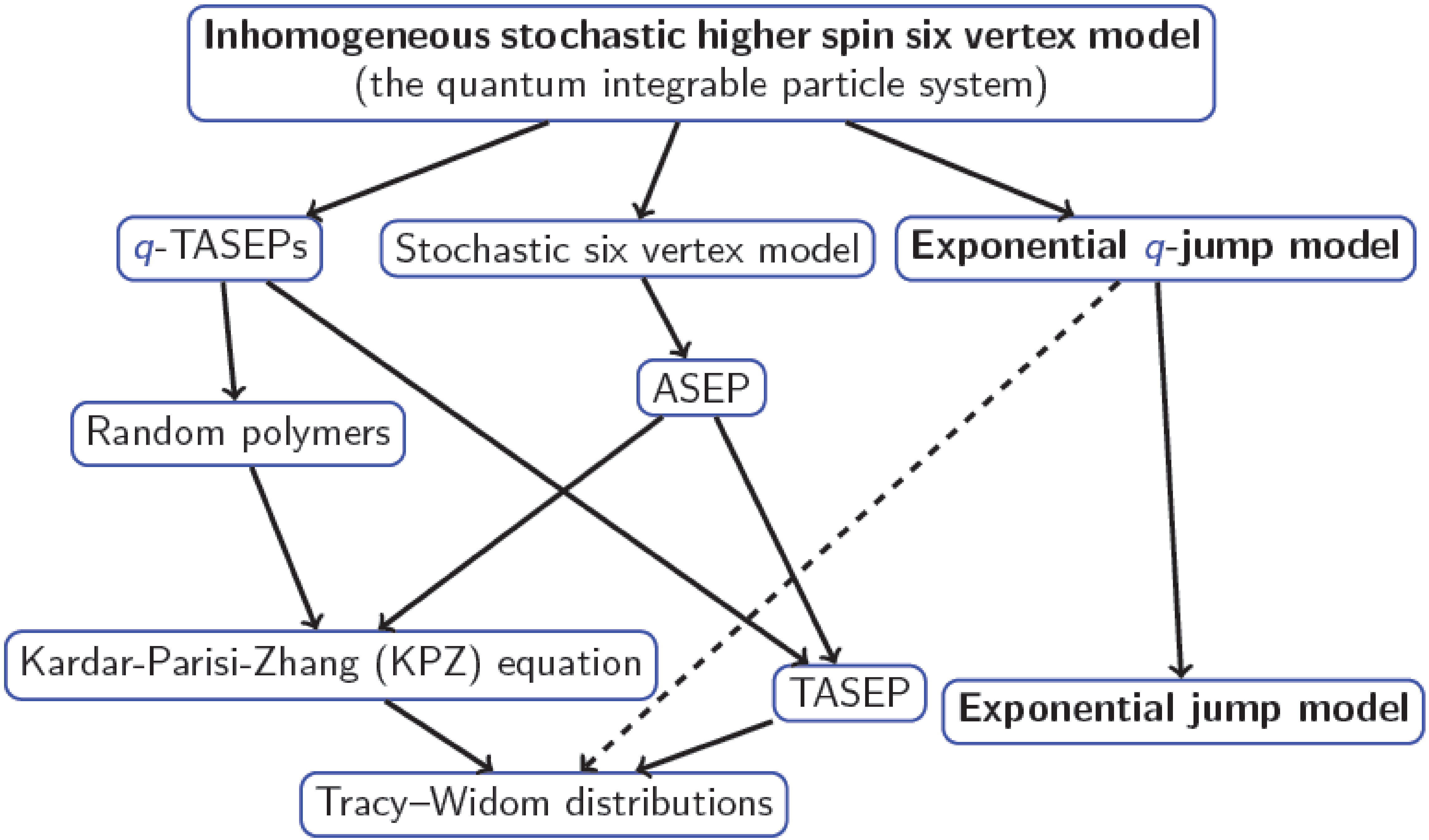
Theorem [B.-Petrov '16] For any $x_1 \geqslant \dots \geqslant x_\ell$, $y \geq 1$,

$$\mathbb{E}\, q^{h(x_1,y)+\dots+h(x_\ell,y)} = \frac{q^{\ell(\ell-1)/2}}{(2\pi i)^\ell} \oint_{\gamma_1}\cdots\oint_{\gamma_\ell} \prod_{1\le A<B\le \ell} \frac{z_A-z_B}{z_A-qz_B} \cdot \prod_{i=1}^{\ell}\left(\prod_{j=1}^{x_i-1}\frac{\xi_j - s_j z_i}{\xi_j - s_j^{-1} z_i}\cdot\prod_{k=1}^{y}\frac{1-q u_k z_i}{1-u_k z_i}\cdot\frac{dz_i}{z_i}\right)$$





$h(x,y)$ = number of paths to the right or through (x,y).



Inhomogeneous stochastic higher spin six vertex model
(the quantum integrable particle system)
q-TASEPs
Stochastic six vertex model
Exponential q-jump model
ASEP
Random polymers
Kardar-Parisi-Zhang (KPZ) equation
TASEP
Exponential jump model
Tracy–Widom distributions

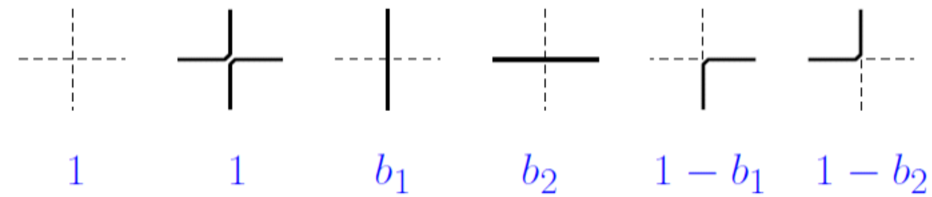
The six vertex case – an asymptotic corollary

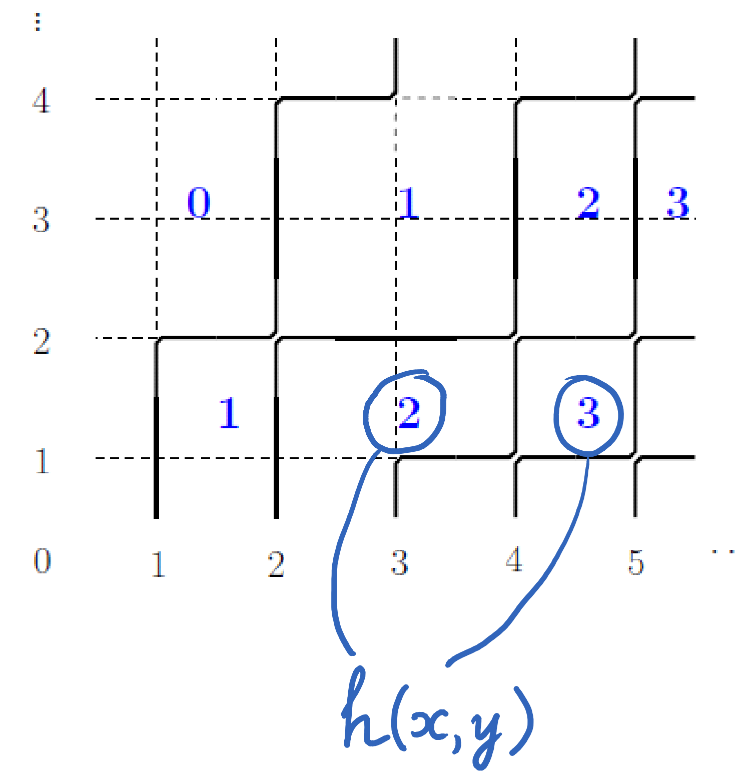
Theorem [B-Corwin-Gorin '14]
Assume $b_1 > b_2$. Then for $\frac{1-b_1}{1-b_2} < \frac{x}{y} < \frac{1-b_2}{1-b_1}$

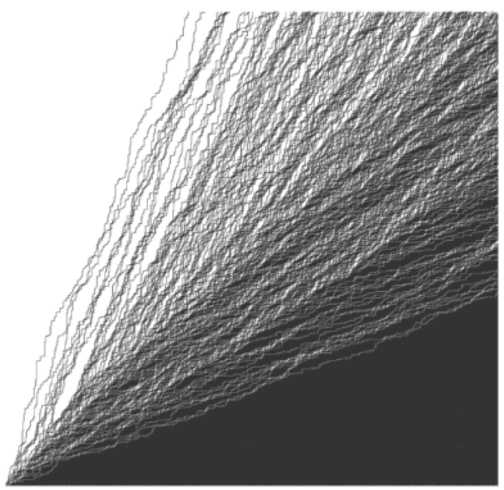
$$\lim_{L\to\infty} \mathbb{P}\left\{ \frac{h(Lx, Ly) - L\cdot H(x,y)}{L^{1/3}\cdot \sigma(x,y)} \leq -s \right\} = F_{GUE}(s)$$

where $H(x,y) = \frac{\left(\sqrt{x(1-b_2)} - \sqrt{y(1-b_1)}\right)^2}{x-y}$, $\sigma(x,y)$ is explicit,

$F_{GUE}(s)$ is the GUE Tracy-Widom distribution.





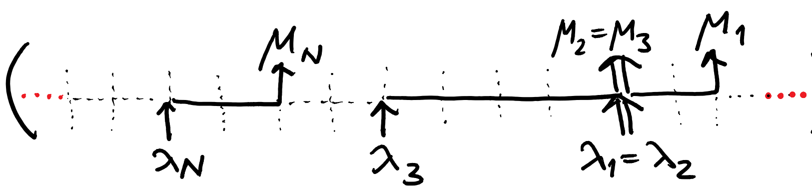


Gwa–Spohn (1992):
This is a member of the KPZ universality class. This class was related to TW in late 1990's.

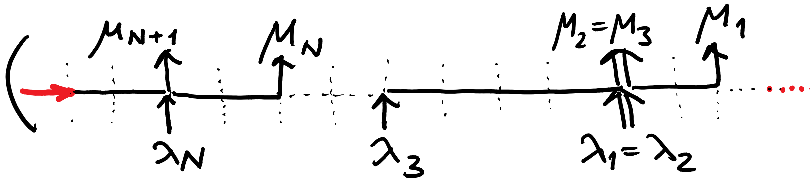
New applications so far

- (2+1)d dynamics that preserves the 6-vertex Gibbs measures on a torus (hypothetically, in (2+1)d AKPZ class)
  [B.-Bufetov '15]
- Unusual phase transitions in the inhomogeneous setting
  [B.-Petrov, in preparation]
- Baik-Ben Arous-Peche type phase transition in ASEP
  [B.-Aggarwal, in preparation]
- Fluctuations along characteristics in equilibrium ASEP
  [Aggarwal, in preparation]

Nuts and bolts – symmetric functions

$$A(u)\, e_\lambda = \sum_\mu \mathrm{weight}_u\Big($$ 

$$\Big)\, e_\mu$$

$$B(u)\, e_\lambda = \sum_\mu \mathrm{weight}_u\Big($$ 

$$\Big)\, e_\mu$$

$$B(u_1)B(u_2)\cdots B(u_M)\, e_\lambda =: \sum_\mu F_{\mu/\lambda}(u_1,\dots,u_M)\, e_\mu, \qquad F_\mu := F_{\mu/\varnothing}$$

$$A(v_1)A(v_2)\cdots A(v_N)\, e_\lambda =: \sum_\mu G_{\mu/\lambda}(v_1,\dots,v_N)\, e_\mu, \qquad G_\mu := G_{\mu/(0^N)}$$

These are symmetric rational functions. Also,

$$F_\mu(u_1,\dots,u_M) = \frac{(1-q)^M}{\prod_{i=1}^{M}(1-su_i)} \cdot \sum_{\sigma\in S_M} \sigma\left(\prod_{i<j} \frac{u_i-qu_j}{u_i-u_j} \cdot \prod_{i=1}^{M}\left(\frac{u_i-s}{1-su_i}\right)^{\mu_i} \right)$$

and there is a similar formula for G_μ (homogeneous system).

Cauchy identities

The commutation relation

$$B(u_1)\,\overline{D}(u_2) = \frac{u_1 - u_2}{qu_1 - u_2}\,\overline{D}(u_2)\,B(u_1)$$

is equivalent to the skew Cauchy identity

$$\sum_{\varkappa} G^{c}_{\varkappa/\lambda}(v)\,F_{\varkappa/\mu}(u) = \frac{1-quv}{1-uv}\sum_{\nu} F_{\lambda/\nu}(u)\,G^{c}_{\mu/\nu}(v)$$

with $G^{c}_{\alpha/\beta} = \frac{c(\alpha)}{c(\beta)}\,G_{\alpha/\beta}$, $\quad c(\gamma) = \prod_{k\geq 0}\frac{(s^2;q)_{g_k}}{(q;q)_{g_k}}$ for $\gamma = 0^{g_0}1^{g_1}2^{g_2}\ldots$.

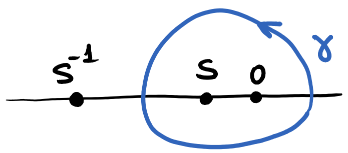
Iterations show that $\left(F_\lambda(u)\right)$ are eigenfunctions of $\left[G_{\mu/\nu}\right]$, and give the usual Cauchy identity

$$\sum_{\mu} F_{\mu}(u_1,\ldots u_M)\,G^{c}_{\mu}(v_1,\ldots,v_N) = \prod_{i=1}^{M}\frac{1-q^i}{1-su_i}\cdot\prod_{\substack{i=1,\ldots,M\\ j=1,\ldots,N}}\frac{1-qu_iv_j}{1-u_iv_j}.$$

Orthogonality

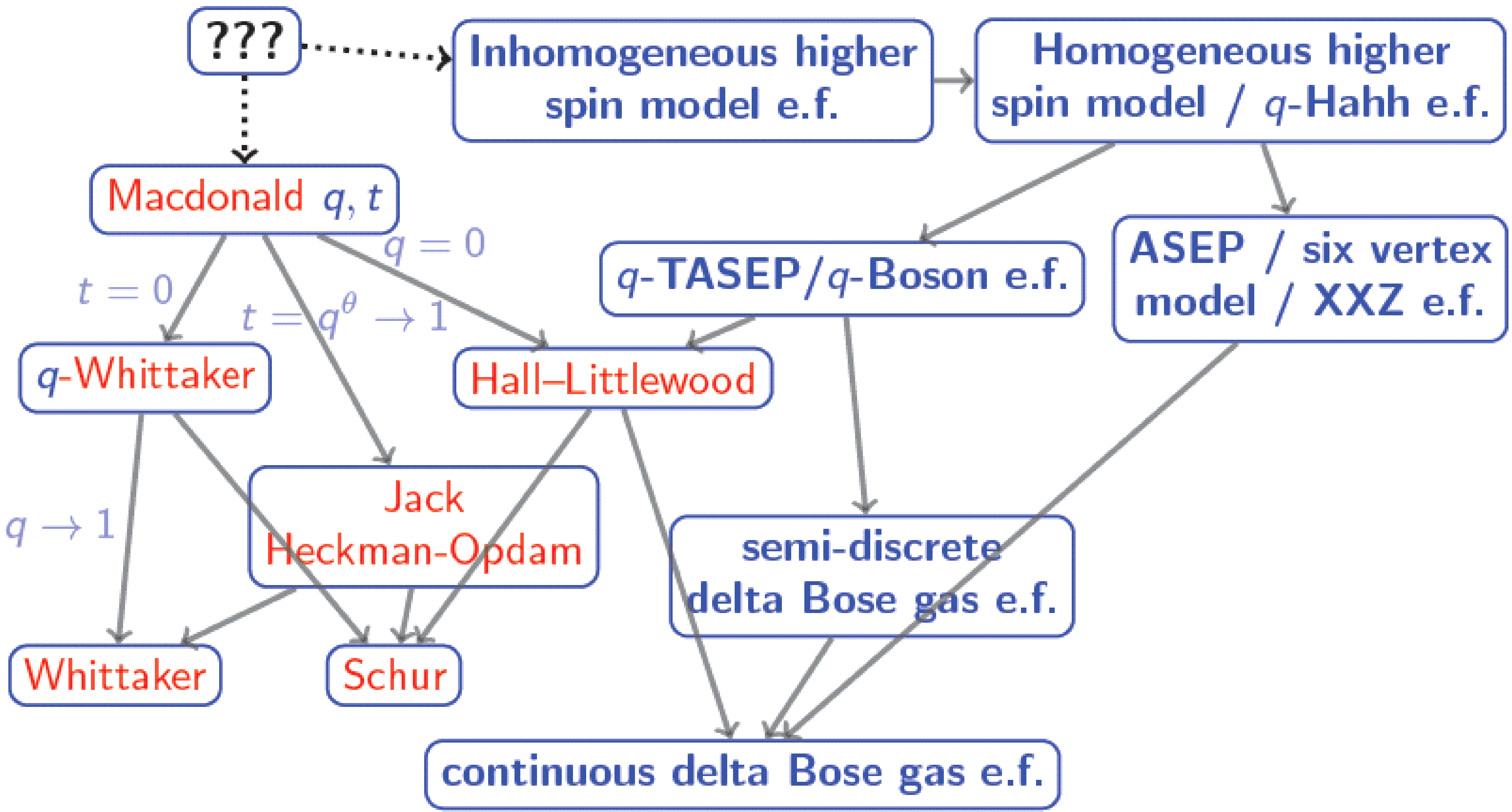
Theorem [Povolotsky '13, B.-Corwin-Petrov-Sasamoto '14-15, B.-Petrov '16]

$$\frac{c(\lambda)}{(2\pi i)^k (1-q)^k\, k!} \oint_\gamma \cdots \oint_\gamma \prod_{1\le A\ne B\le k} \frac{u_A-u_B}{u_A-qu_B}\, F_\lambda(u_1,\dots,u_k)\, F_\mu(u_1^{-1},\dots,u_k^{-1}) \prod_{i=1}^{k} \frac{du_i}{u_i} = \mathbb{1}_{\lambda=\mu}.$$



This orthogonality relation and the Cauchy identities are two basic ingredients that are needed to prove the q-moment formula.

This relation also described a selection rule for "rapidities" u that turn the "off-shell Bethe vectors" F(u) into a complete orthogonal basis.


???
Inhomogeneous higher spin model e.f.
Homogeneous higher spin model / q-Hahn e.f.
Macdonald q, t
$q = 0$
$t = 0$
$t = q^{\theta} \to 1$
q-TASEP/q-Boson e.f.
ASEP / six vertex model / XXZ e.f.
q-Whittaker
Hall–Littlewood
$q \to 1$
Jack Heckman-Opdam
semi-discrete delta Bose gas e.f.
Whittaker
Schur
continuous delta Bose gas e.f.

Summary

- The higher spin six vertex model allows domains for which the partition functions are simple products.
- A specialization of spectral parameters on a part of such a domain gives a Markovian ("stochastic") model.
- For the stochastic model, a large set of observables can be explicitly averaged, leading to asymptotic analysis.
- The key tool is a new family of symmetric rational functions whose properties are derived directly from the Yang-Baxter eq.