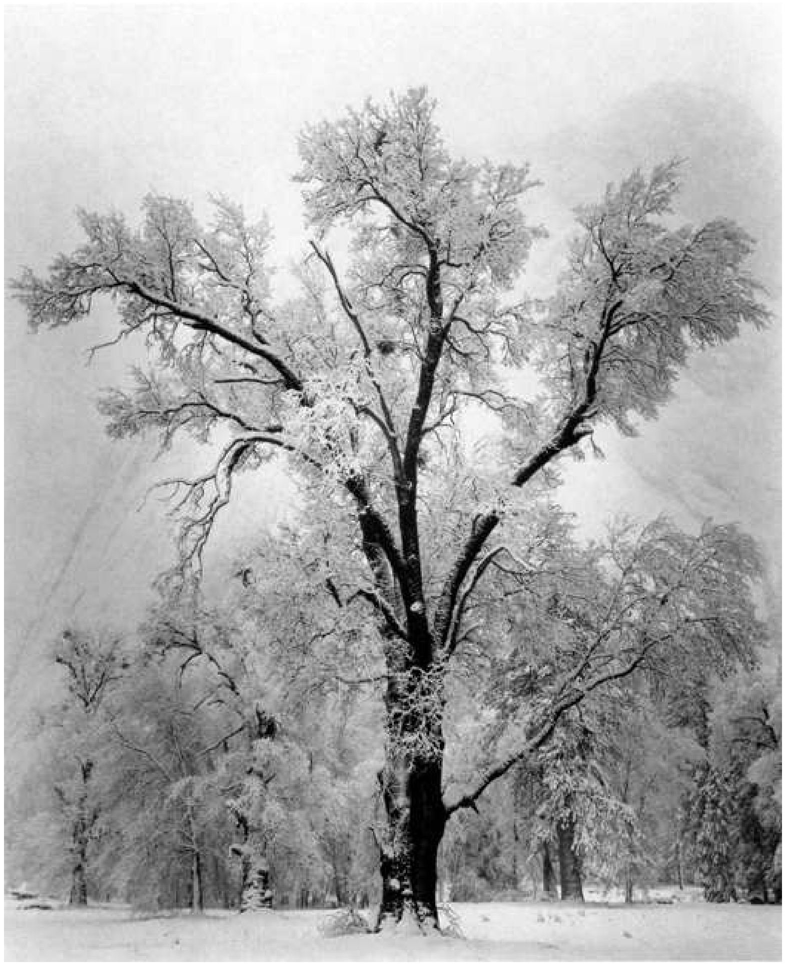
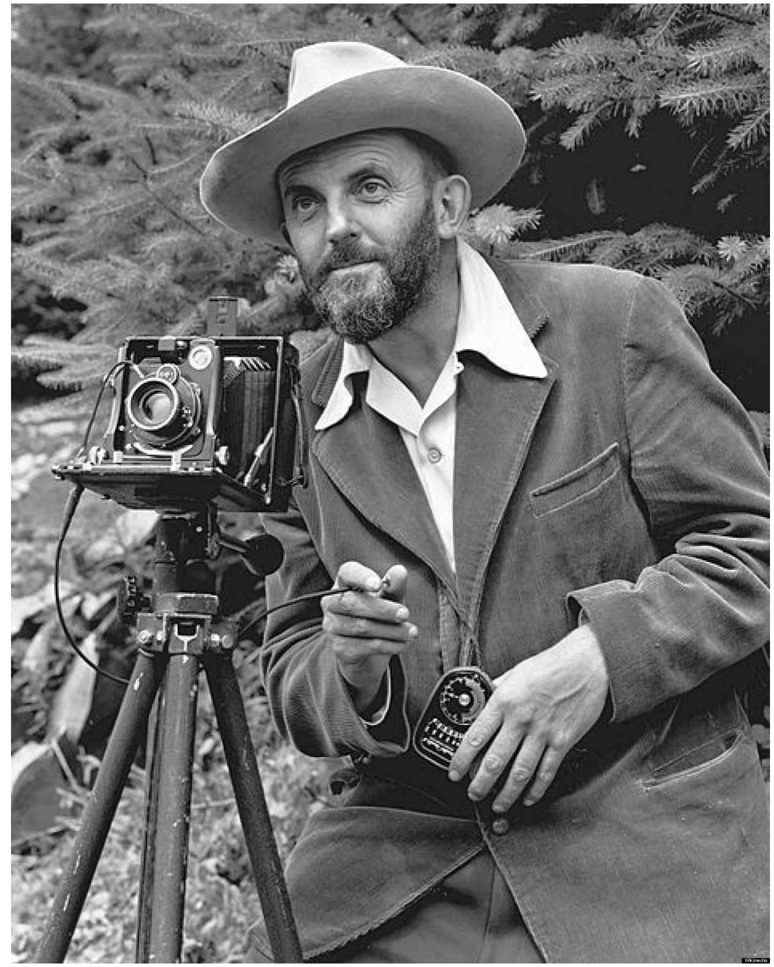
Snowflakes and Trees

Christopher Bishop, Stony Brook

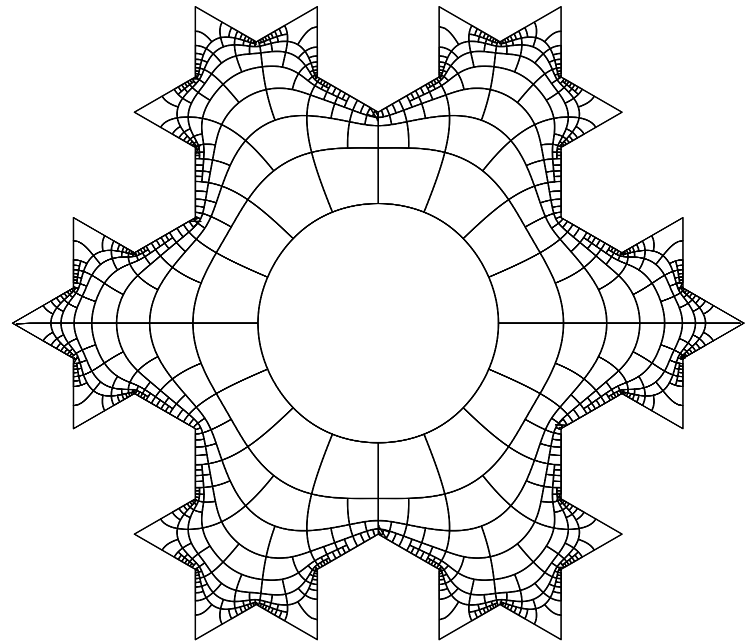
Everything is Complex – March 6–11 , 2016 Saas-Fee, Switzerland

www.math.sunysb.edu/~bishop/lectures

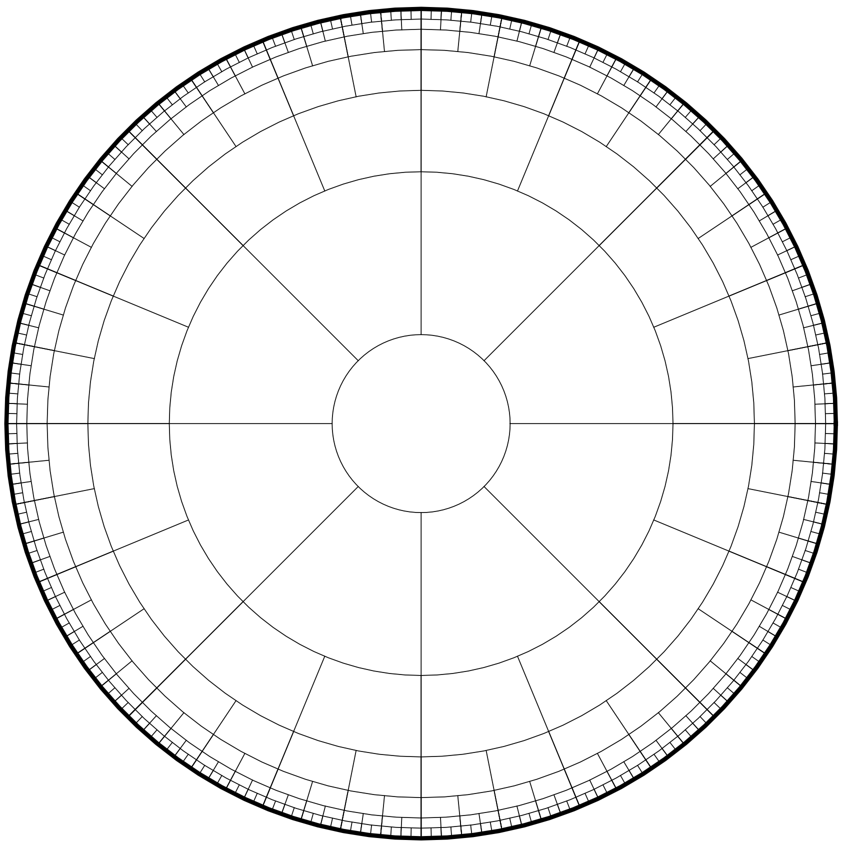
 

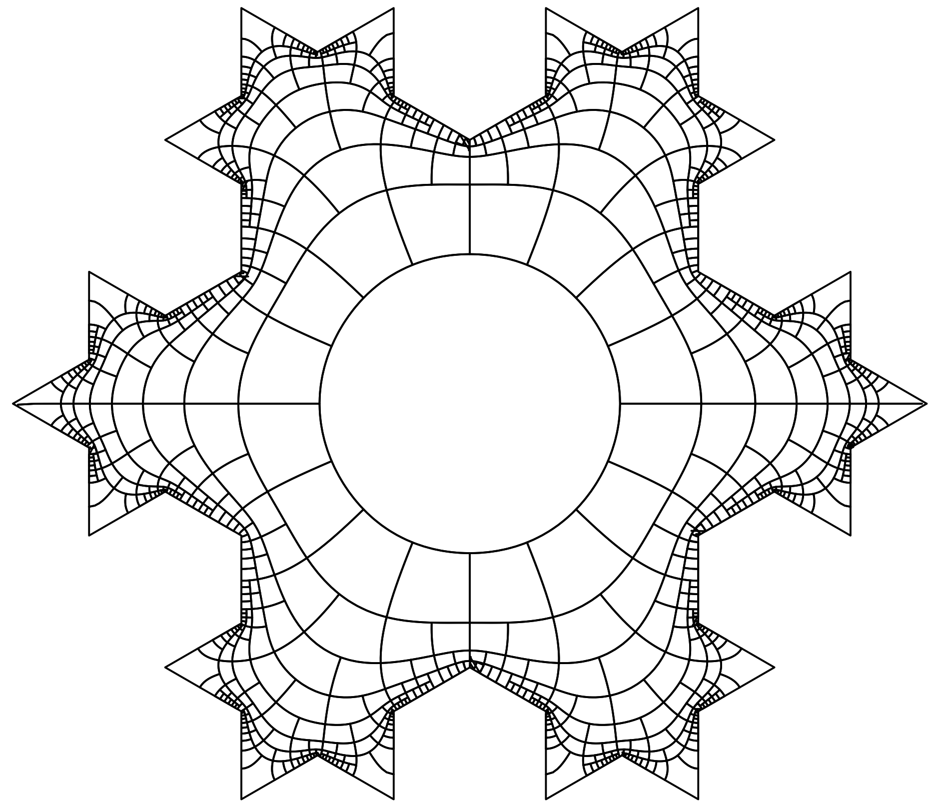
Ansel Adams

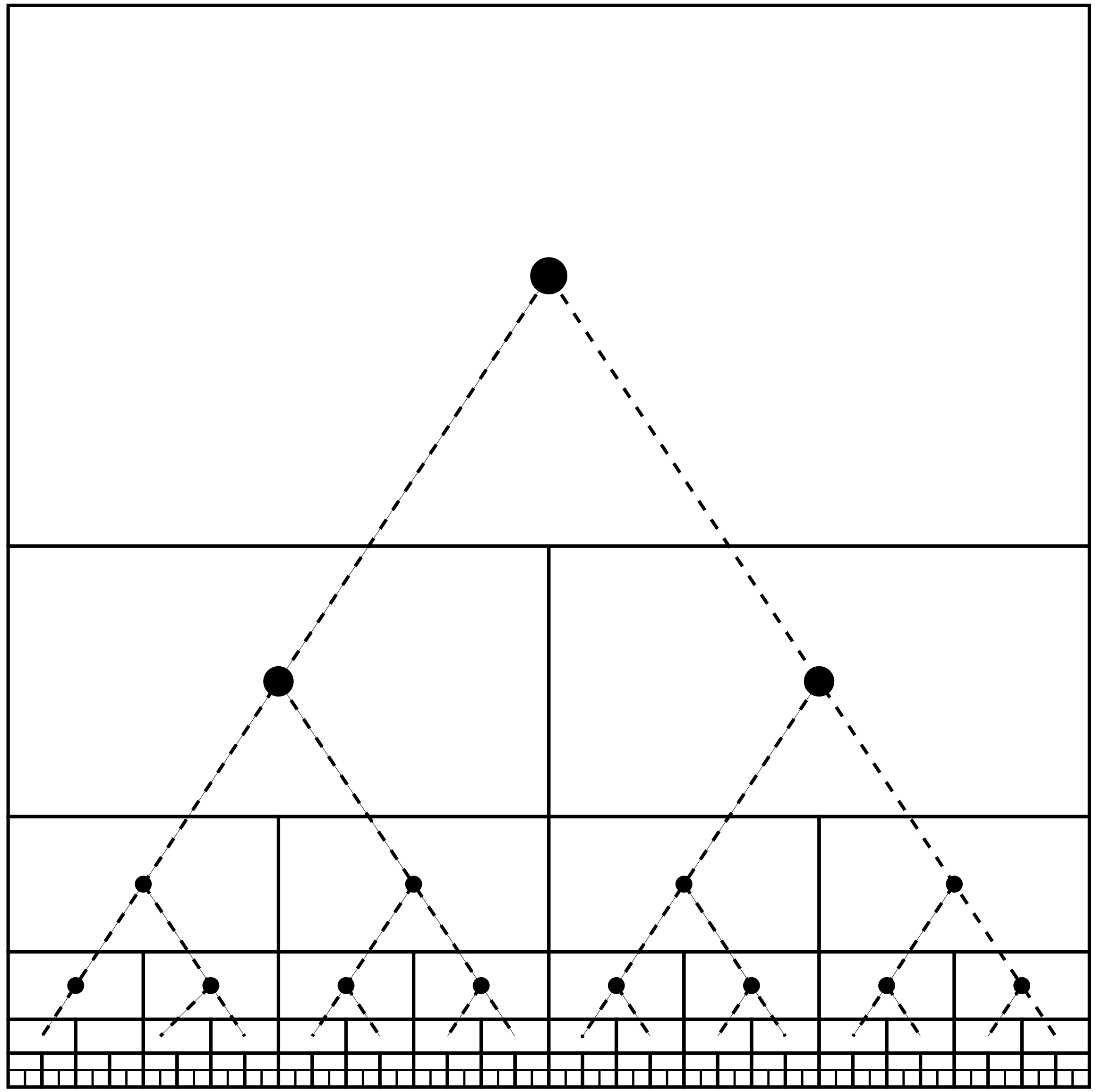
Pyotr Ilyich Tchaikovsky

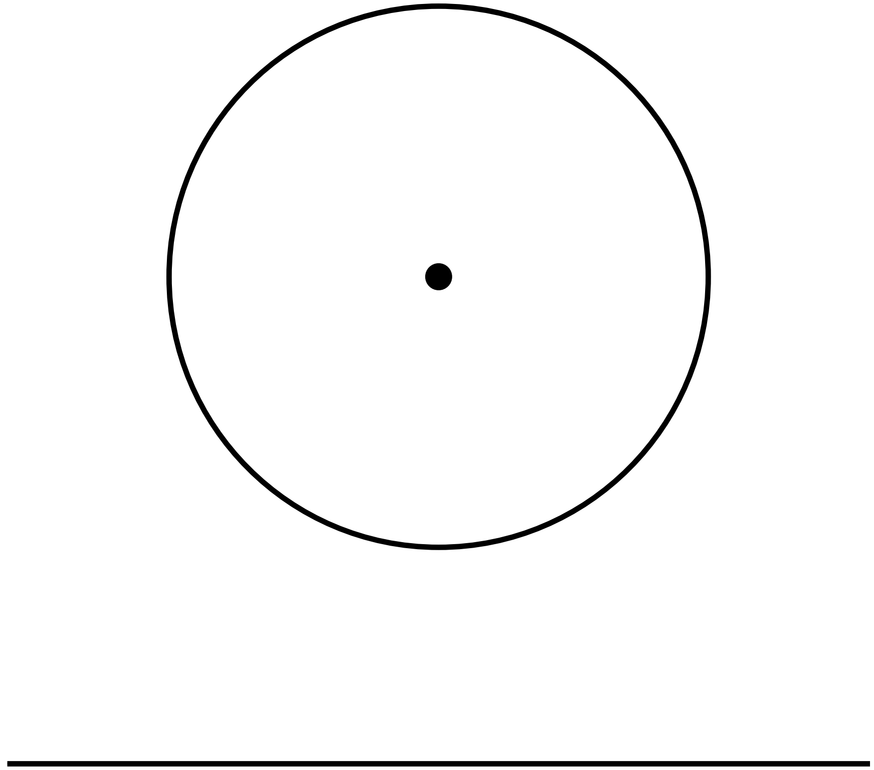


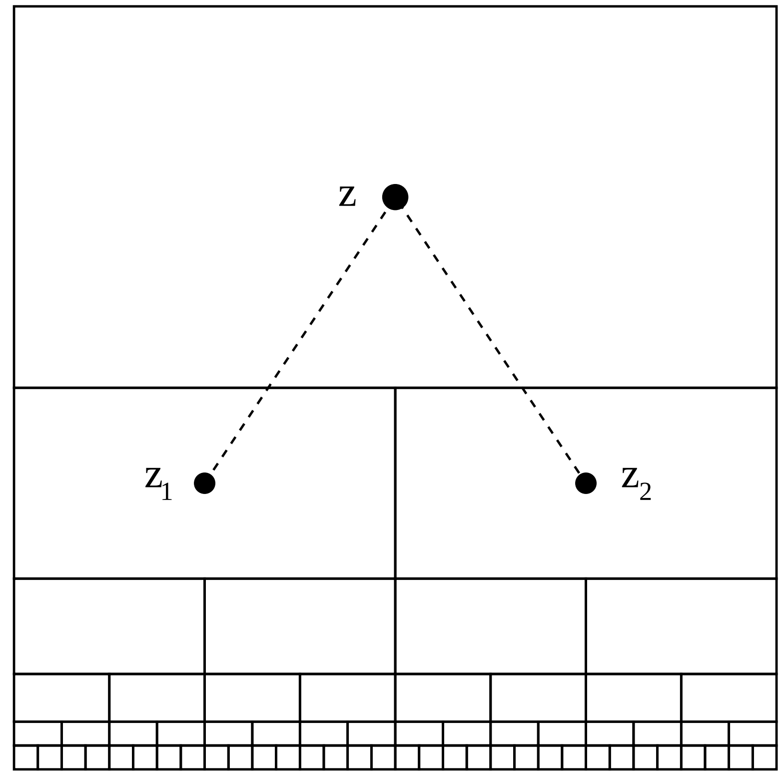
Nikolai G. Makarov





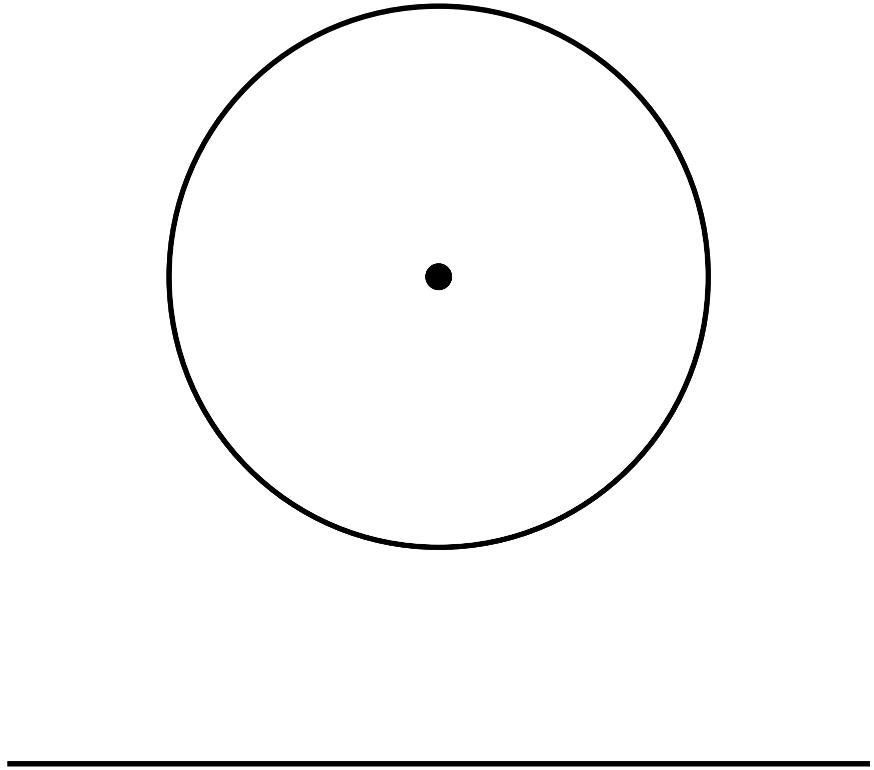


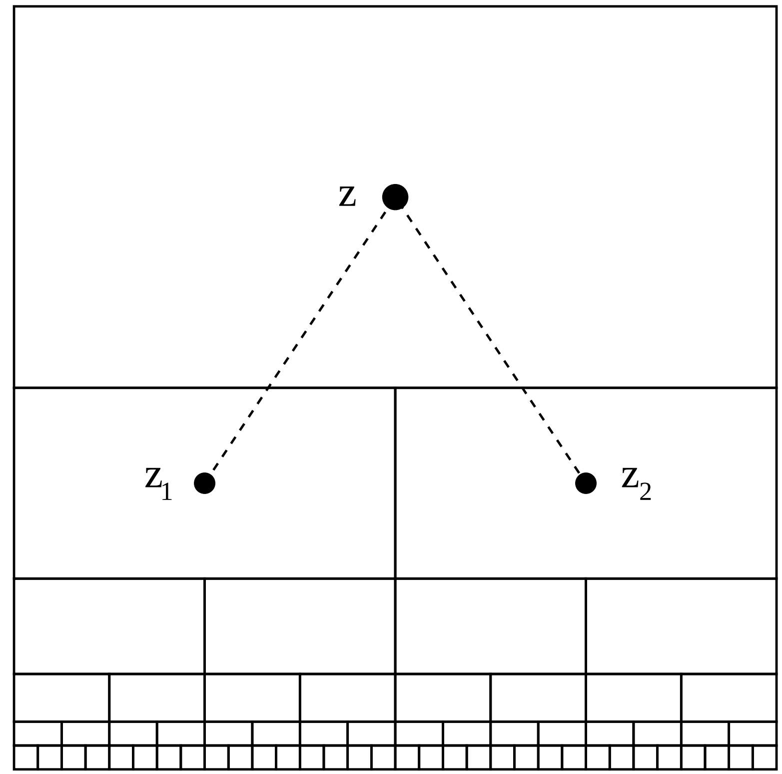




$$u(z) = \frac{1}{2\pi} \int u(z + re^{i\theta}) d\theta \qquad\qquad u(z) = \frac{1}{2}(u(z_1) + u(z_2))$$

"tree version" of mean value property means u is a dyadic martingale.





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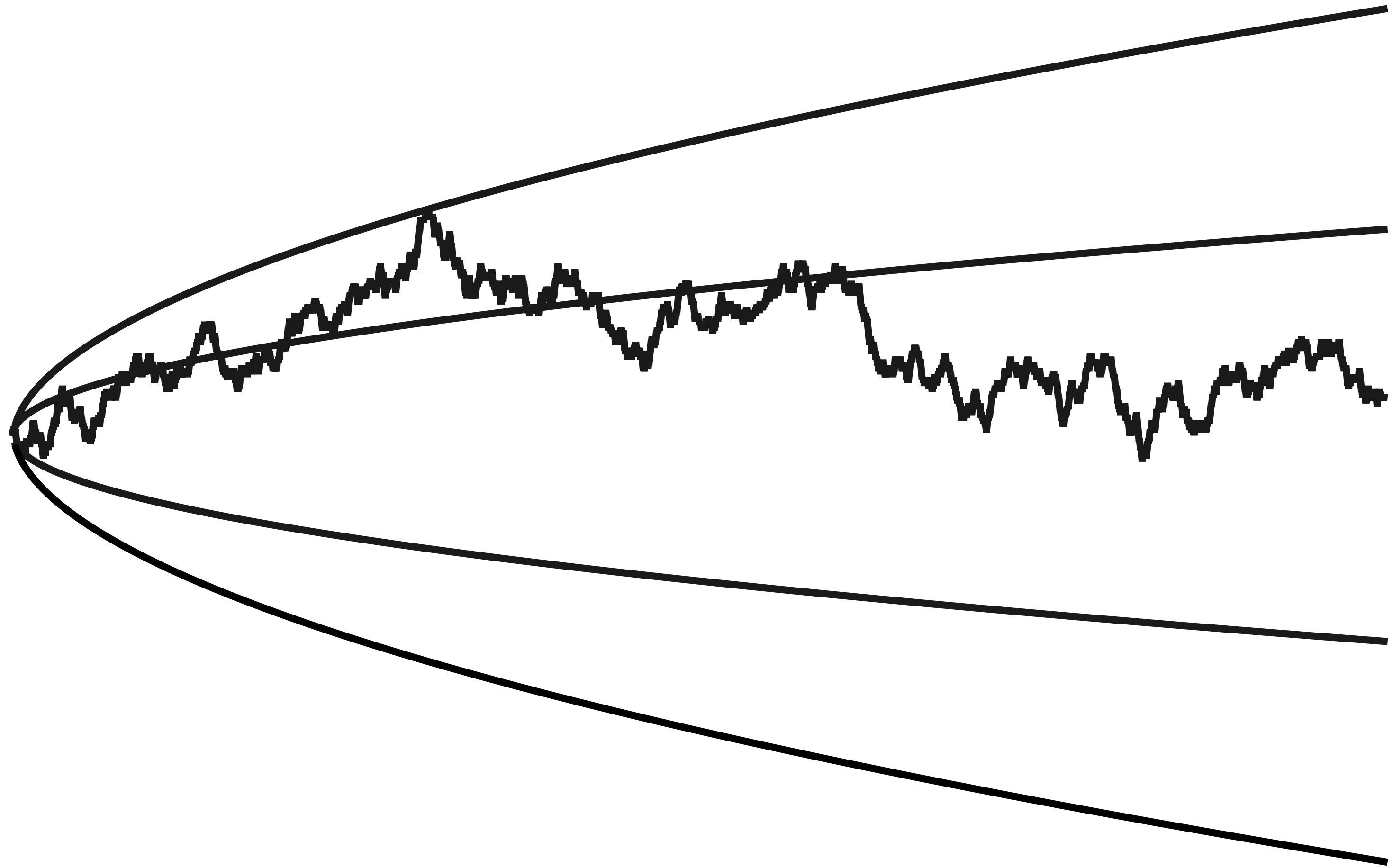
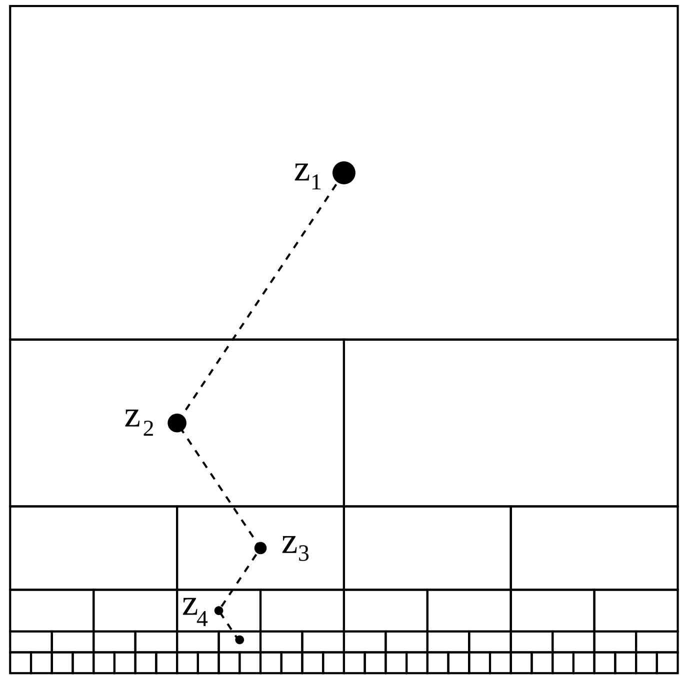
If true, then u behaves like a random walk, satisfies LIL.

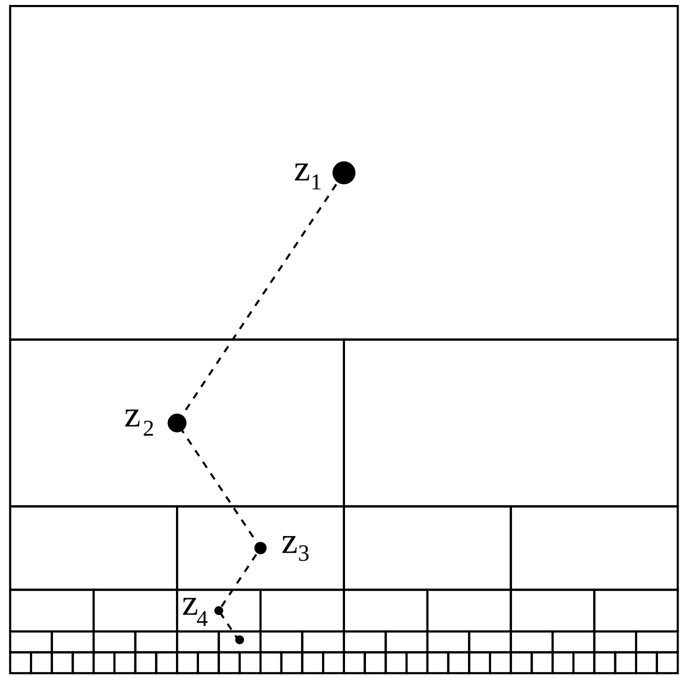


Law of the iterated logarithm: $|S_n| \leq \sqrt{2n \log \log n}$



$$1 - |z_n| = 2^{-n} \quad \Rightarrow \quad n = \log \frac{1}{1 - |z_n|}$$

$$|u(z_n)| \leq C \sqrt{\log \frac{1}{1 - |z_n|} \log \log \log \frac{1}{1 - |z_n|}}$$



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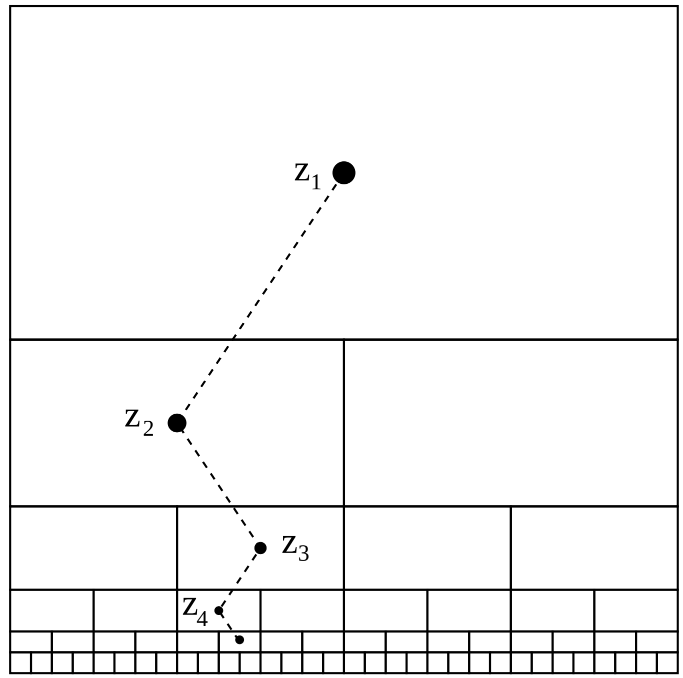
Makarov proved this is really true!



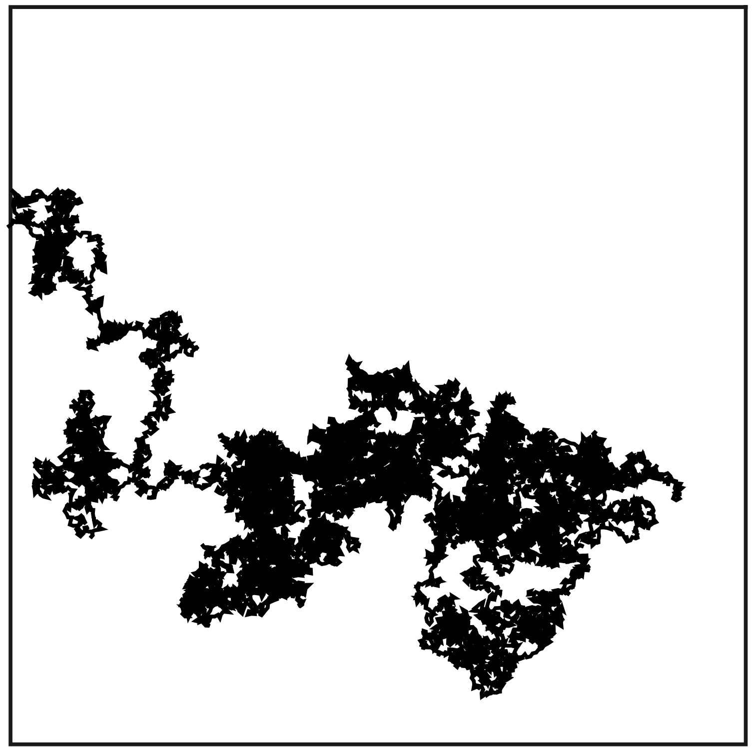
Makarov's LIL: If u is harmonic and Bloch,

$$|u(z)| \leq C(\log \frac{1}{1-|z|} \log\log\log \frac{1}{1-|z|})^{1/2}$$

Bloch means $|\nabla u(z)| = O(1/(1-|z|))$.

Applying to $u = \log|f'|$, f conformal, gives LIL for harmonic measure.

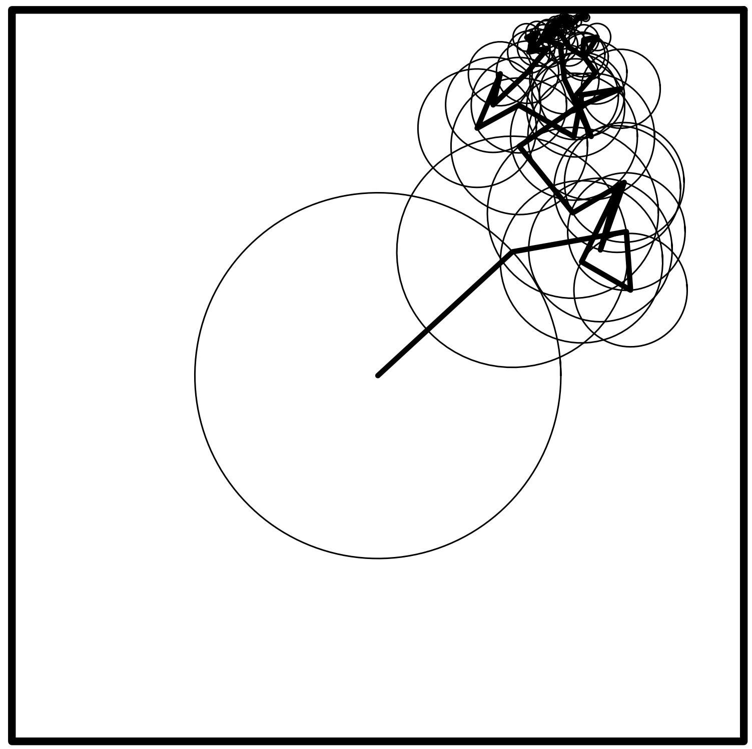
Harmonic measure = hitting distribution of Brownian motion



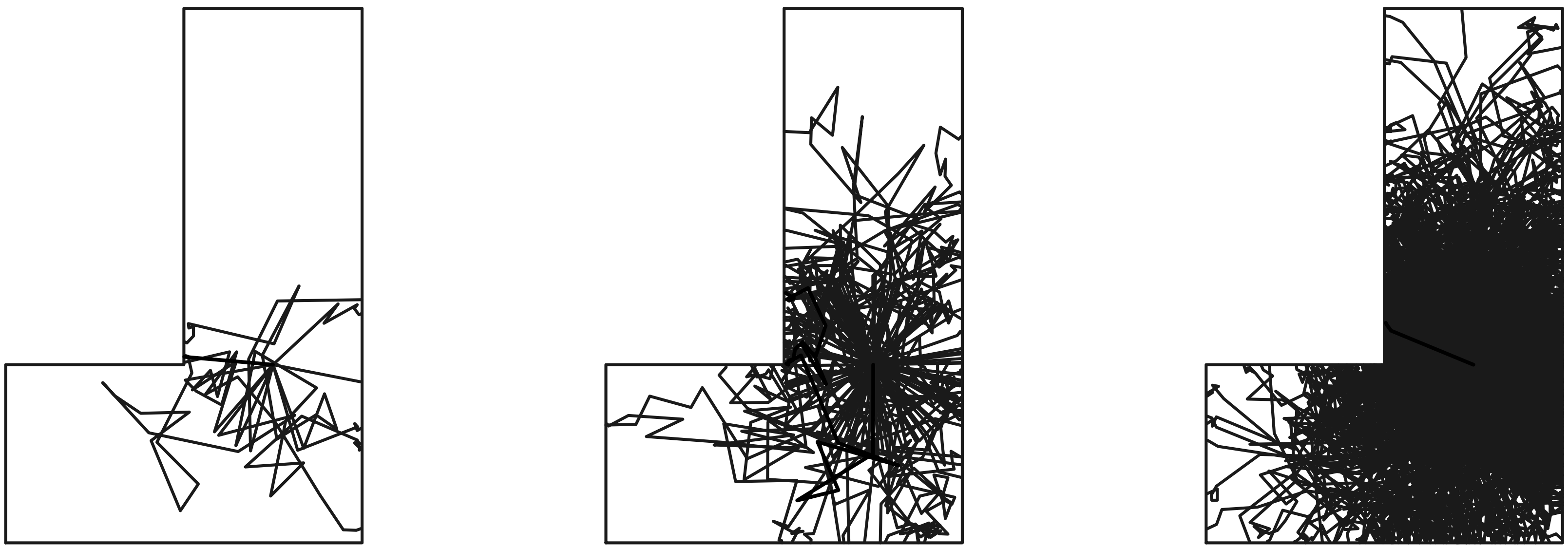
$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Usually we drop z, Ω, just write $\omega(E)$.

Harmonic measure = hitting distribution of Brownian motion

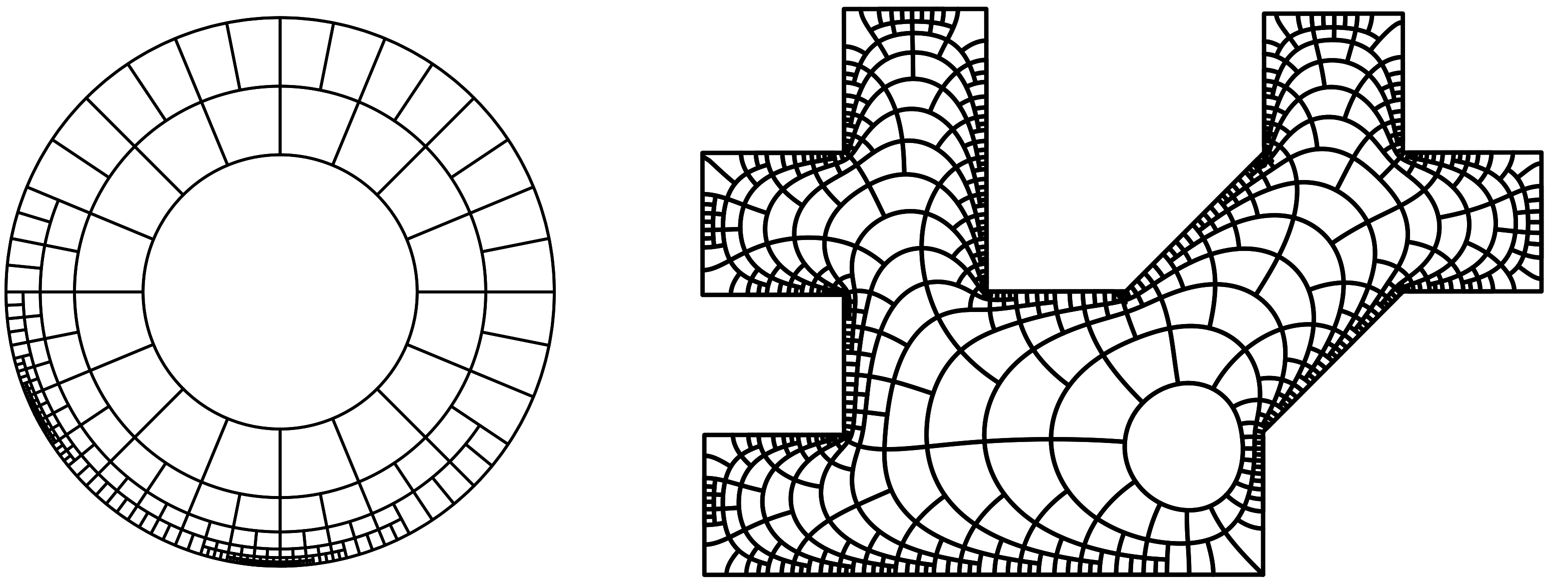


Harmonic measure also equals distribution of the discrete process that steps half-way to boundary each time (faster to simulate).



Step half-way to boundary, $N = 10, 100, 1000$.

By conformal invariance, harmonic measure equals image of length on unit circle under Riemann map. Powerful tool in 2 dimensions.



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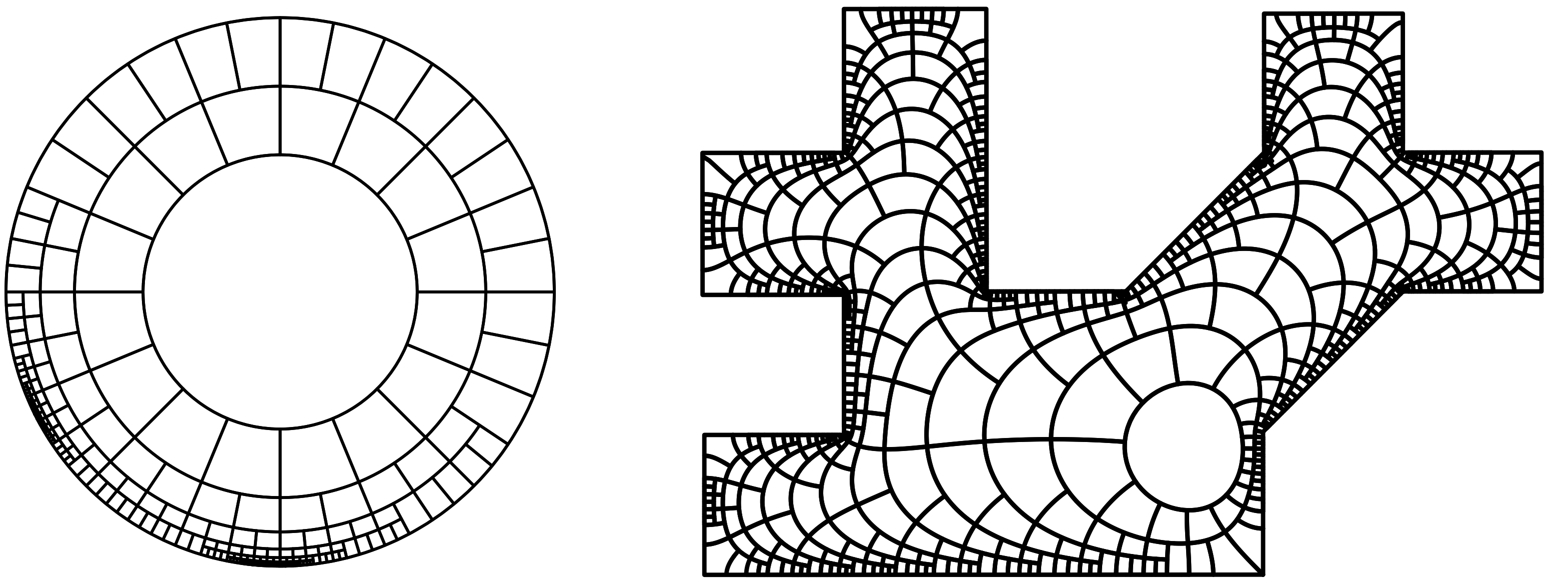
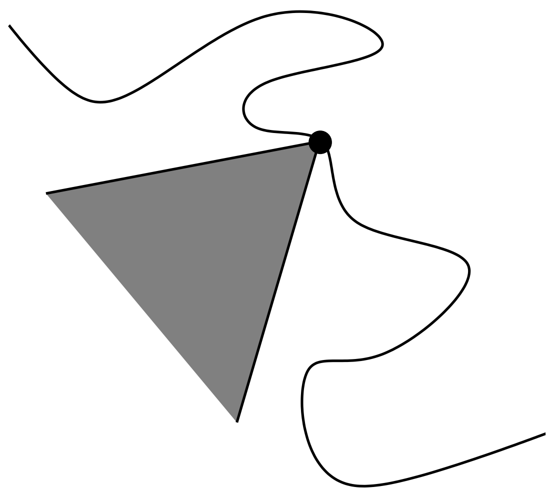


For simply connected domains Makarov proved:

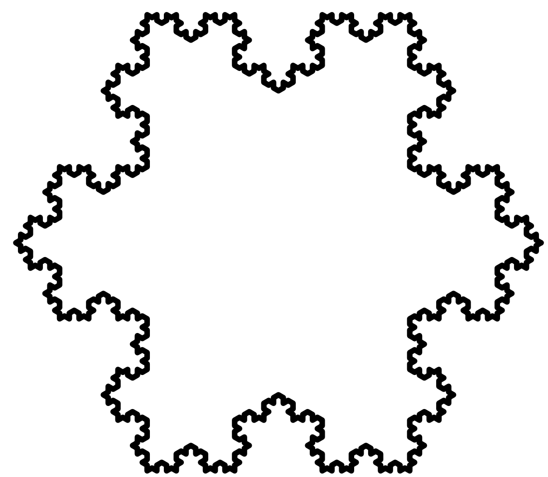
- Harmonic measure gives full mass to some set of dimension 1.
- Harmonic measure gives zero mass to sets of dimension < 1.
- Computed sharp guage function (LIL) and where $\omega \ll \Lambda_1$ or $\omega \perp \Lambda_1$.

F. and M. Riesz (1916): $\Lambda_1 \ll \omega \ll \Lambda_1$ on cone points.

Makarov (1984): $\limsup_{r \to 1} \frac{\omega(D(x,r))}{r} = \infty, \quad \omega$-a.e. off cone points.



A cone point



Cone points have zero length

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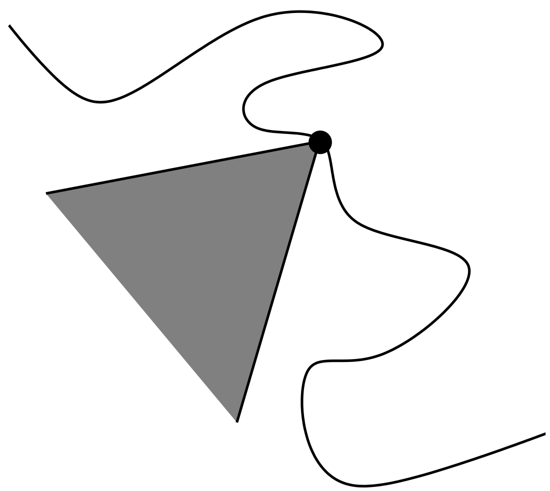

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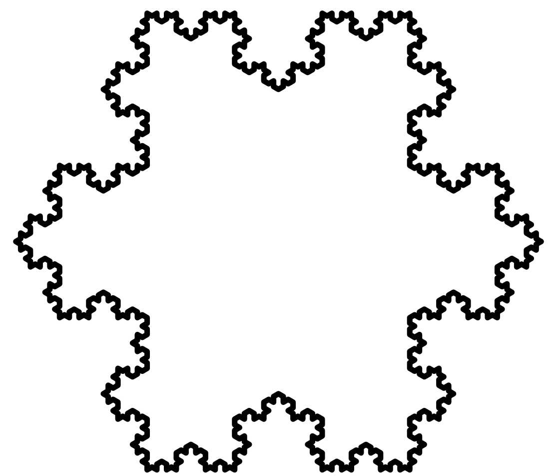

Cone points have zero length

Sunhi Choi (2004): $\liminf_{r \to 1} \frac{\omega(D(x,r))}{r} = 0, \quad \omega$-a.e. off cone points.

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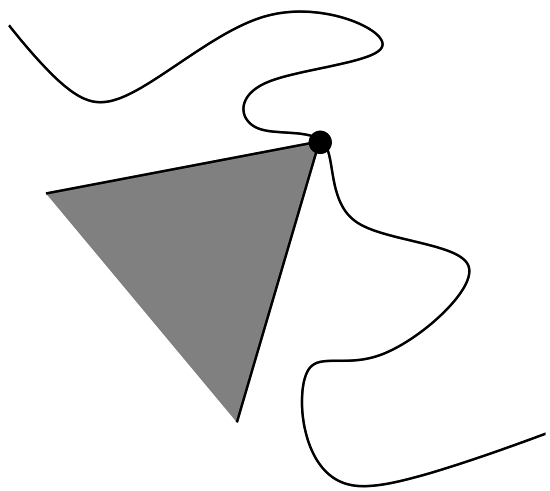
Makarov (1984): $\limsup_{r\to 1} \frac{\omega(D(x,r))}{r} = \infty$, ω-a.e. off cone points.



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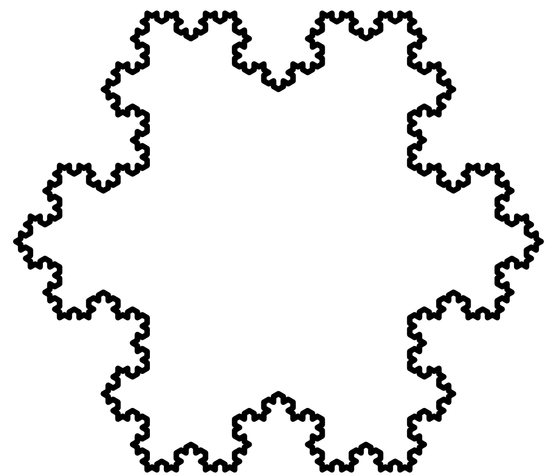
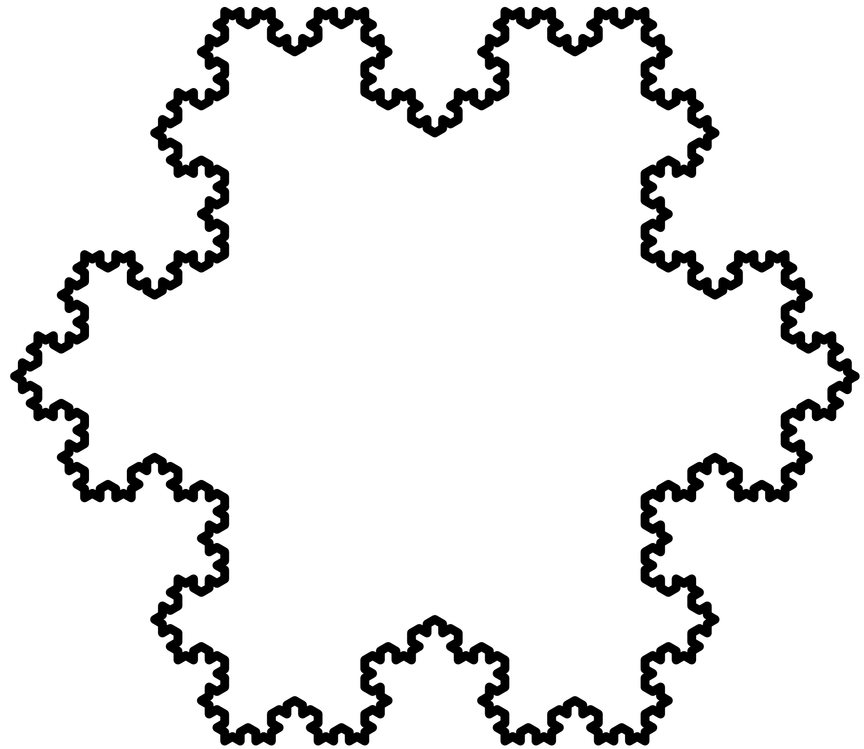


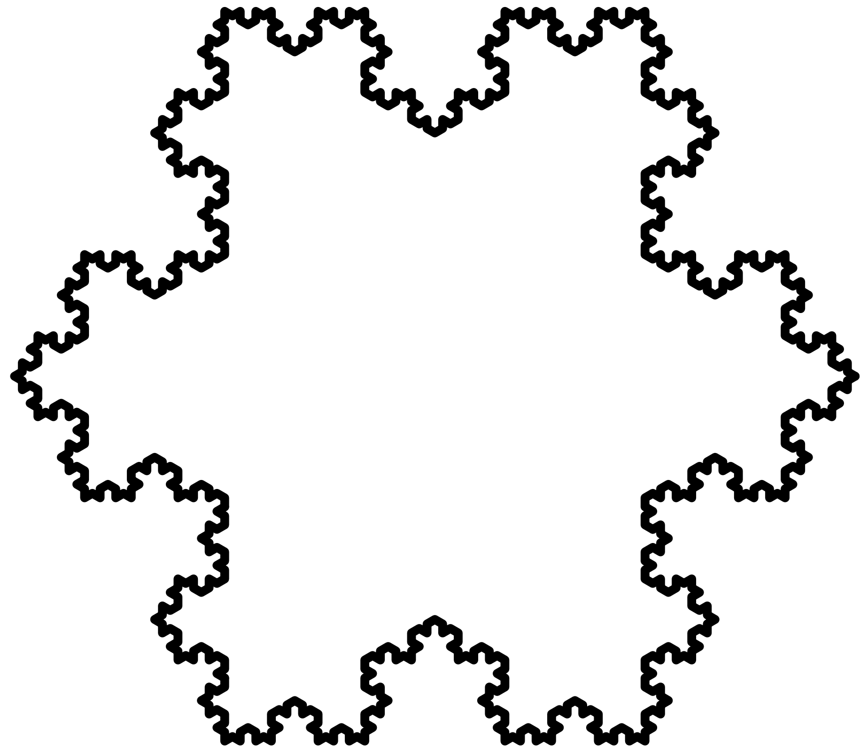
Cone points have zero length

For general domains (not simply connected), is $\limsup_{r\to 1} \frac{\omega(D(x,r))}{r} = \infty$, ω-a.e. off cone points?



For snowflakes $\omega \perp \Lambda_1$

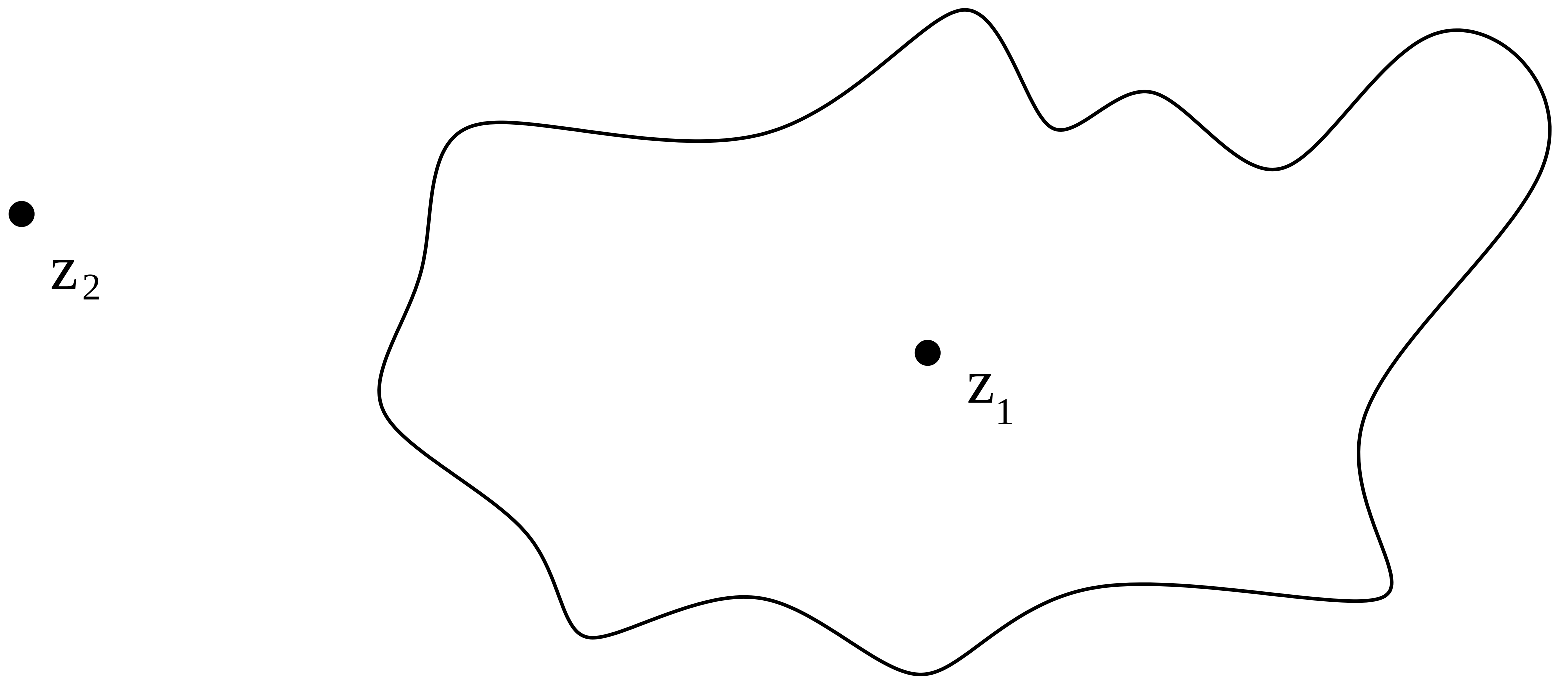
(Almost all Brownian particles absorbed by set of zero 1-measure)



For snowflakes $\omega \perp \Lambda_1$
(Almost all Brownian particles absorbed by set of zero 1-measure)

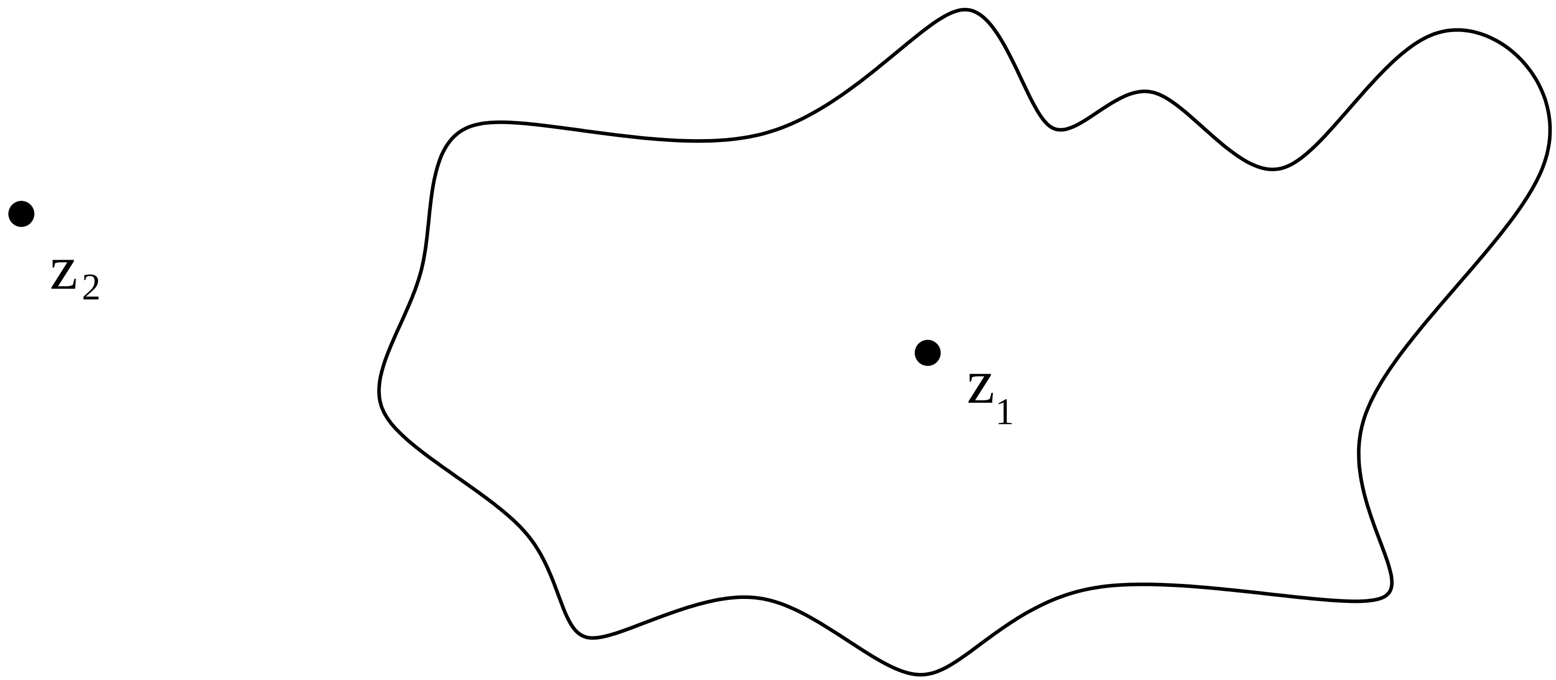
Same is true for outside domain. Same set of zero length?

For a closed curve γ, define ω_1, ω_2 for different sides.



Basic problem: how is geometry of γ reflected in ratio ω_1/ω_2?

For a closed curve γ, define ω_1, ω_2 for different sides.



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For example, if $\omega_1 = \omega_2$ must Γ be a circle?

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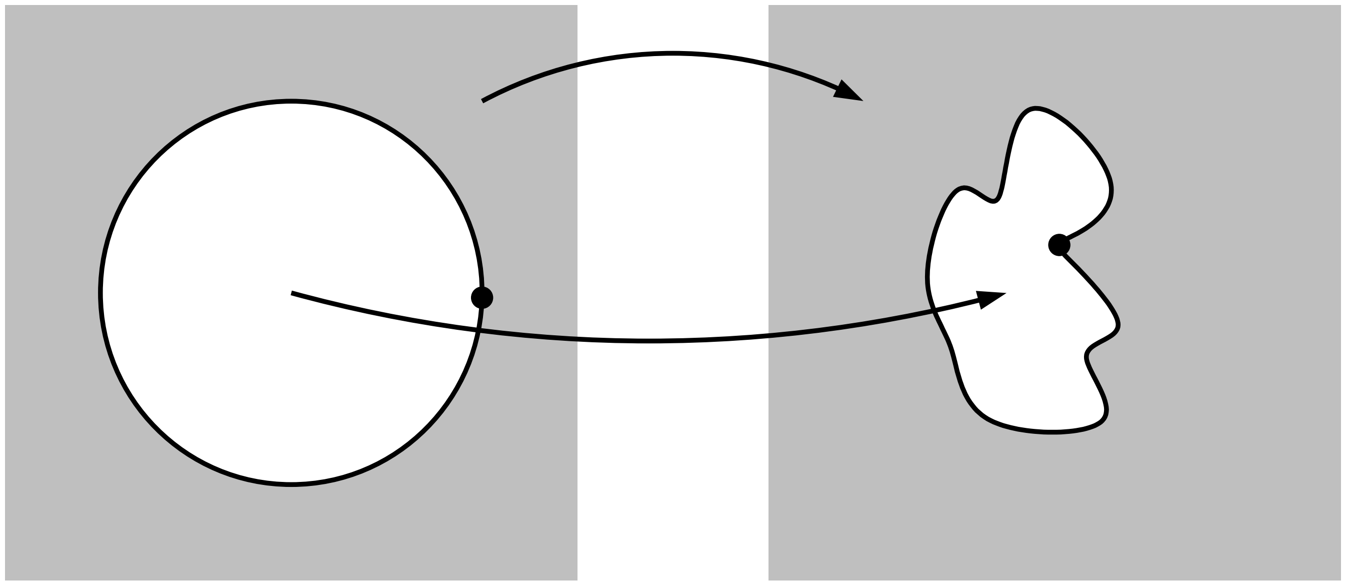

Conformally map inside to inside, outside to outside.

$\omega_1 = \omega_2$ implies maps agree on boundary.

Get homeomorphism of plane holomorphic off circle.

Is entire by Morera's theorem. 1-1 implies linear.

What happens if measures are comparable?

$$\frac{1}{C} \leq \frac{\omega_1}{\omega_2} \leq C$$

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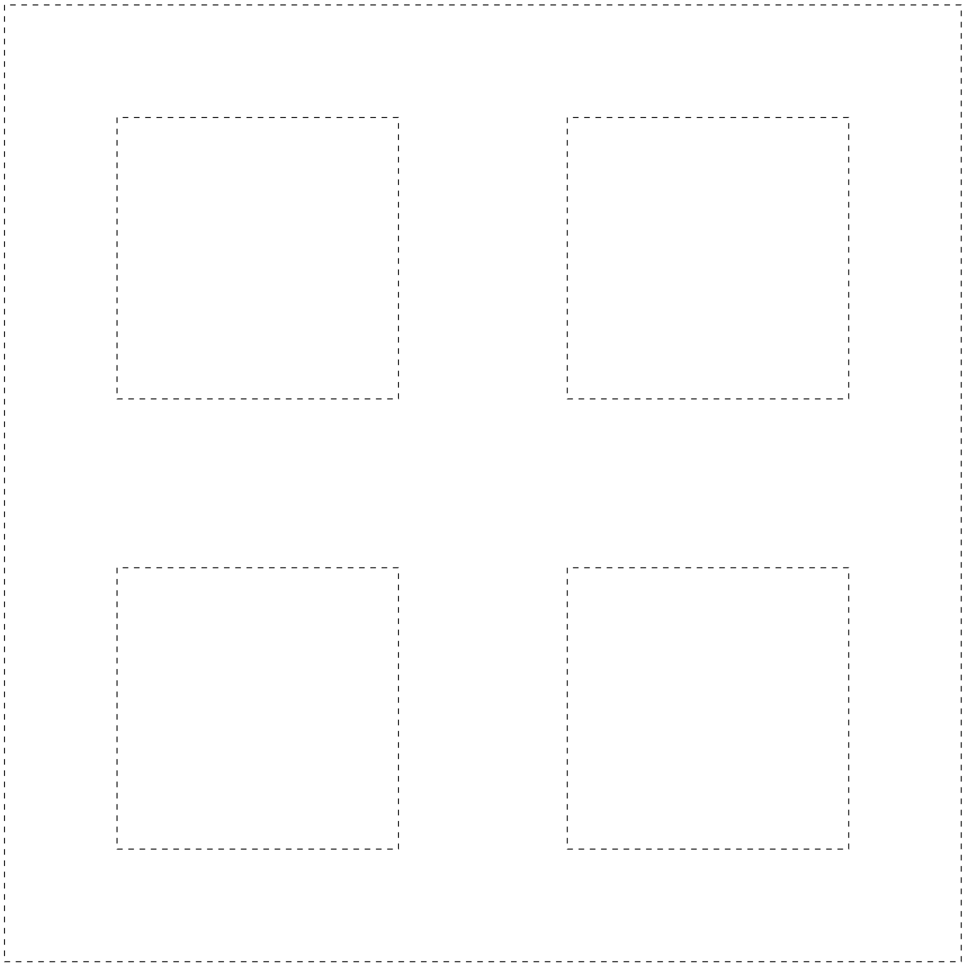
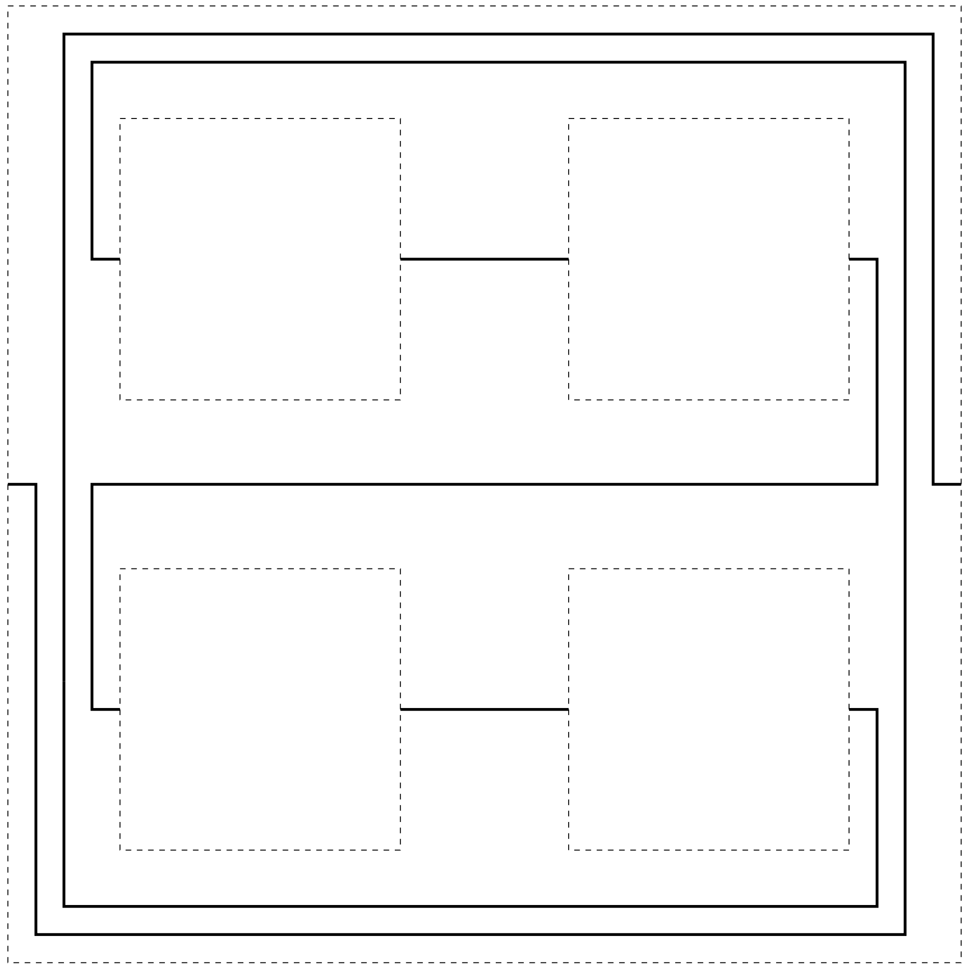
- If $C < \infty$, γ is a quasi-circle (Beurling and Ahlfors). Converse false.
- If $C = 1 + \epsilon$, then γ is rectifiable (G. David).
- For C is large, γ need not be rectifiable (Semmes).

For $\omega_1 \sim \omega_2$ with large C, γ need not be rectifiable.



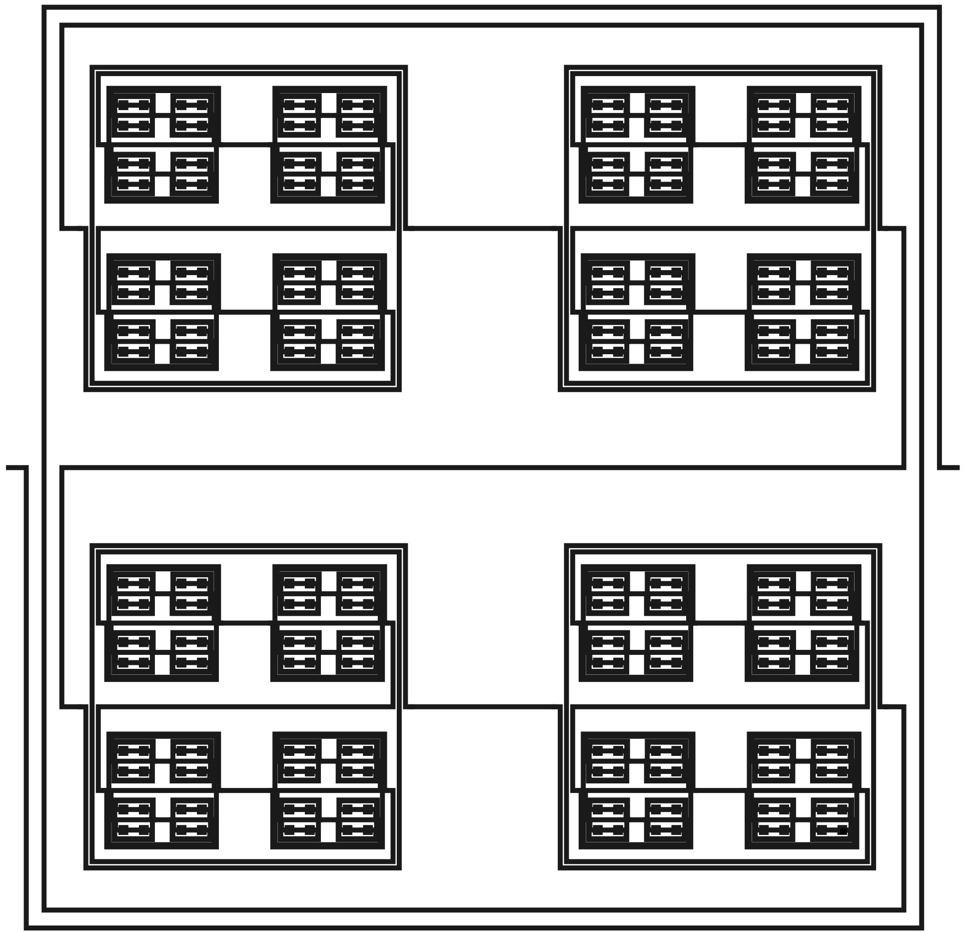
Building blocks for a Cantor set.

For $\omega_1 \sim \omega_2$ with large C, γ need not be rectifiable.



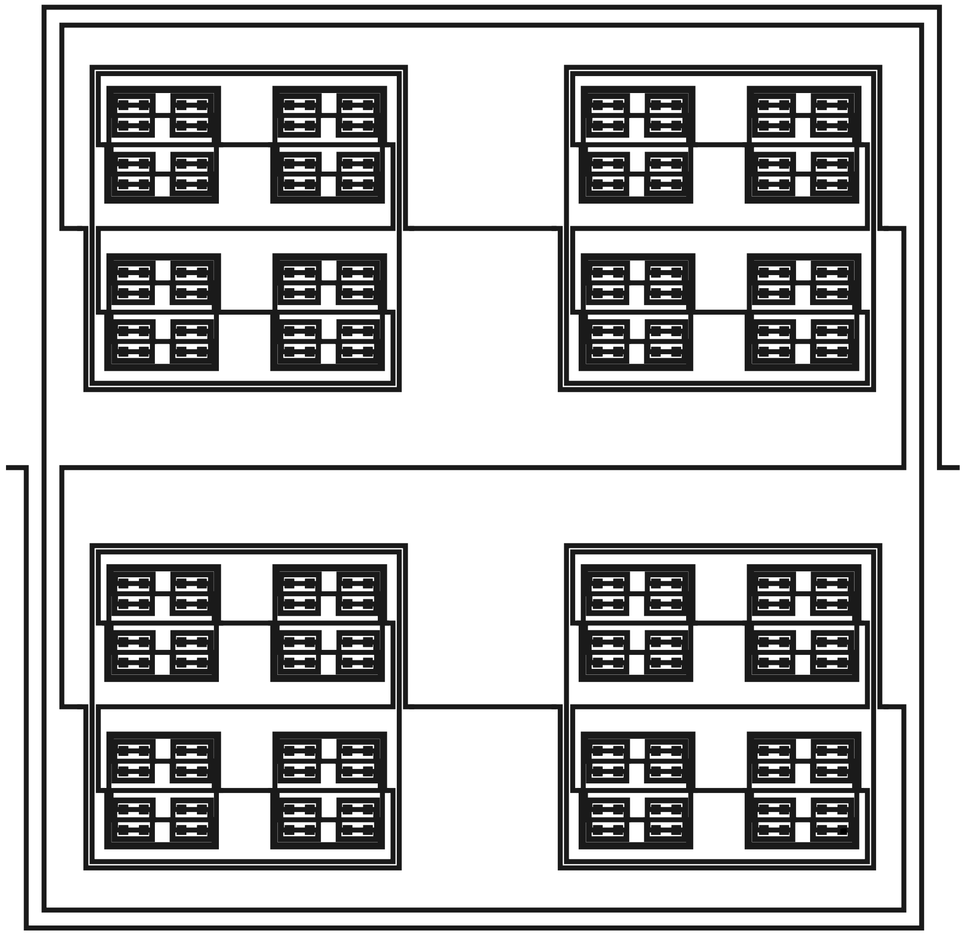
Building blocks for a curve through the Cantor set.

For $\omega_1 \sim \omega_2$ with large C, γ need not be rectifiable.

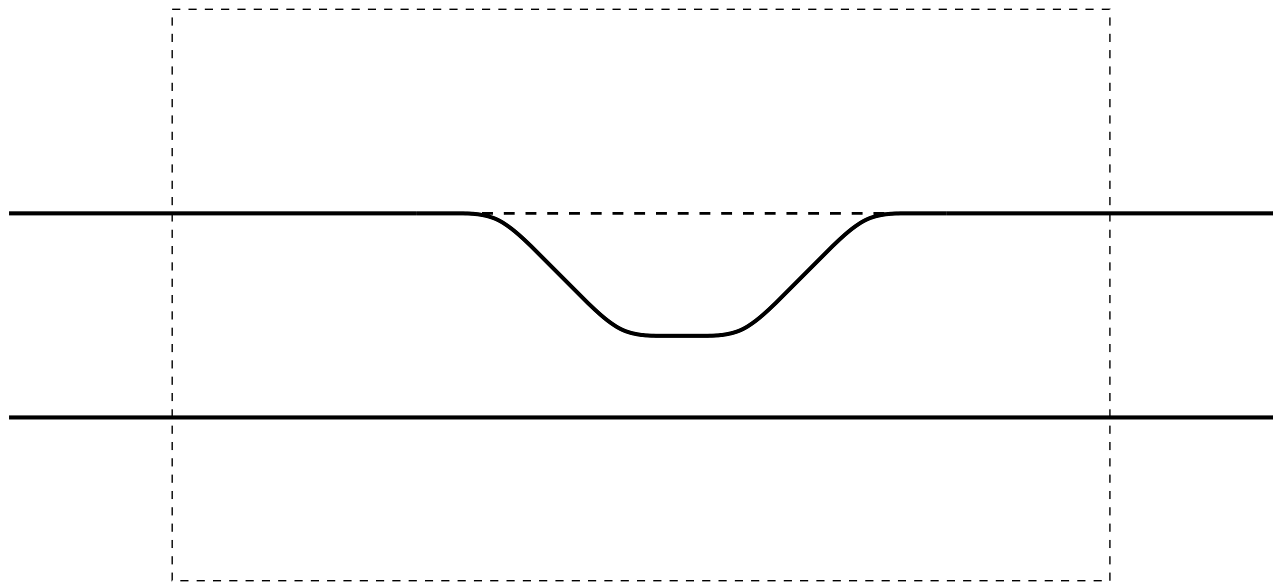


Garnett and O'Farrell: $\dim(\gamma) > 1$ and $\omega_1 \ll \omega_2 \ll \omega_1$.

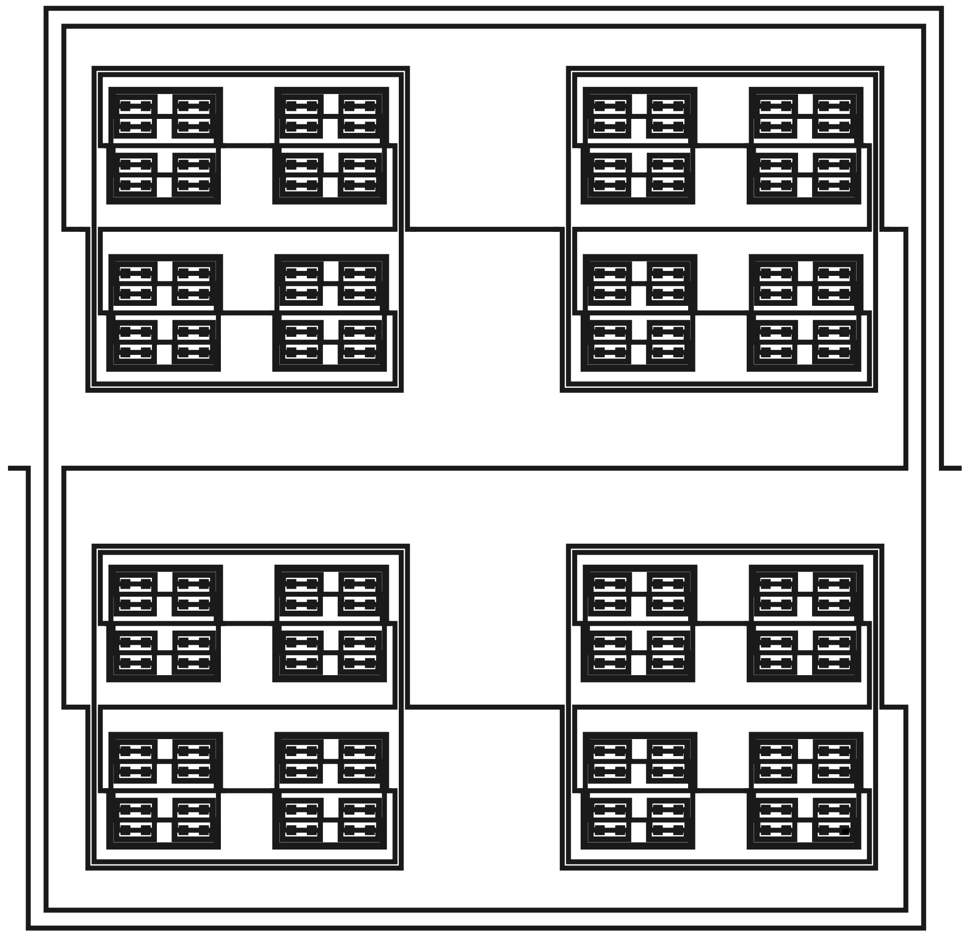
For $\omega_1 \sim \omega_2$ with large C, γ need not be rectifiable.



Consider "tube" along bottom edge that leads to next level.



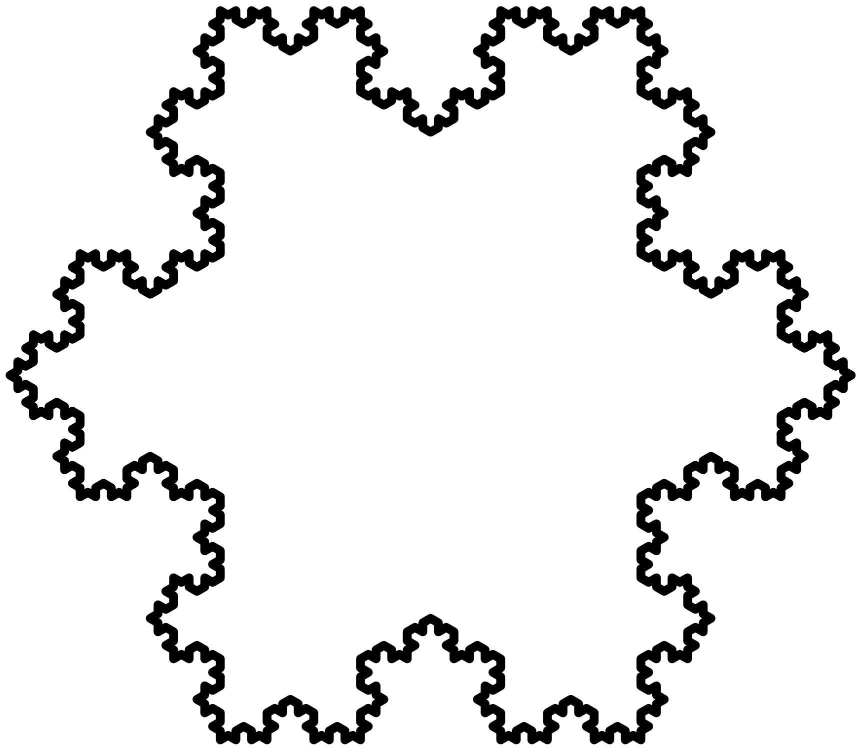
By “pinching” tube can reduce harmonic measure on one side only.



By pinching at all levels, can get $\frac{1}{C}\omega_1 \le \omega_2 \le C\,\omega_1$.

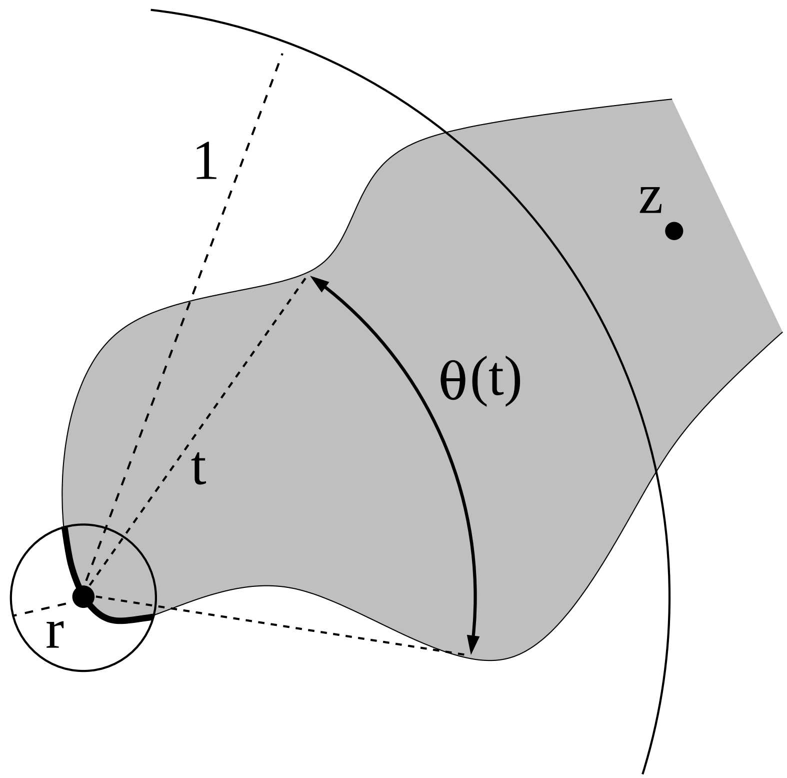
Can we have $\omega_1 \perp \omega_2$ for some curves?

Can we have $\omega_1 \perp \omega_2$ for some curves? Yes.

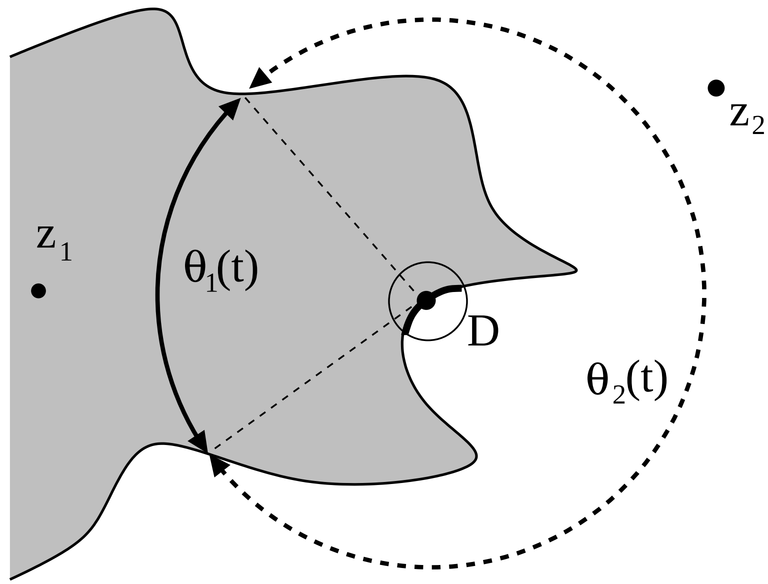


The Ahlfors estimate: θ is angle measure

$$\omega(D(x,r)) \leq C \exp(-\pi \int_r^1 \frac{1}{\theta(t)} \frac{dt}{t})$$



Apply Ahlfors estimate to both sides at once:



$$\theta_1 + \theta_2 \leq 2\pi \qquad \Rightarrow \qquad \frac{1}{\theta_1} + \frac{1}{\theta_2} \geq \frac{2}{\pi}$$

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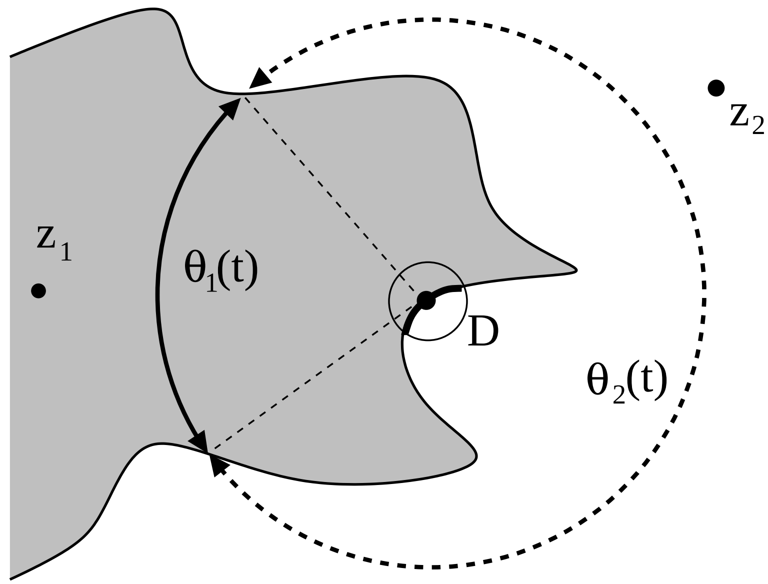

$$\begin{aligned}\omega_1(D)\omega_2(D) &\le C\exp(-\pi\int_r^1(\frac{1}{\theta_1(t)}+\frac{1}{\theta_2(t)})dt)\\ &\le C\exp(-\pi\int_r^1\frac{2}{\pi}\frac{dt}{t})\\ &\le Cr^2\end{aligned}$$

By Makarov's limsup theorem, ω_1 almost everywhere

$$\limsup_{r \to 0} \frac{\omega_1(D)}{r} = \infty,$$

Then $\omega_1 \omega_2 = O(r^2)$ implies

$$\frac{\omega_2(D)}{r} = O(\frac{r}{\omega_1(D)}) \to 0.$$

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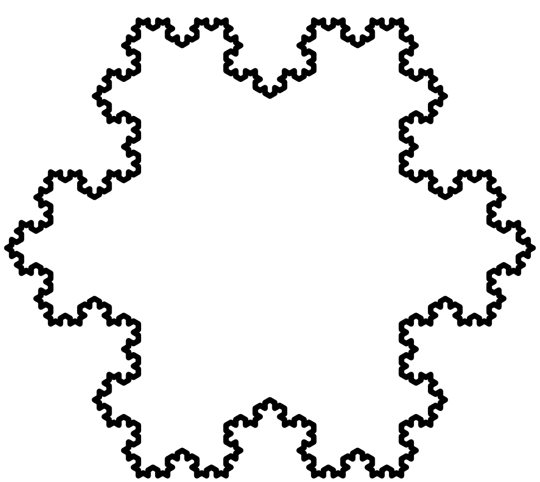
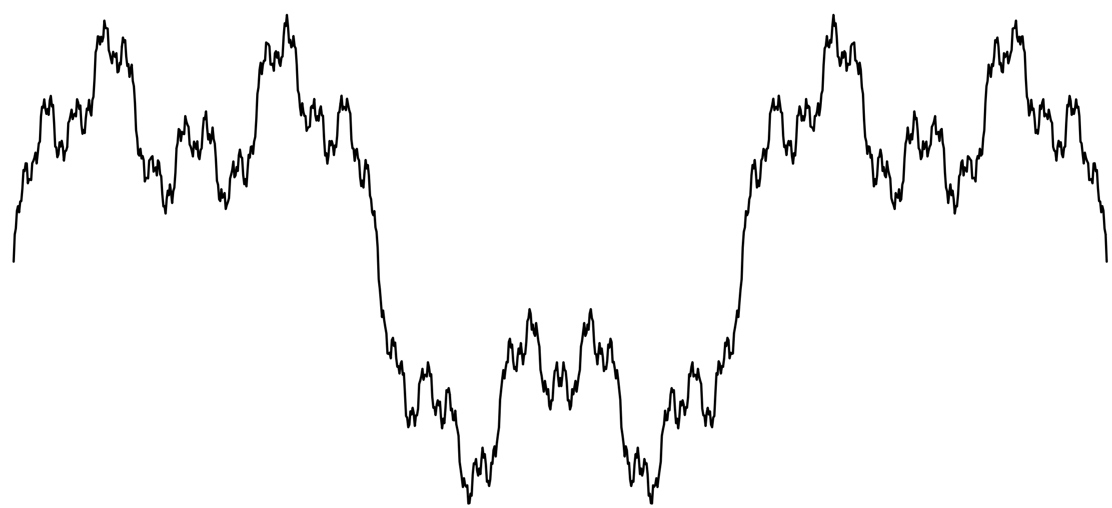
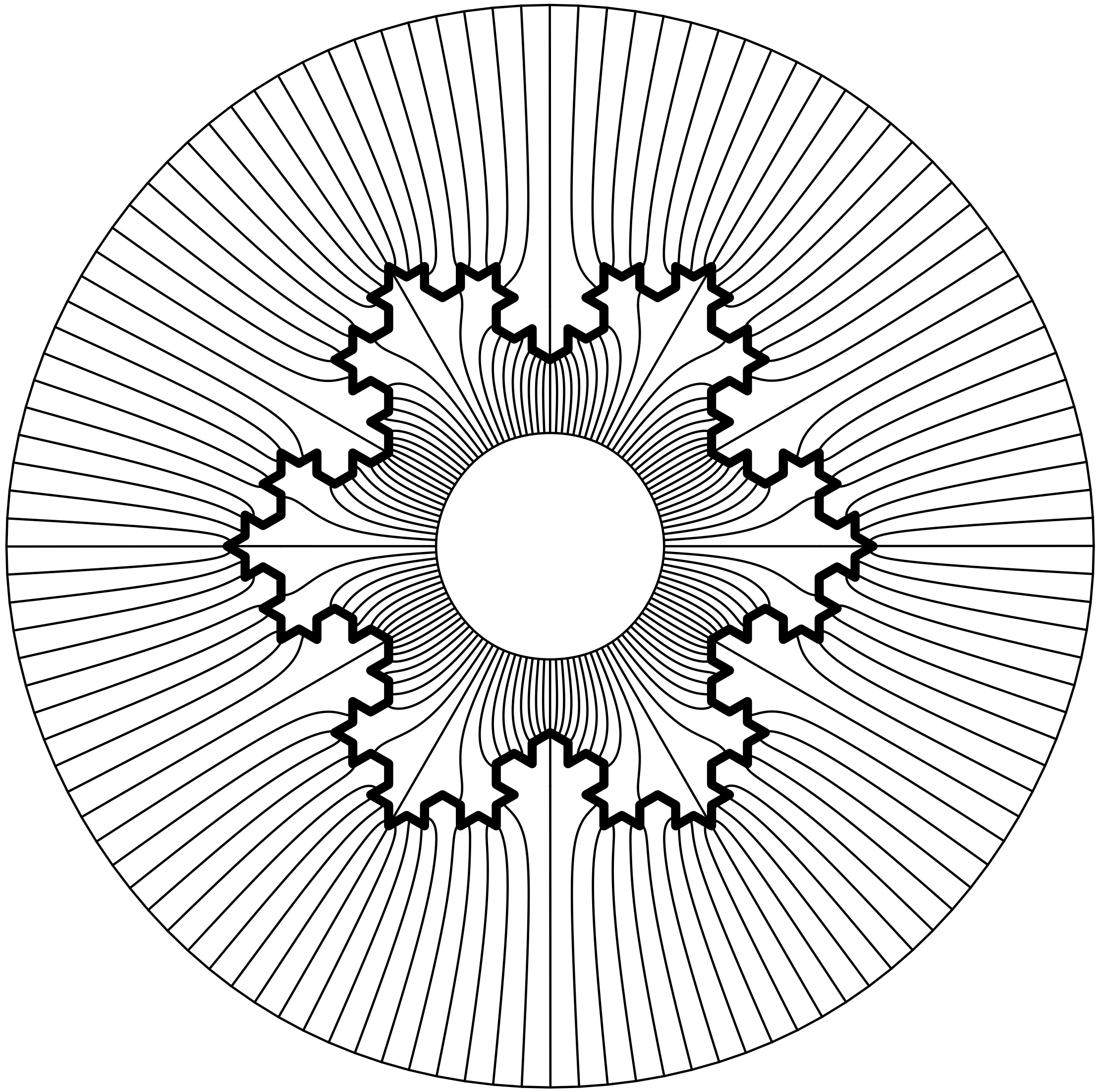
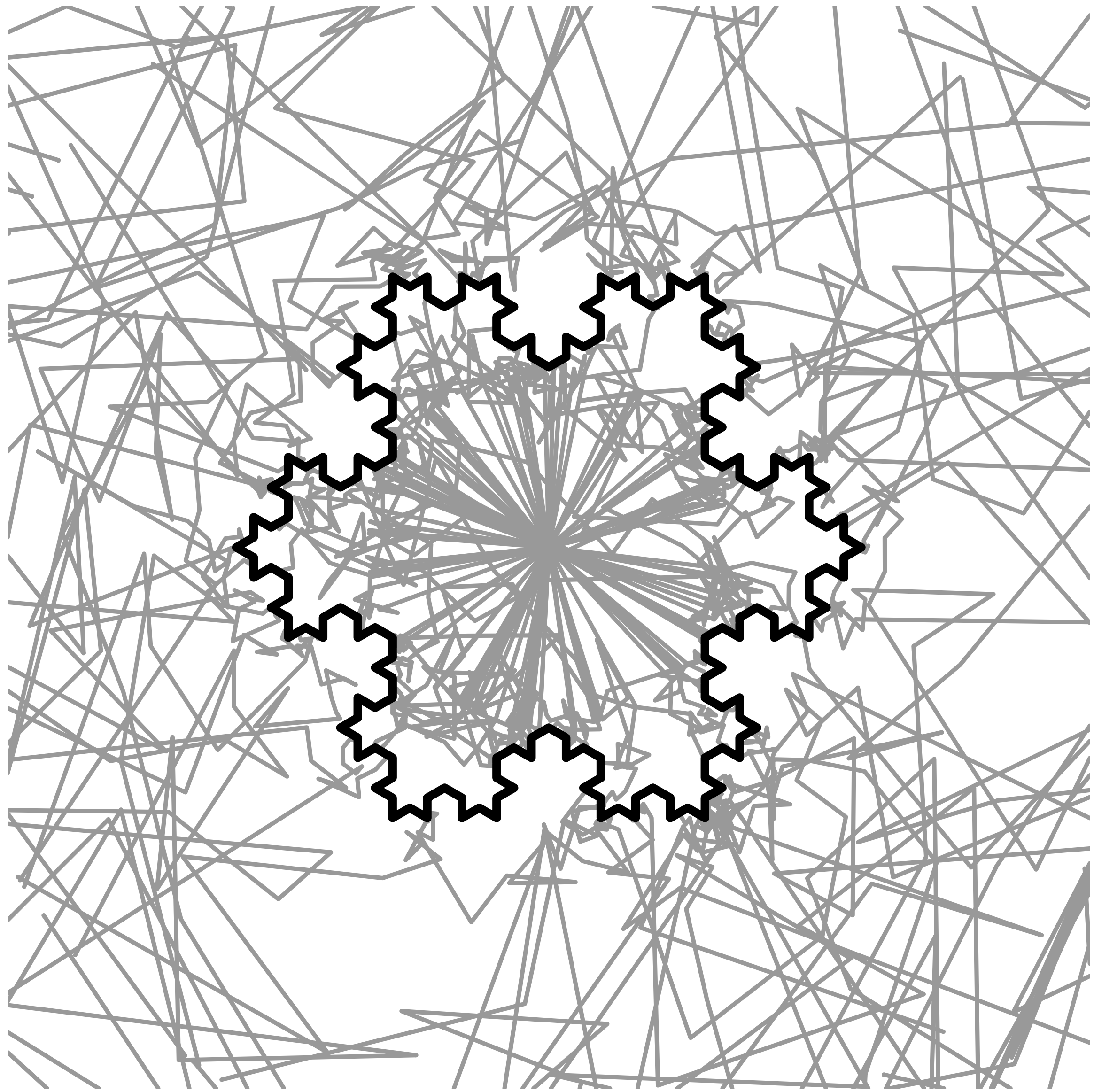
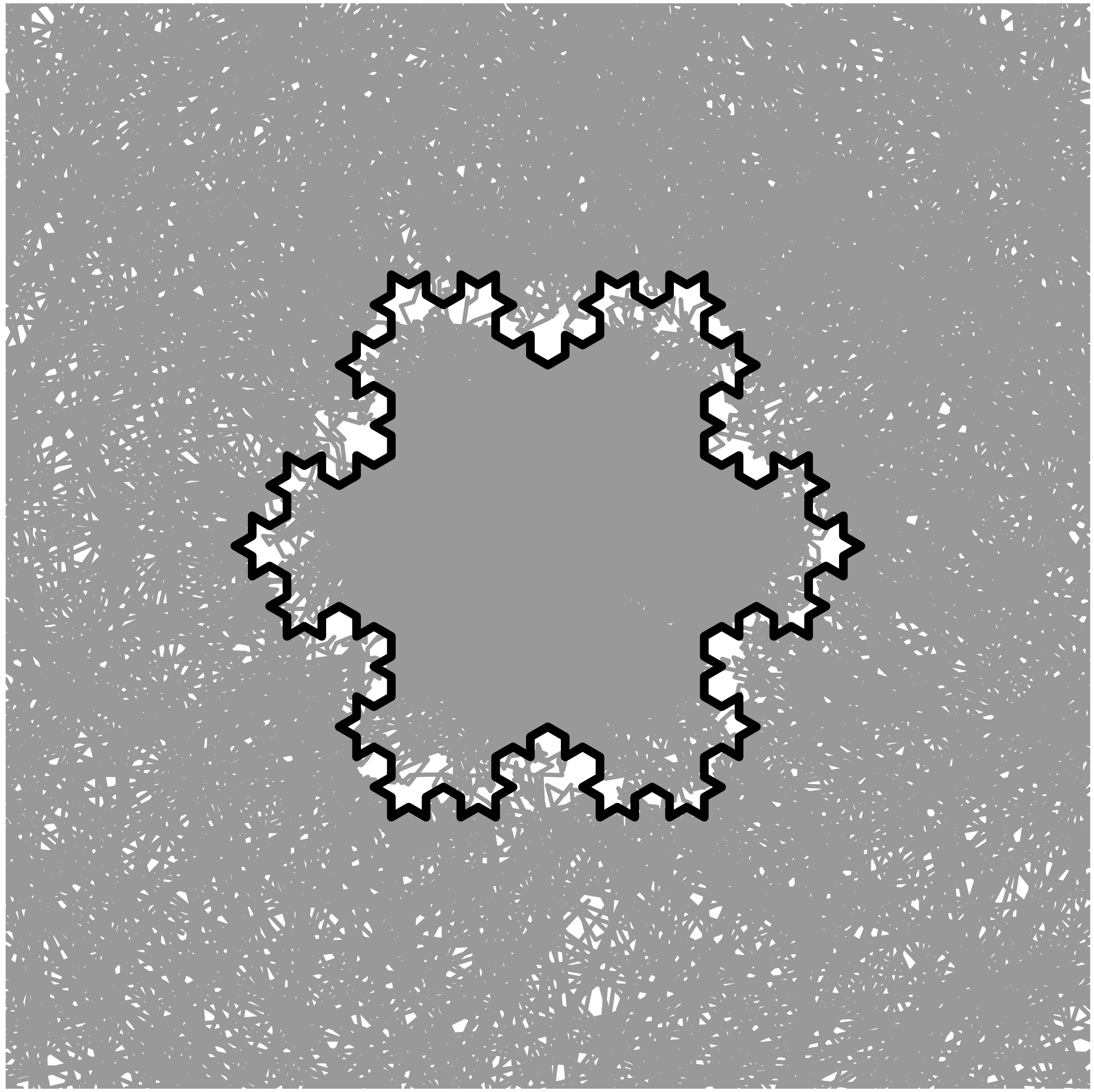
Theorem: If tangents points have zero linear measure, then $\omega_1 \perp \omega_2$.











Recent progress:

Theorem: Suppose $\Omega \subset \mathbb{R}^{n+1}$. If $\omega \ll \Lambda_n$ on $E \subset \partial\Omega$, then E has a ω-full measure n-rectifiable subset. If $\Lambda_n \ll \omega$ on E, then E is n-rectifiable.

"Rectifiability of harmonic measure",

2015 preprint by Azzam, Hofmann, Martell, Mayboroda, Mourgoglou, Tolsa and Volberg.

Absolute continuity $\Rightarrow$ Riesz transforms bounded $\Rightarrow$ Recitifiability

Very recent progress:

Theorem: Suppose $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ are connected, $\partial\Omega_1 = \partial\Omega_2$, and satisfy CDC. Then $\omega_1 \perp \omega_2$ off the tangent points.

"Mutual absolute continuity of interior and exterior harmonic measure implies rectifiability",

2016 preprint by Azzam, Mourgoglou and Tolsa.

CDC = Capacity Density Condition.

Now consider a different kind of “two-sided” harmonic measure problem.

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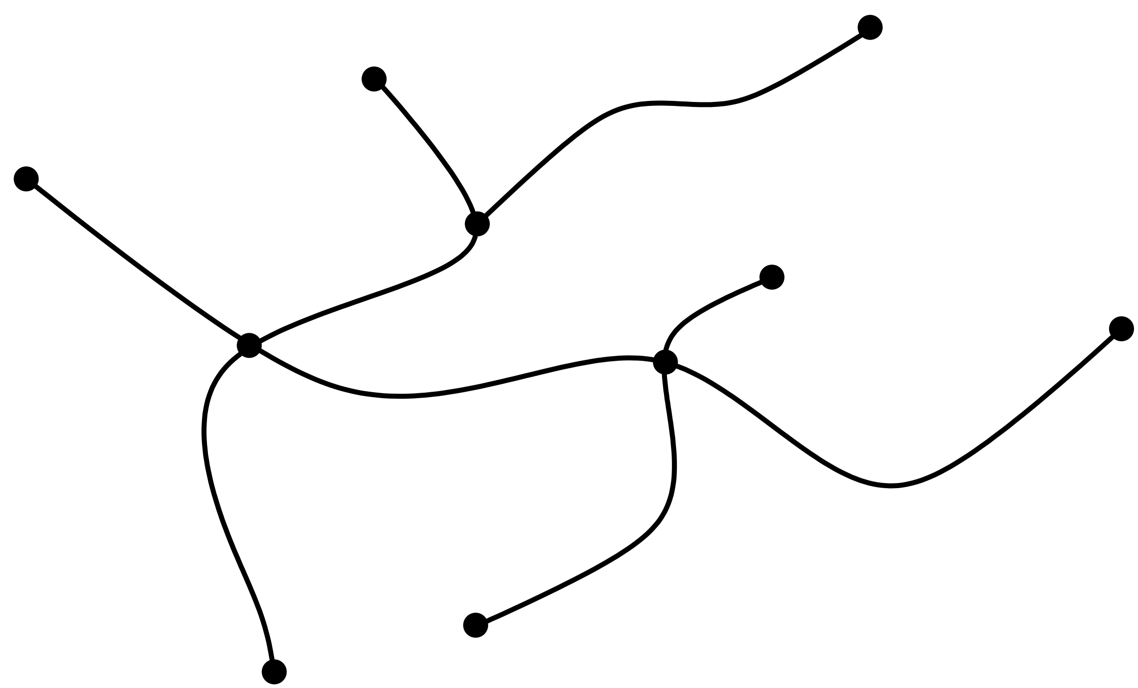
What happens if we replace closed curve by a tree?



Can we make harmonic measure the same on “both sides” of every edge?

A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from ∞
- edge subsets have same measure from both sides

This is also called a "**true tree**".

Does every planar tree have a "**true form**"?

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A line segment is an example.

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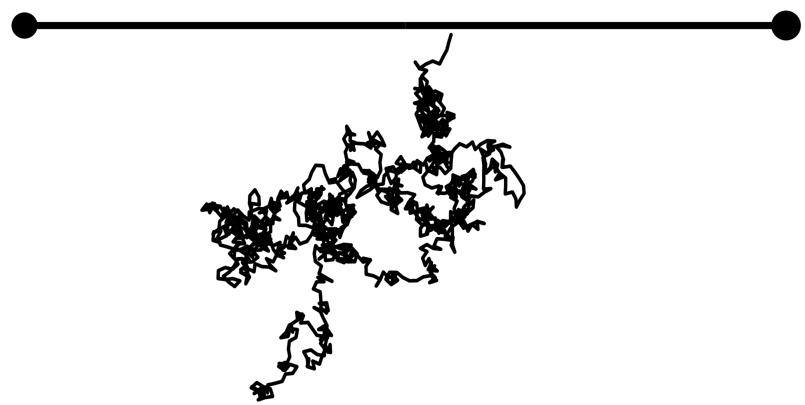
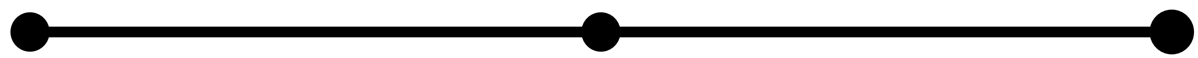
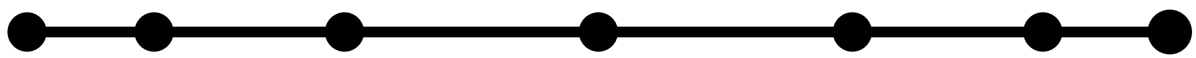
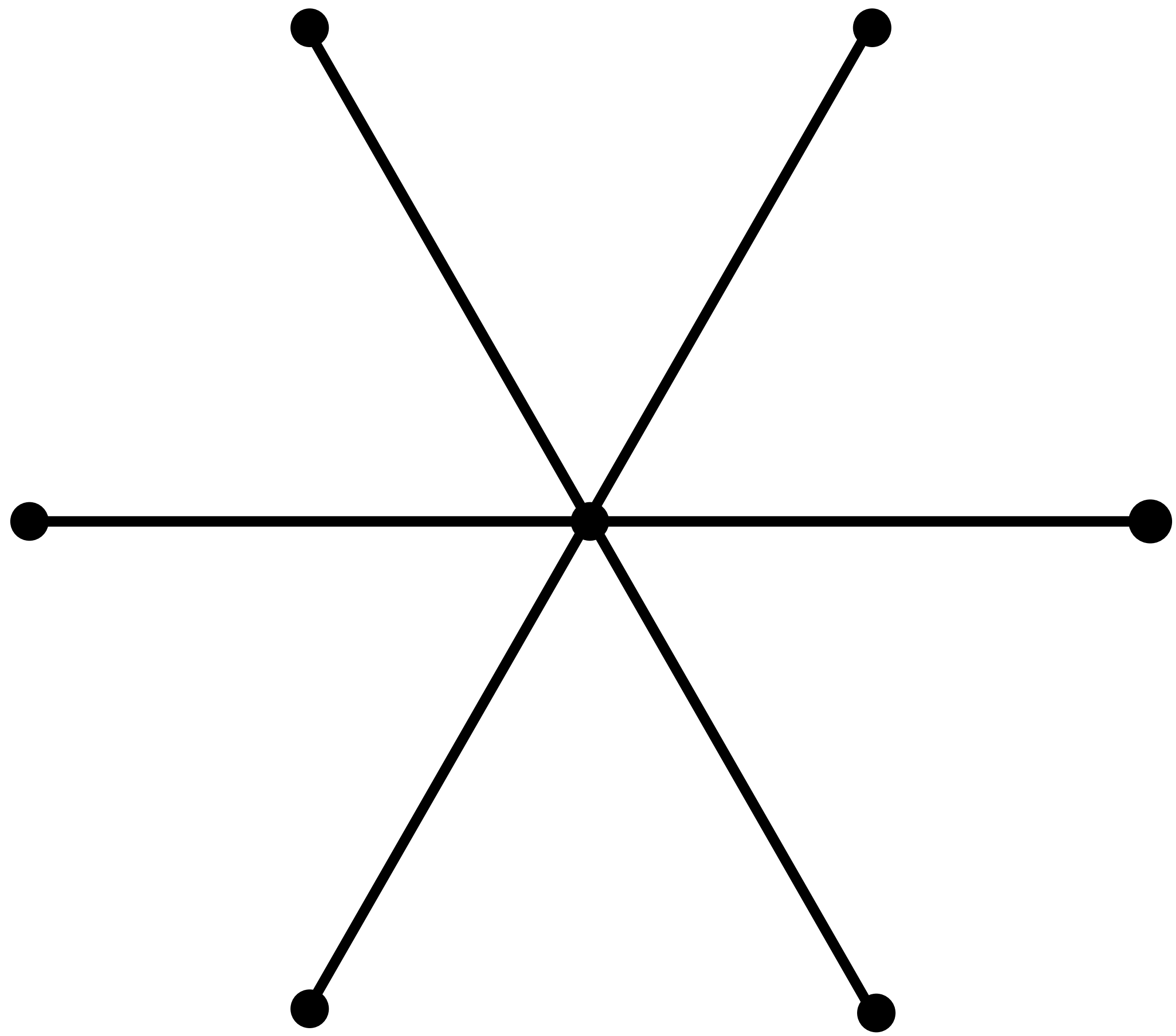
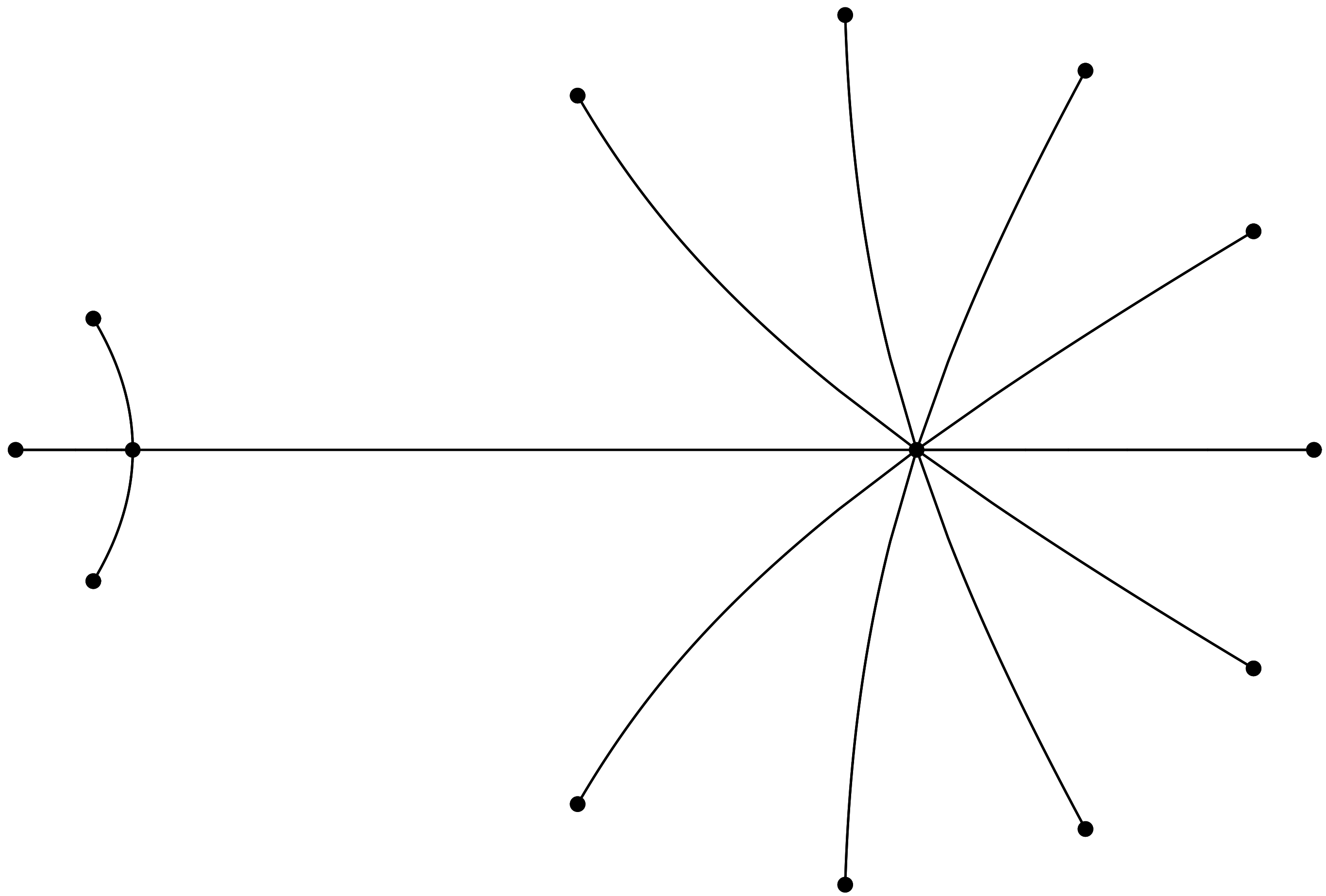
A line segment is an example. Are there others?

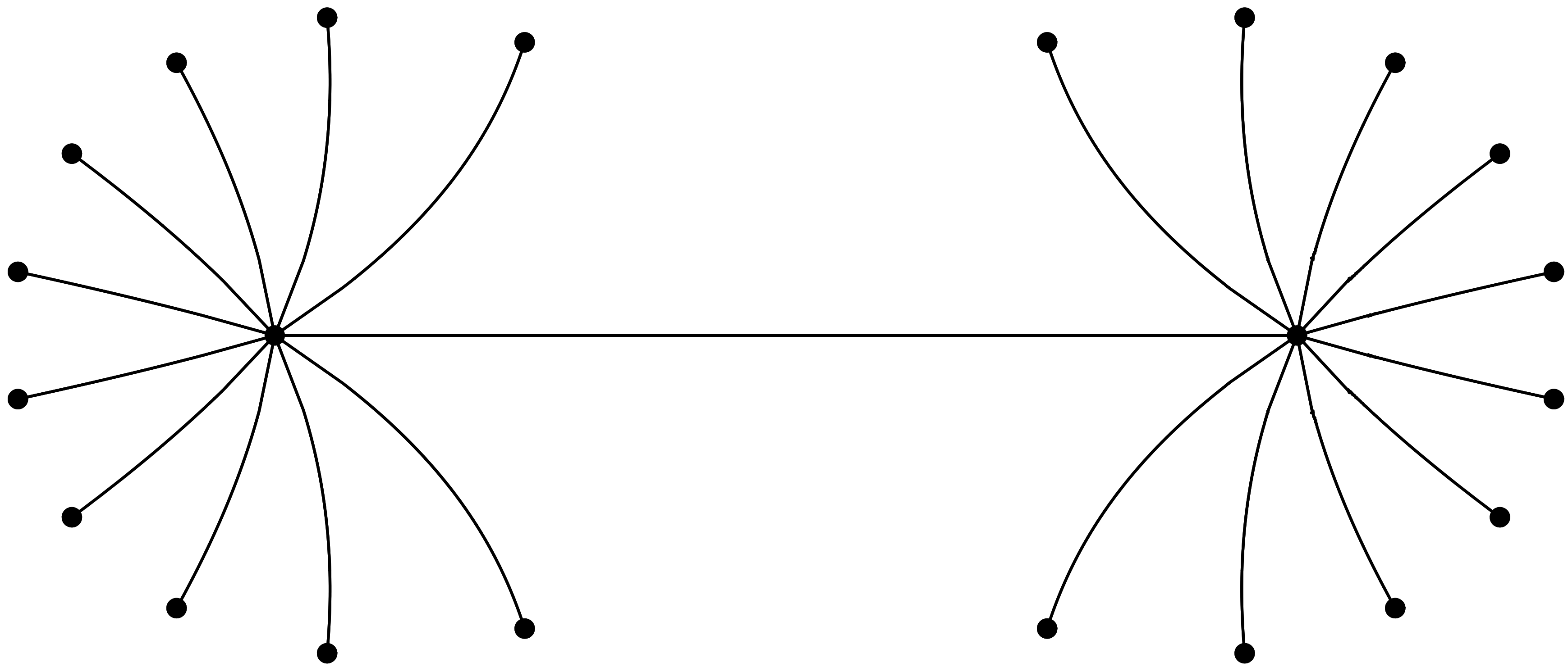












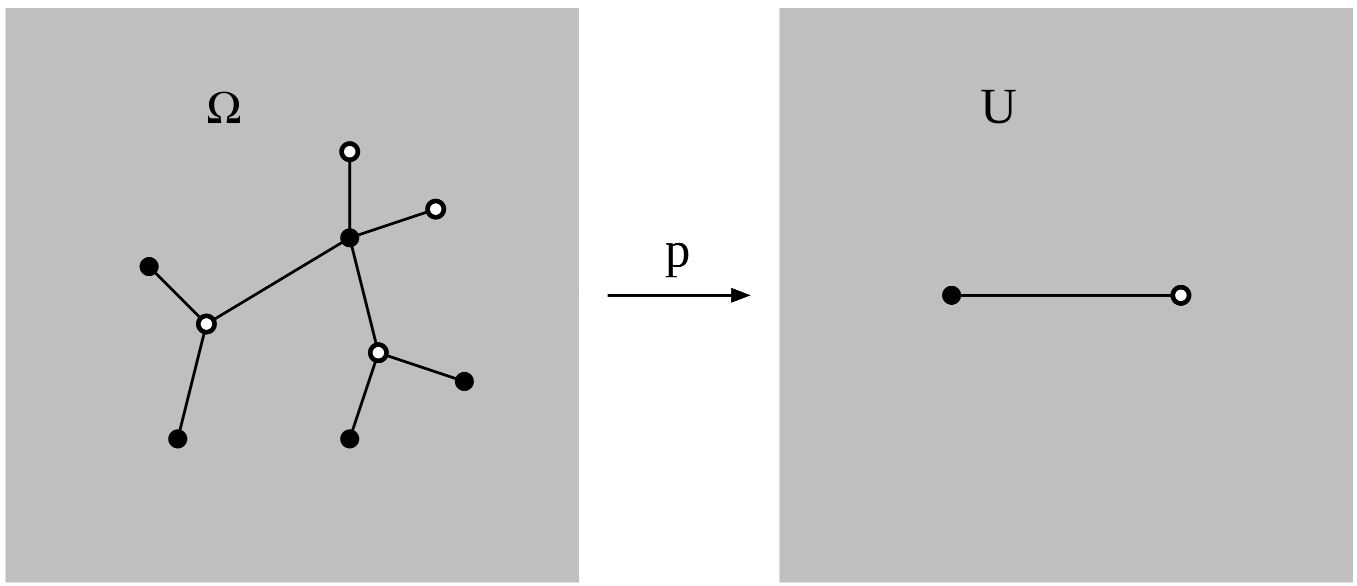
$p =$ polynomial

$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} =$ critical values

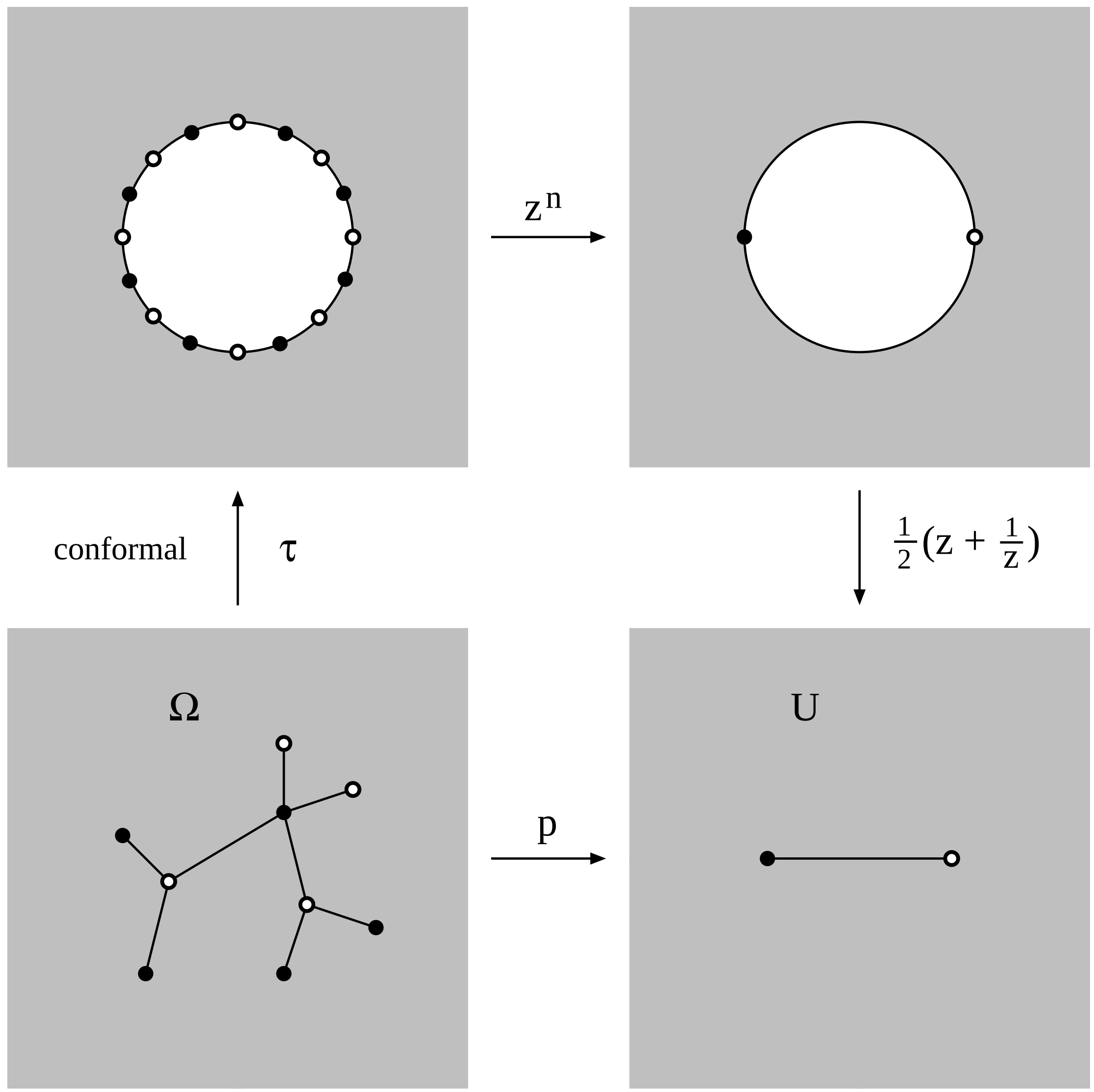
If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.

Balanced trees $\leftrightarrow$ Shabat polynomials

Fact: T is balanced iff $T = p^{-1}([-1,1])$, $p =$ Shabat.



$\Omega = \mathbb{C} \setminus T$ $\qquad$ $U = \mathbb{C} \setminus [-1,1]$

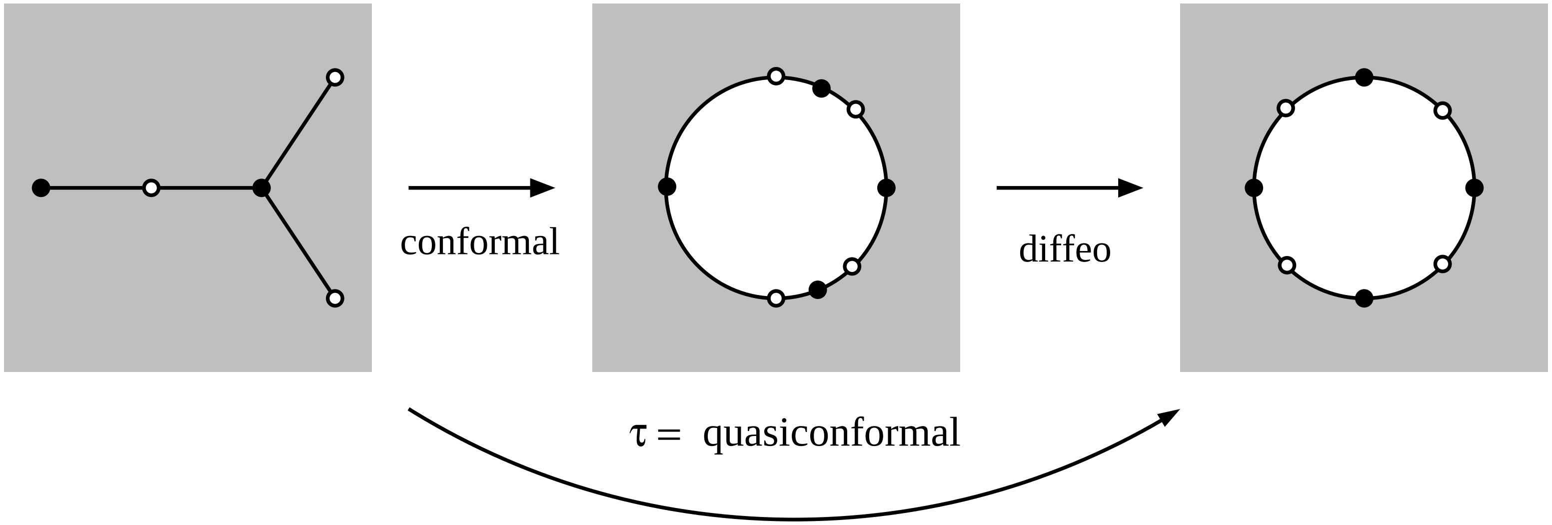


p is entire and n-to-1 $\Rightarrow p =$ polynomial

What if the tree is not balanced?

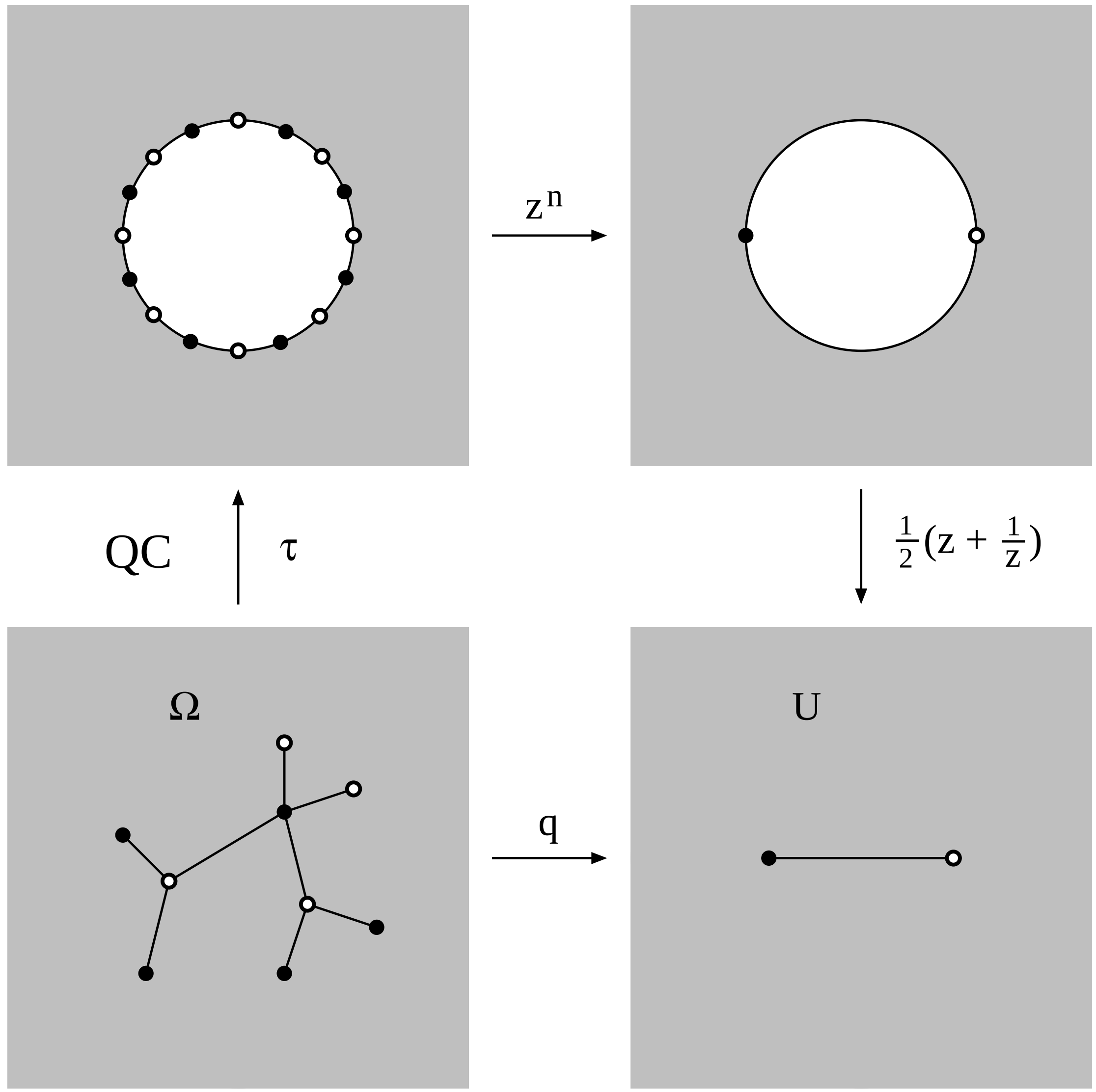
What if the tree is not balanced?

Replace conformal map by quasiconformal map.



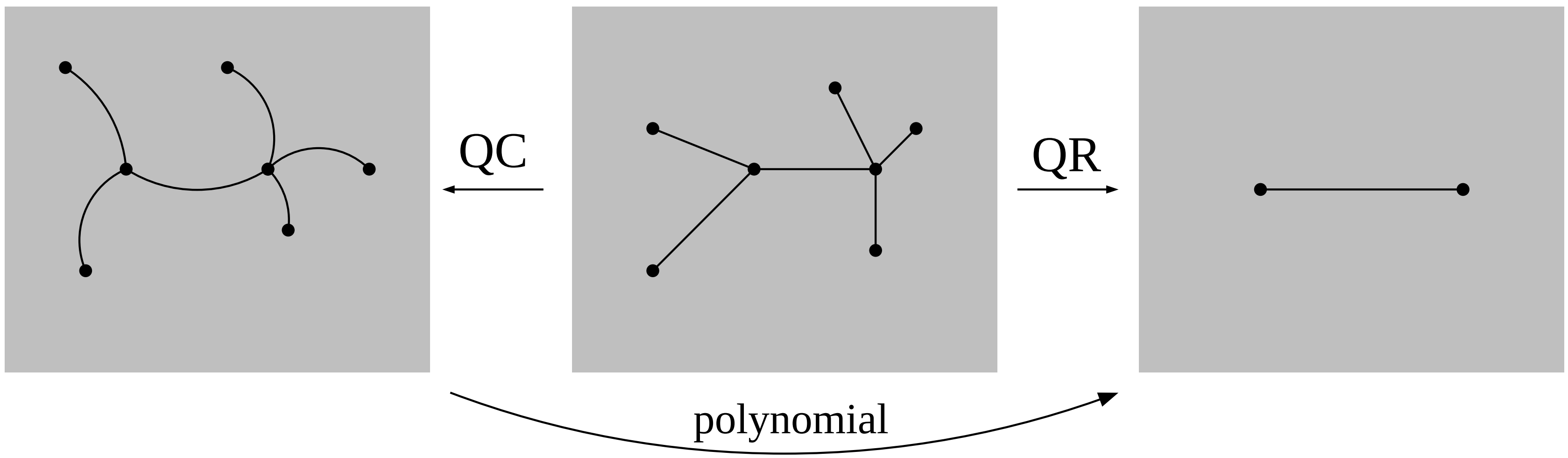
Map $\Omega \to \{|z| > 1\}$ conformally; "equalize intervals" by diffeomorphism.

Composition is quasiconformal. QC constant depends on tree.



τ is quasiconformal on Ω. q is quasi-regular on plane.

Measureable Riemann mapping theorem says there is a QC homeomorphism φ so $p = q \circ \varphi$ is holomorphic.



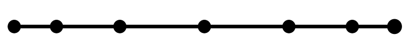
Corollary: Every planar tree has a true form.

"all combinatorics occur"

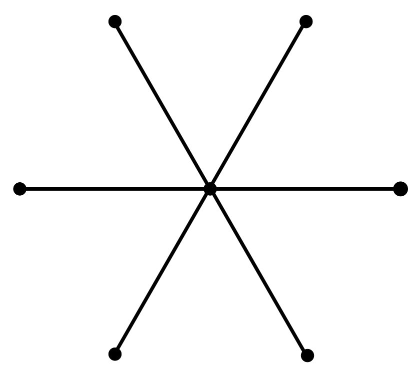
In Grothendieck's theory of *dessins d'enfants*, a finite graph of a topological surface determines a conformal structure and a Belyi function (a meromorphic function to sphere branched over 3 values).

Shabat polynomial is special case of Belyi function.
(branch points $= -1, 1, \infty$).

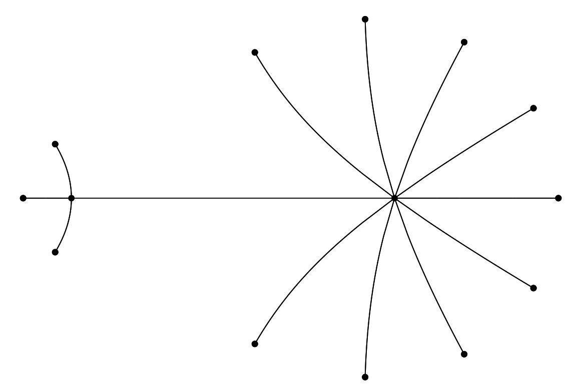
Is the polynomial computable from the tree?



1st type Chebyshev



$p(z) = 2z^n - 1$



$p'(z) = c(z+1)^a(z-1)^b$

Kochetkov, *Planar trees with nine edges: a catalogue*, 2007:
"The complete study of trees with 10 edges is a difficult work, and probably no one will do it in the foreseeable future".

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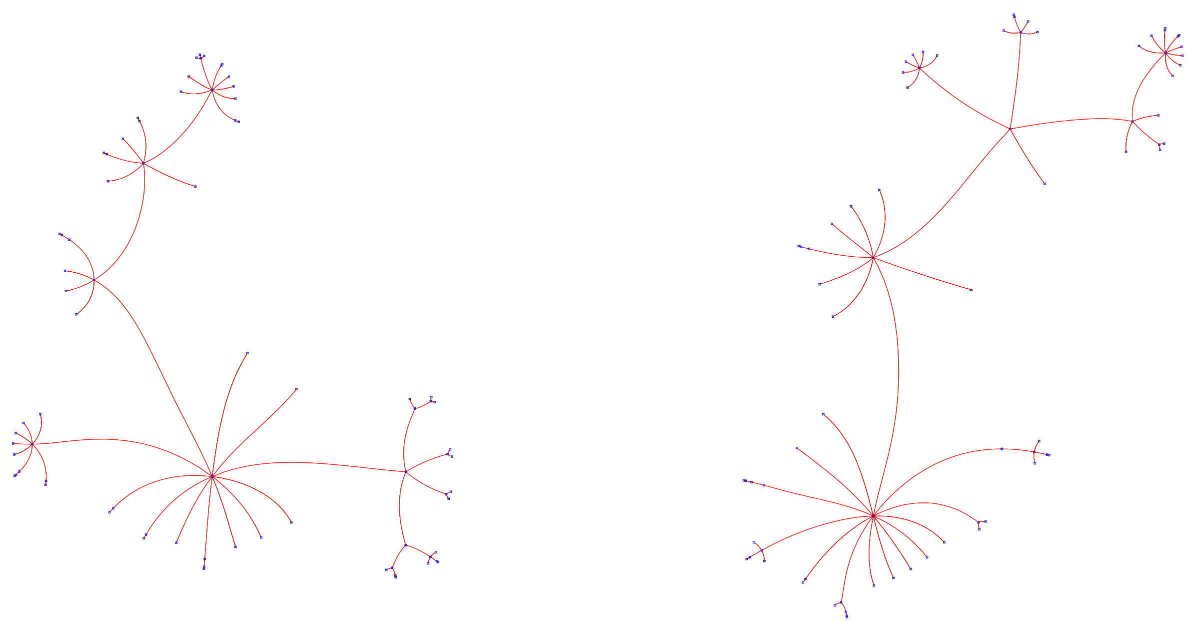
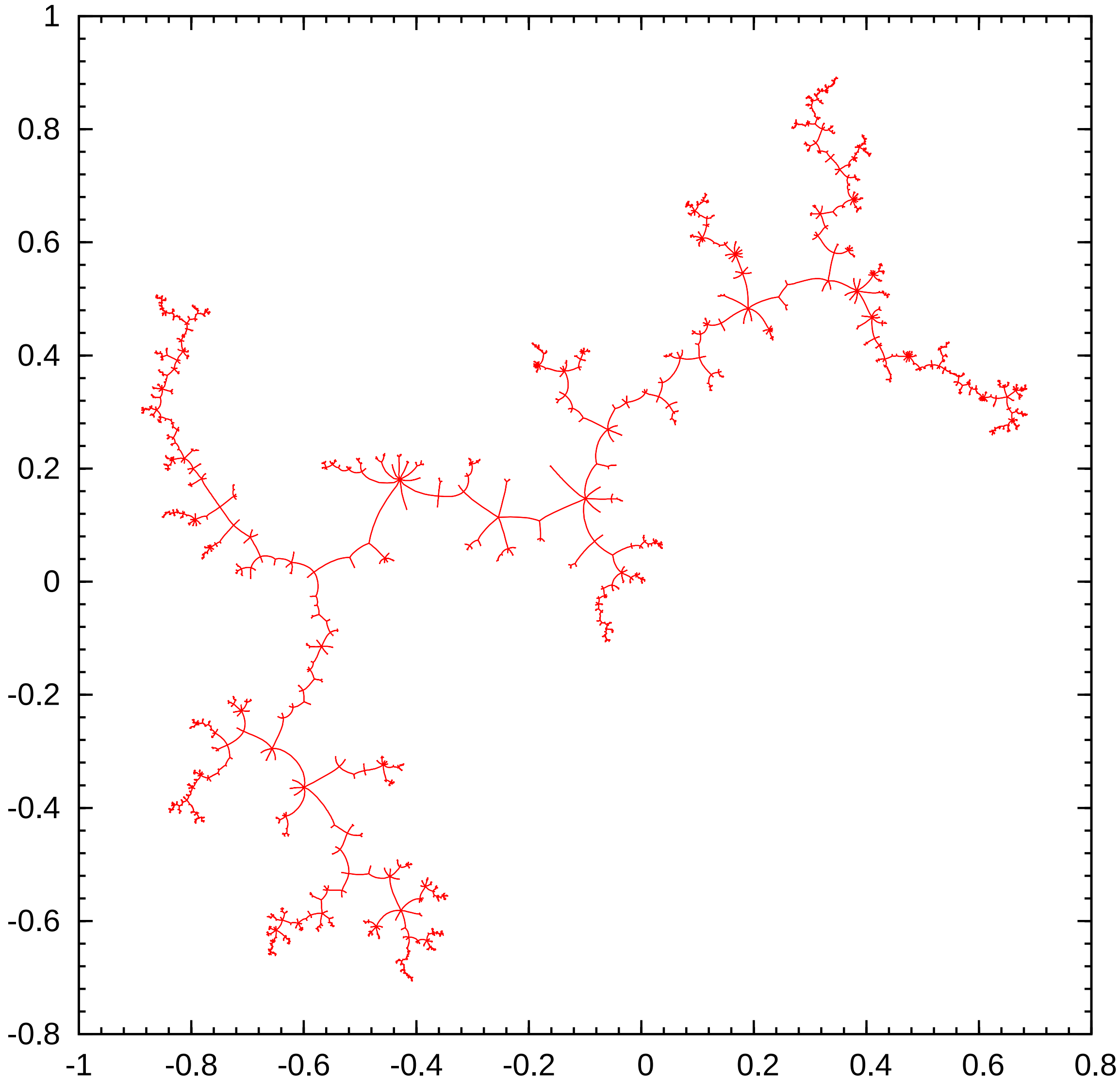
"The complete study of trees with 10 edges is a difficult work, and probably no one will do it in the foreseeable future".

Kochetkov gave "short catalog" of 10-edge trees in 2014.

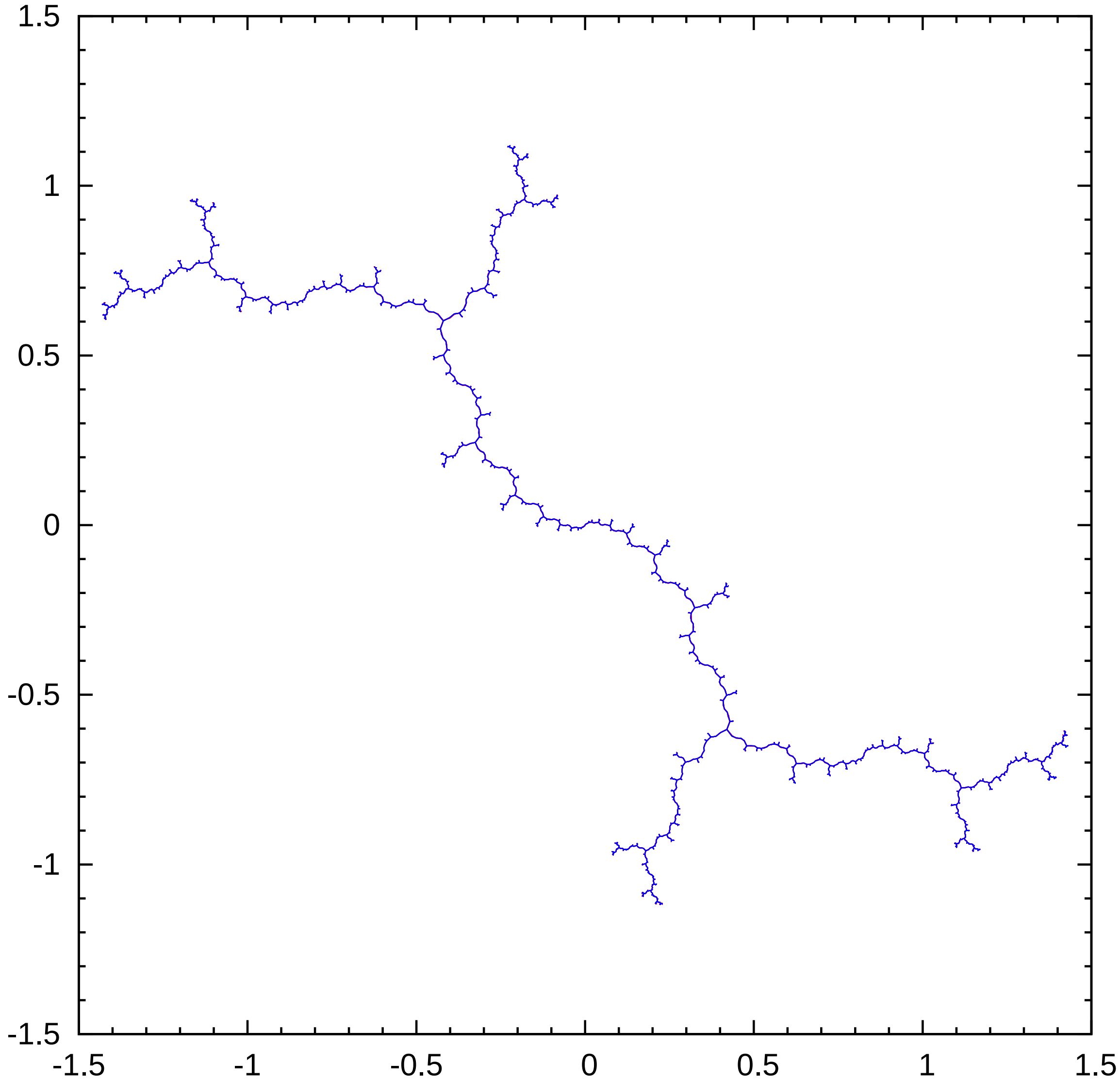
Marshall and Rohde approximated all 95,640 true trees with 14 edges.



They can get 1000's of digits of accuracy. Can such approximations and lattice reduction (e.g., PSLQ) give the exact algebraic coefficients?



Random true tree with 10,000 edges



True Tree with dynamical combinatorics

Every planar tree has a true form.

In other words, all possible **combinatorics** occur.

What about all possible **shapes**?

Every planar tree has a true form.

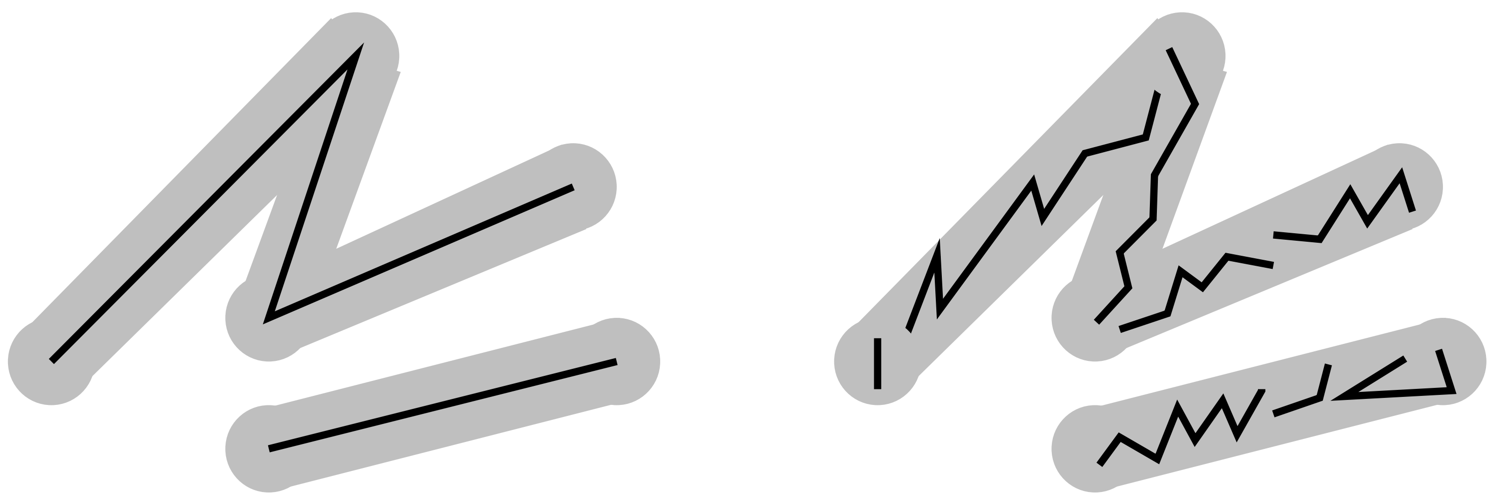
In other words, all possible **combinatorics** occur.

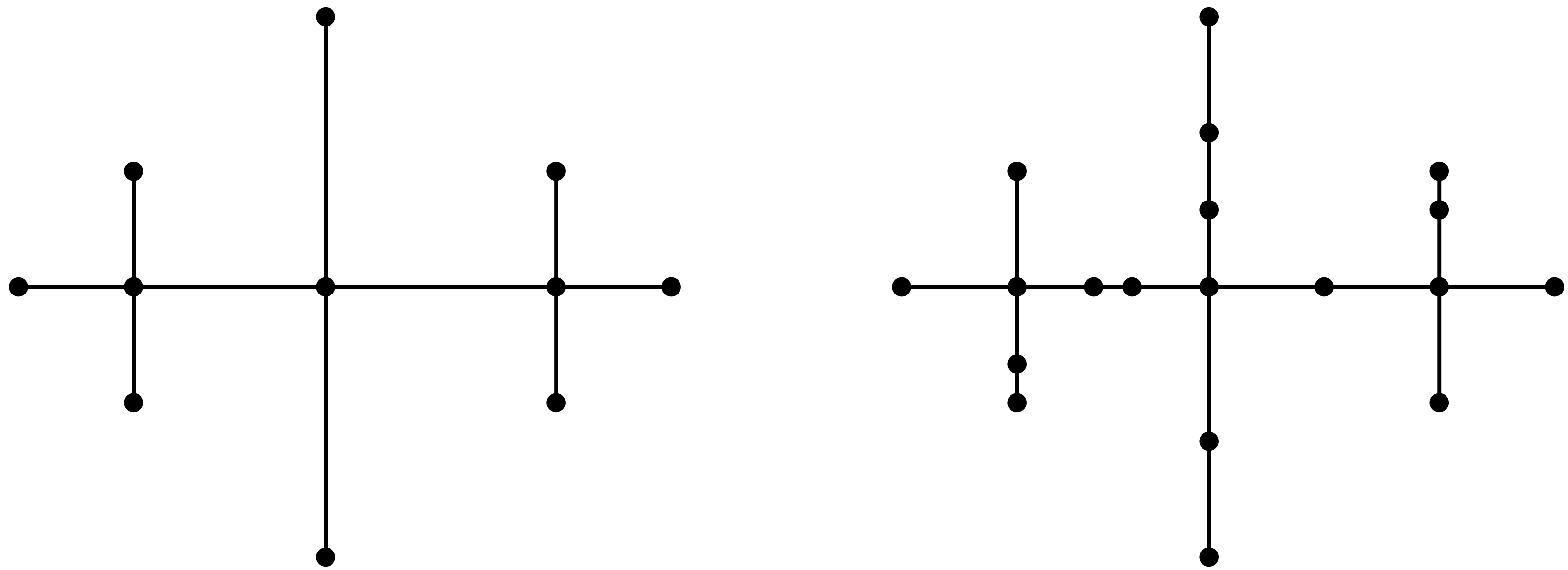
What about all possible **shapes**?

Hausdorff metric: if E is compact,

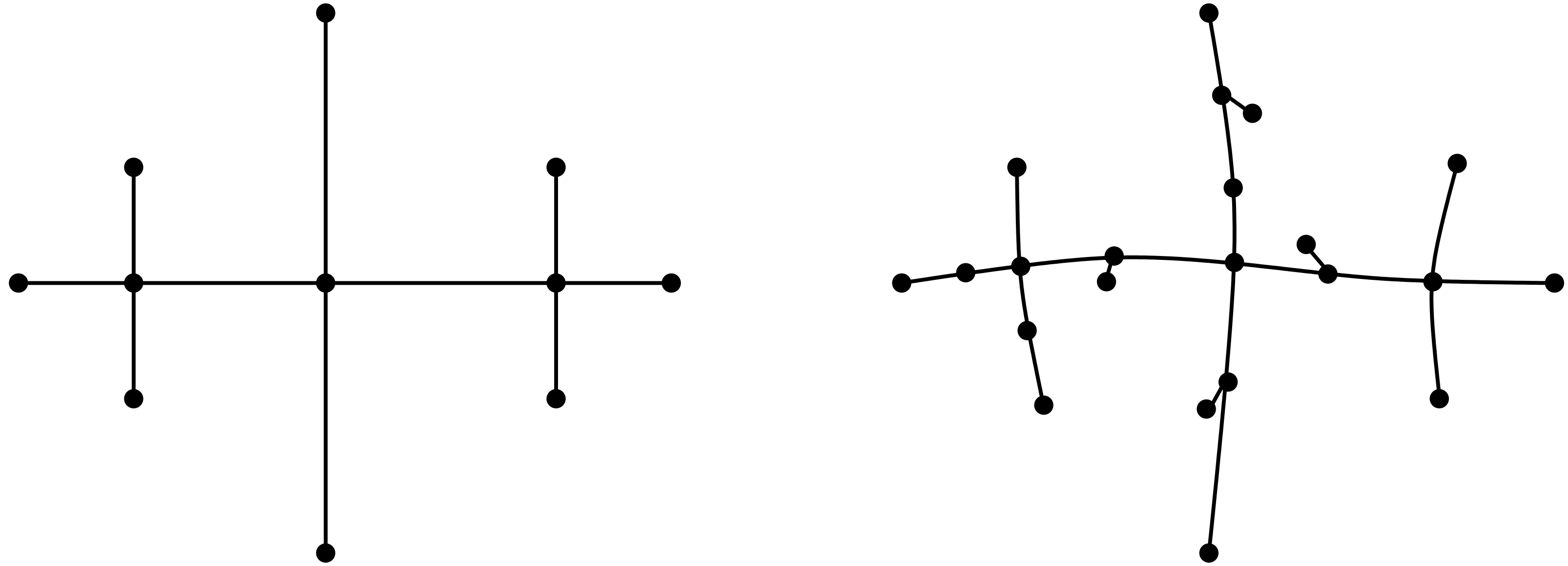
$$E_\epsilon = \{z : \text{dist}(z, E) < \epsilon\}.$$

$$\text{dist}(E, F) = \inf\{\epsilon : E \subset F_\epsilon, F \subset E_\epsilon\}.$$





Different combinatorics, same shape



Different trees, similar shapes

Close in Hausdorff metric

Theorem: Every planar continuum is a limit of true trees.

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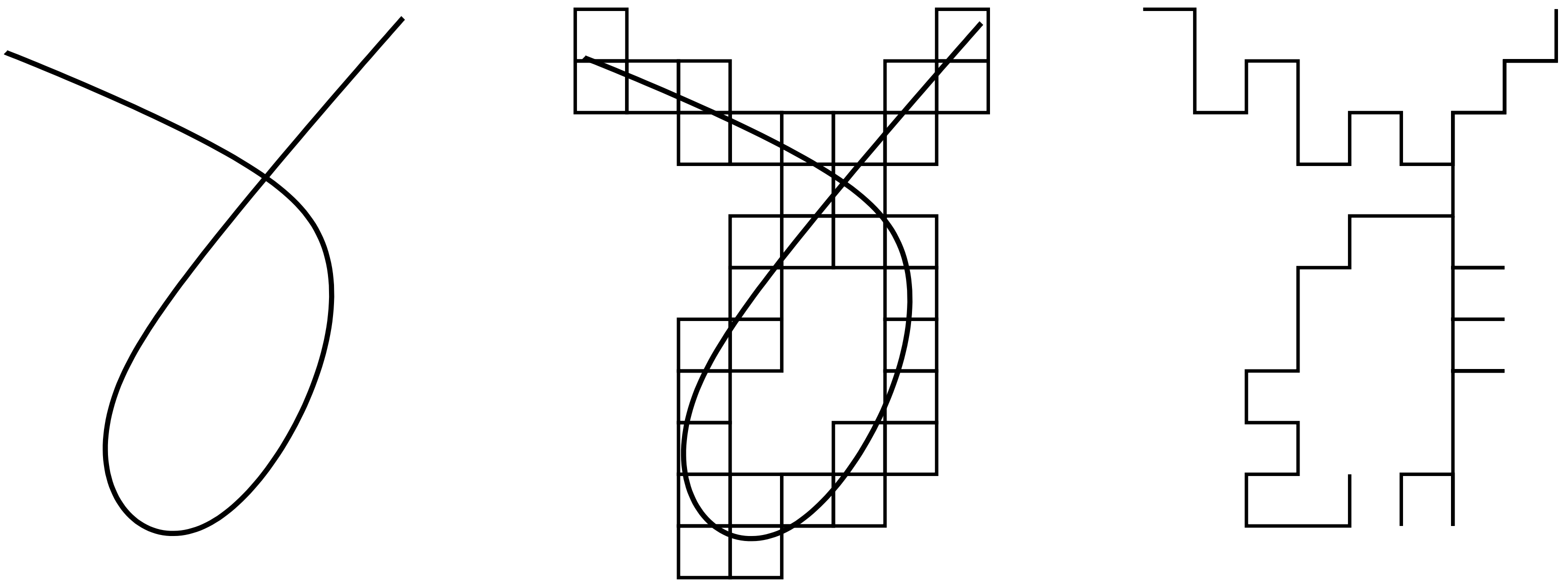
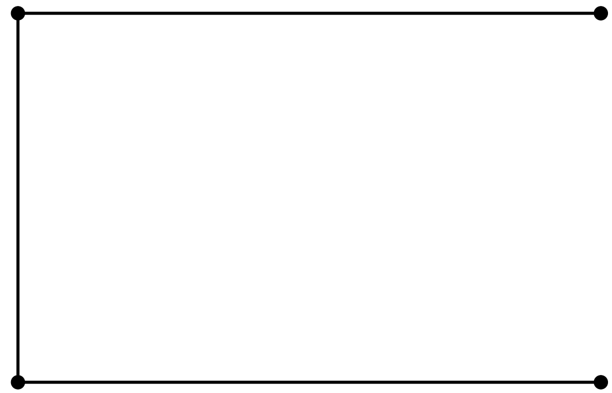
"True trees are dense" or "all shapes occur"

Answers question of Alex Eremenko.

Enough to approximate certain finite trees by true trees.

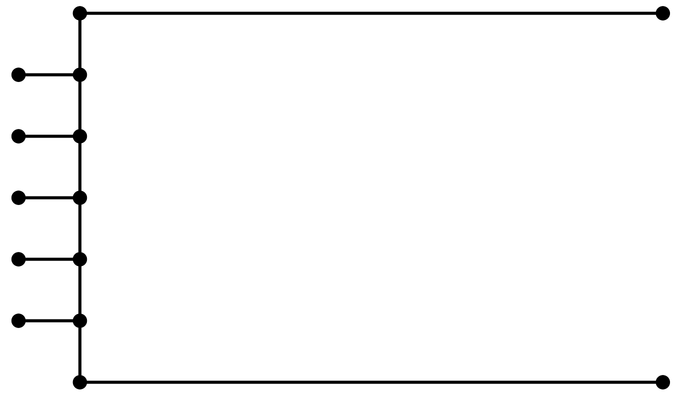


Idea: reduce harmonic measure ratio by adding edges.



Vertical side has much larger harmonic measure from left.

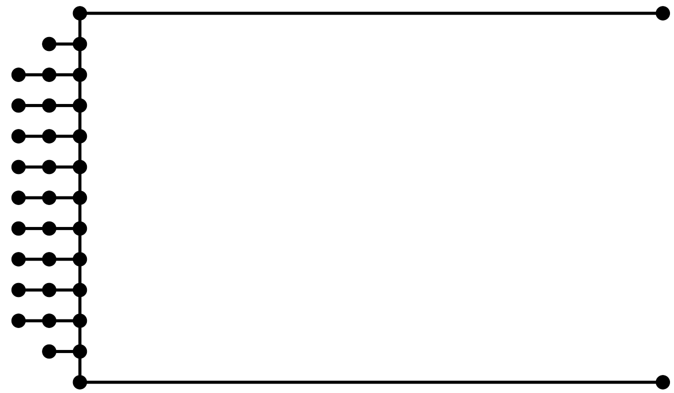
Idea: reduce harmonic measure ratio by adding edges.



"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

Idea: reduce harmonic measure ratio by adding edges.



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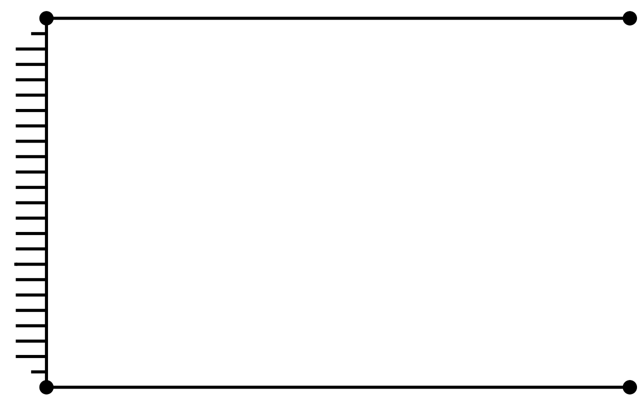
Idea: reduce harmonic measure ratio by adding edges.



"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

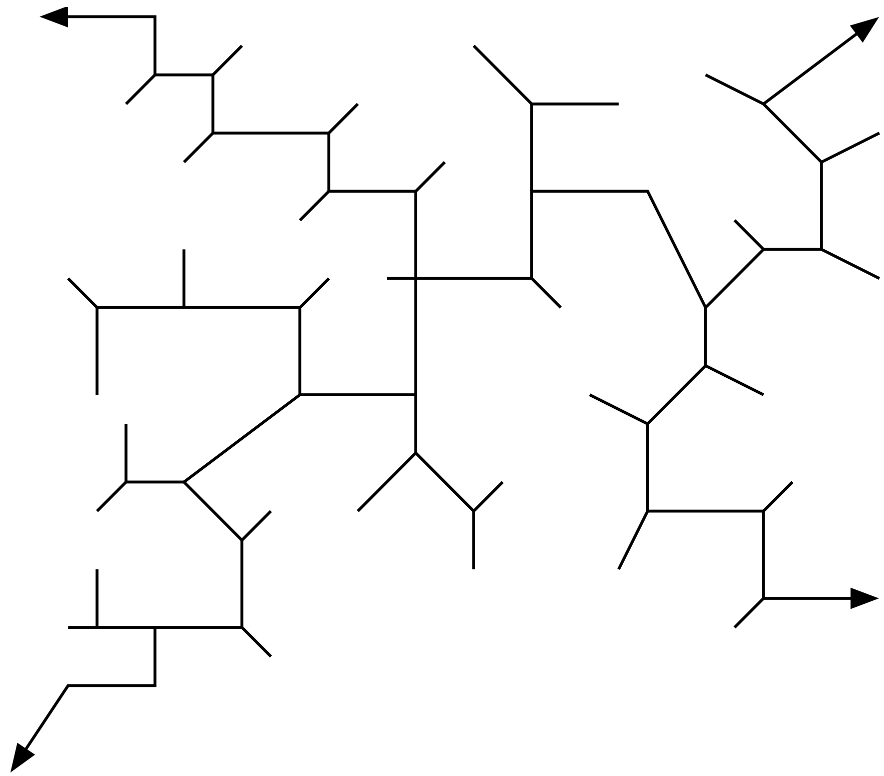
Longer spikes mean more reduction. Spikes can be very short.

Approximately balanced $\Rightarrow$ exactly balanced by MRMT.

QC-constant uniformly bounded. Only non-conformal very near tree.

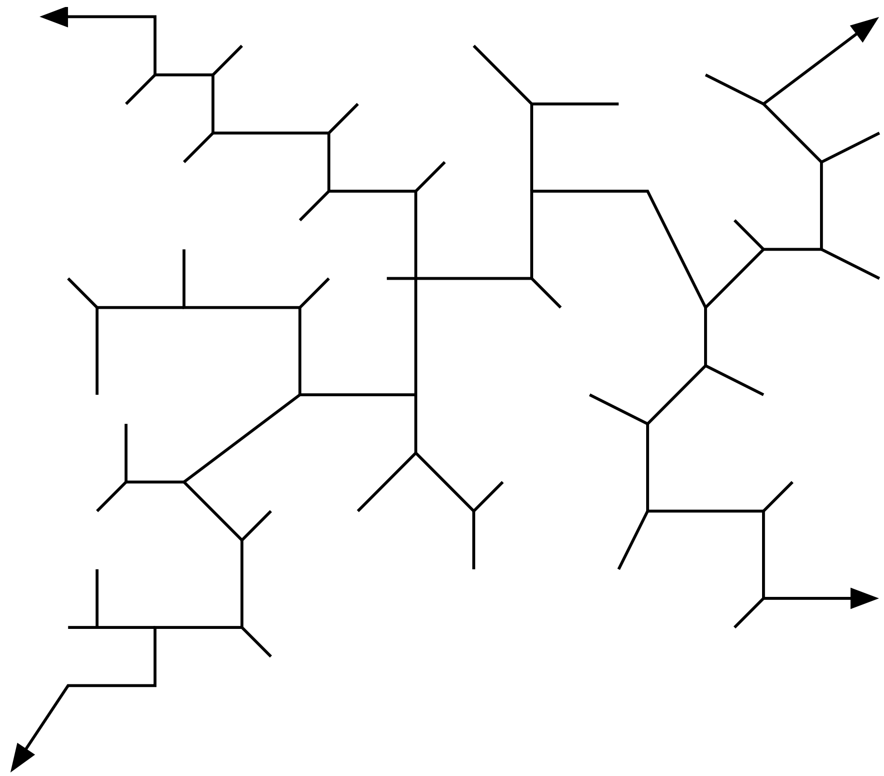
Implies correction map close to identity. This proves theorem.

What about infinite trees?



Is there a theory of *dessins d'adolescents* that relates infinite trees to entire functions with two critical values?

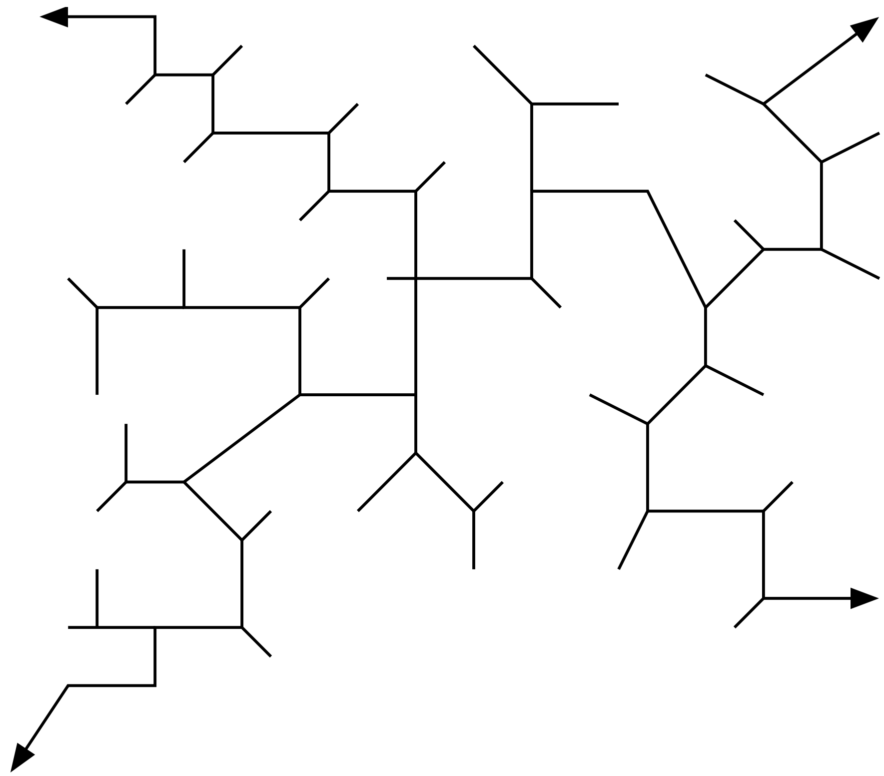
What about infinite trees?



What does "balanced" mean now?

Harmonic measure from ∞ doesn't make sense.

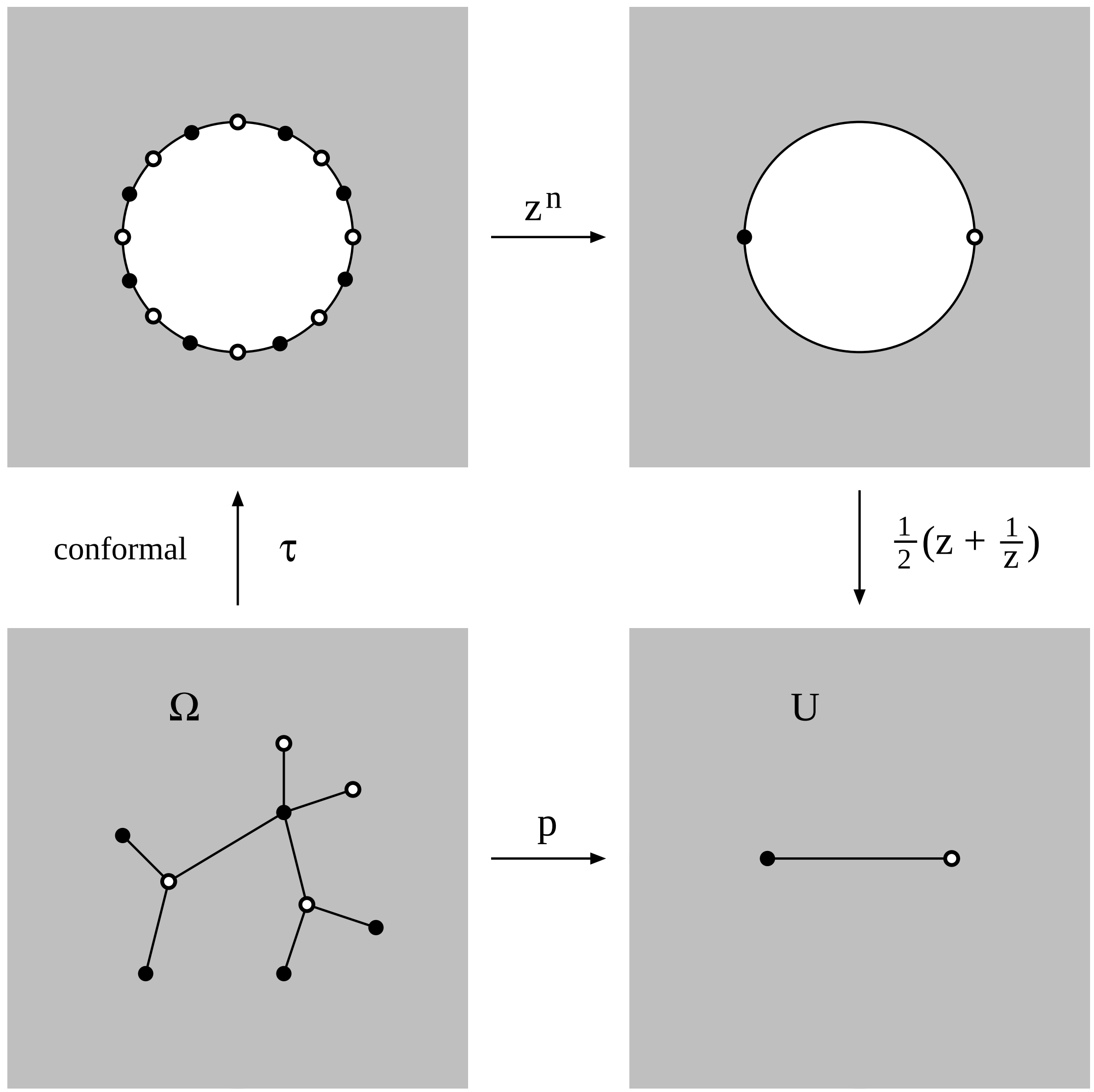
What about infinite trees?



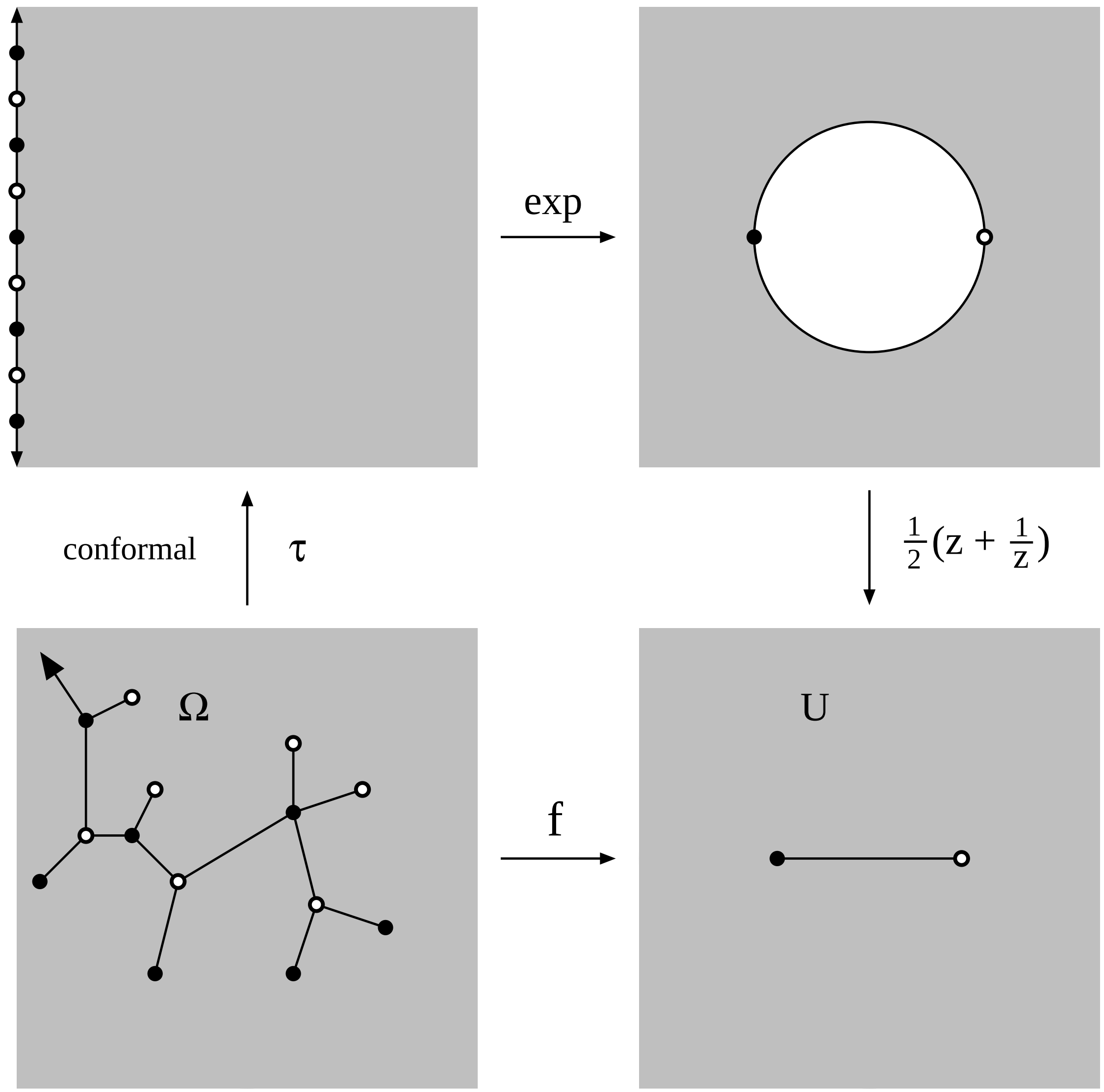
Main difference:

$\mathbb{C}\setminus$ finite tree = one annulus

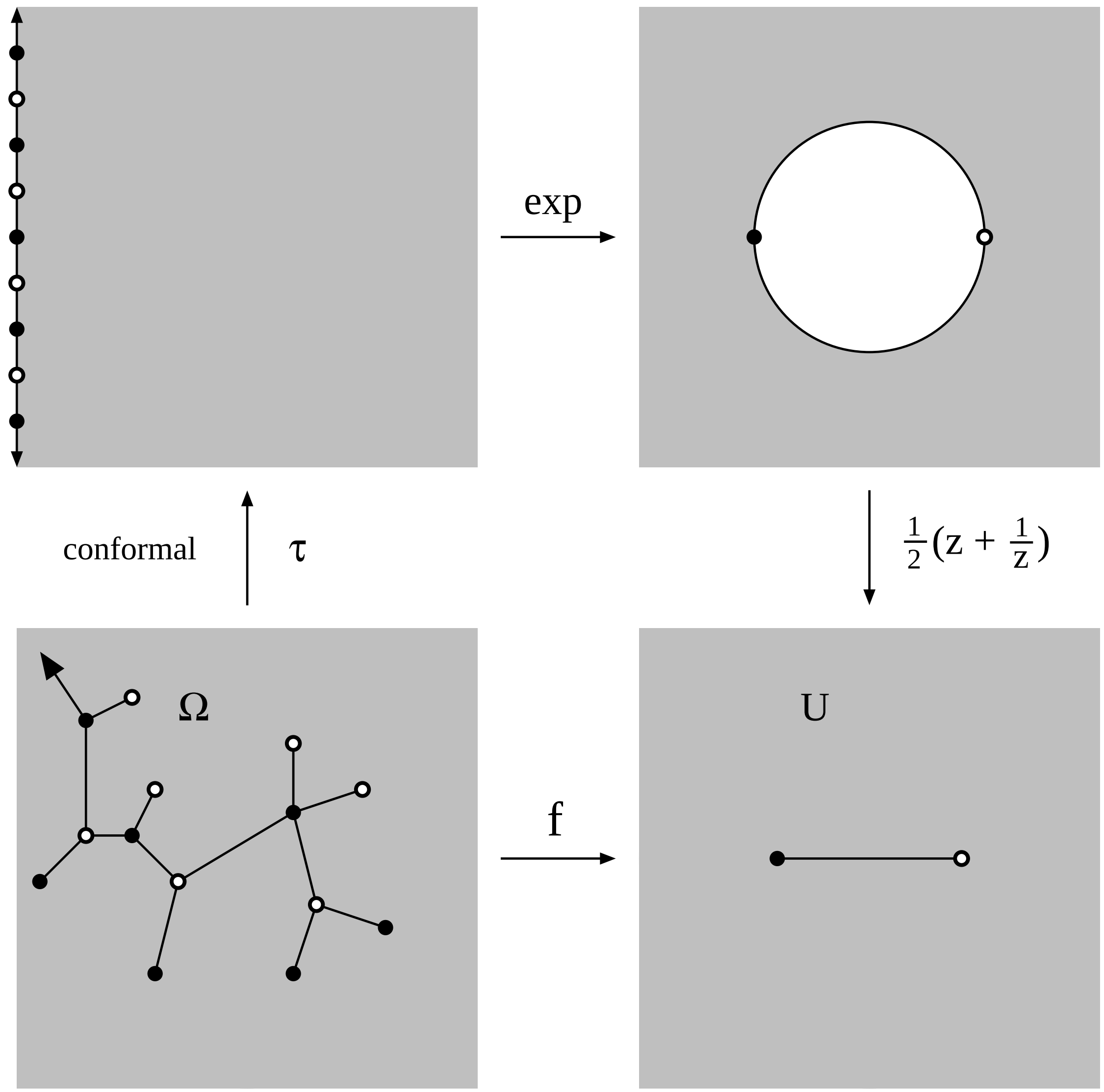
$\mathbb{C}\setminus$ infinite tree = many simply connected components



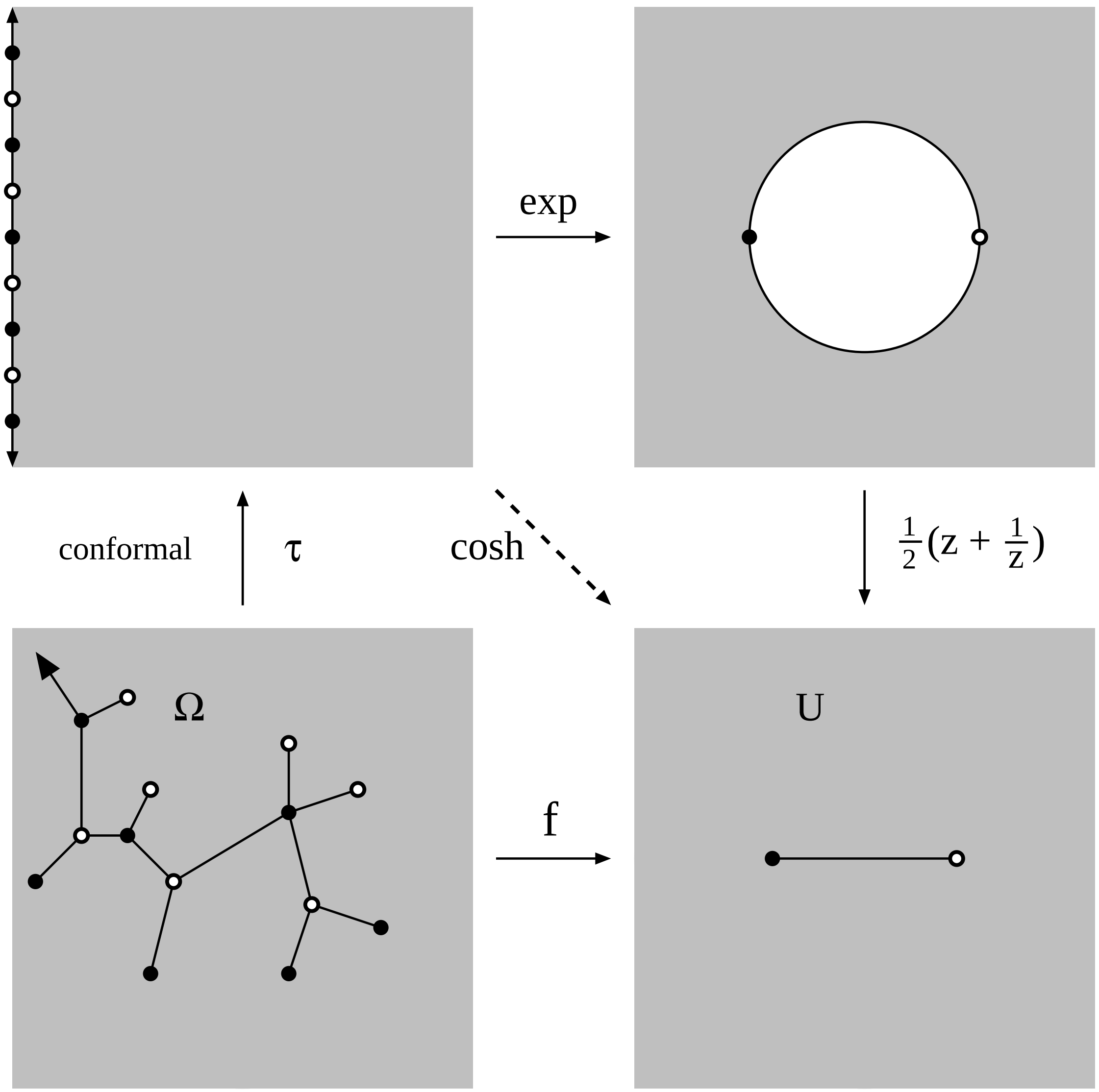
Recall finite case. Infinite case is very similar.



τ maps components of $\Omega = \mathbb{C} \setminus T$ to right half-plane.



Pullback length to tree. Every side gets τ-length π.

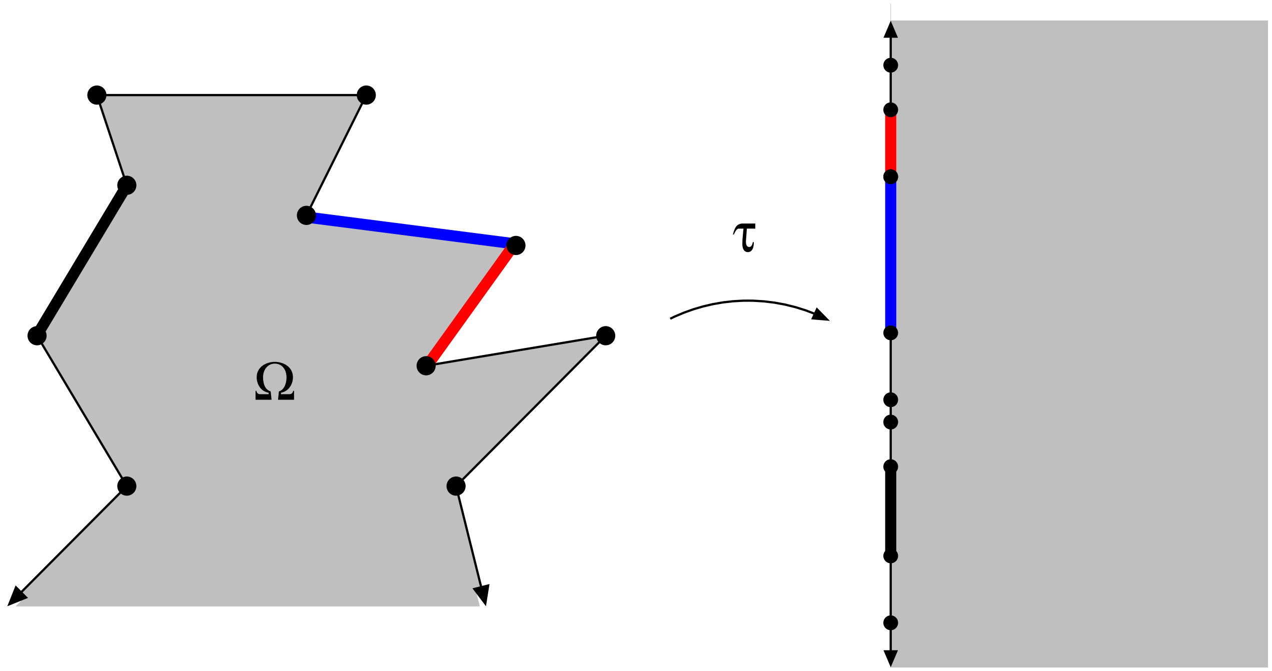


Balanced tree $\Leftrightarrow f = \cosh \circ \tau$ is entire, $\mathrm{CV(f)} = \pm 1$.

For general tree T, define:

τ is conformal from components of $\Omega = \mathbb{C} \setminus T$ to RHP=$\{x > 0\}$.

τ-length is pull-back if length on imaginary axis to sides of T.



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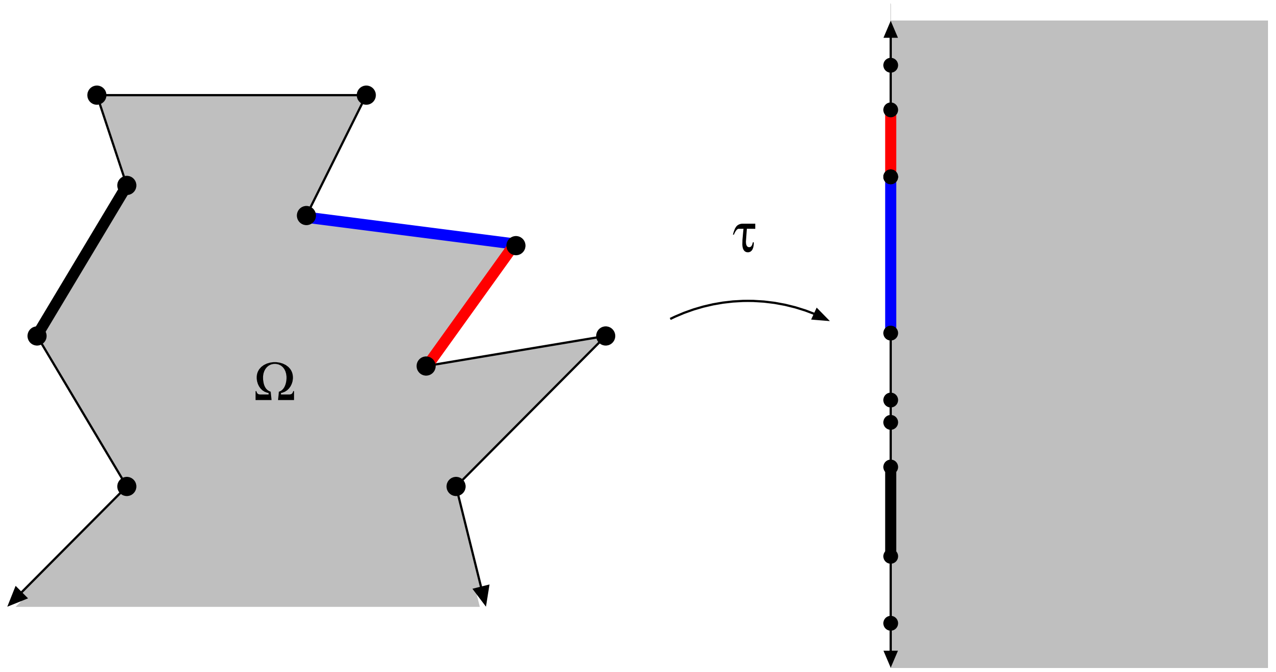
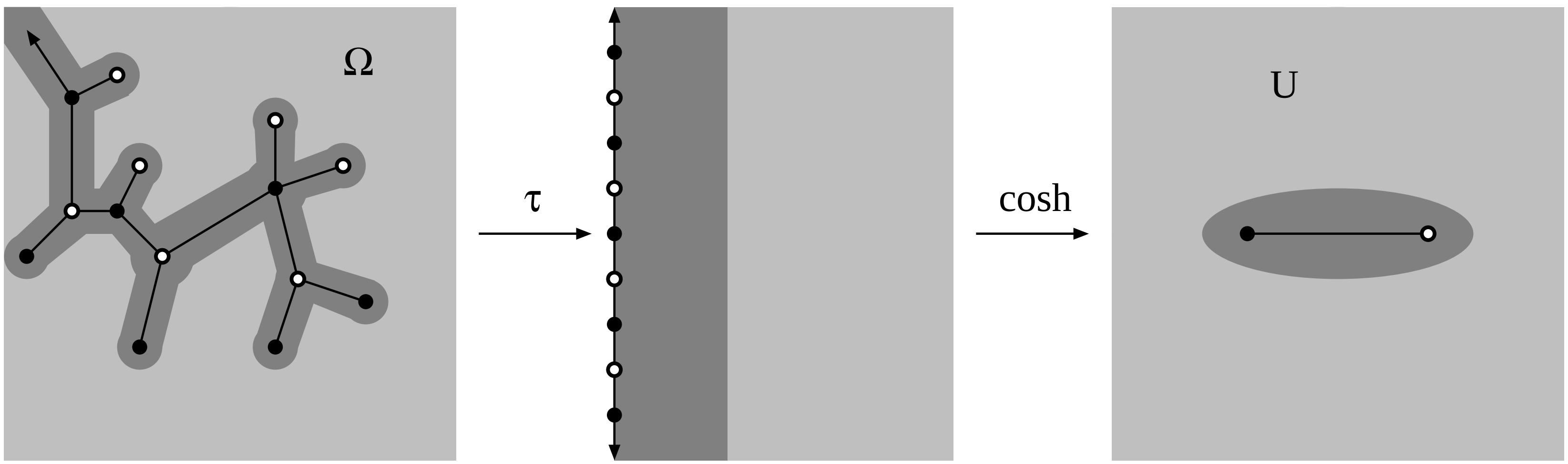


We make two assumptions about components of Ω.

1. Adjacent sides have comparable τ-length (local, bounded geometry)

2. All τ-lengths are $\geq \pi$ (global, smaller than half-plane)

QC Folding Thm: If (1) and (2) hold, then there is a quasi-regular g such that $g = \cosh \circ \tau$ off $T(r)$ and $\mathrm{CV}(g) = \pm 1$.



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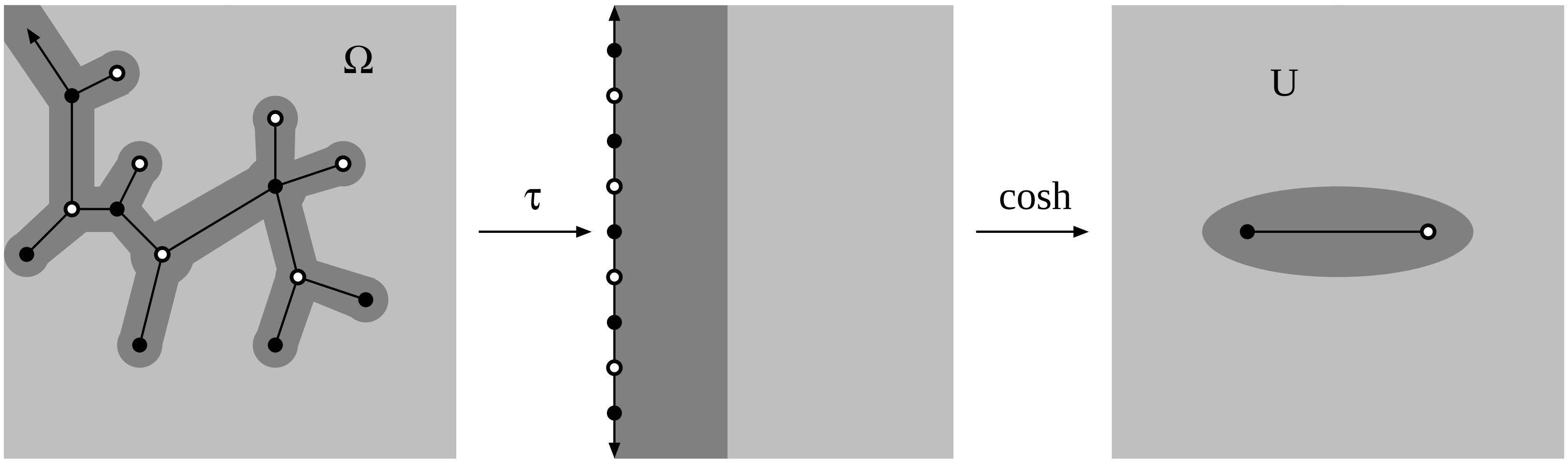

$T(r)$ is a "small" neighborhhod of the tree T.

QR constant depends only on constants in (1).

Cor: There is an entire function $f = g \circ \varphi$ with $\mathrm{CV}(f) = \pm 1$ so that $f^{-1}([-1, 1])$ approximates the shape of T.

What is $T(r)$? If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \le r \cdot \mathrm{diam}(e)\}$$

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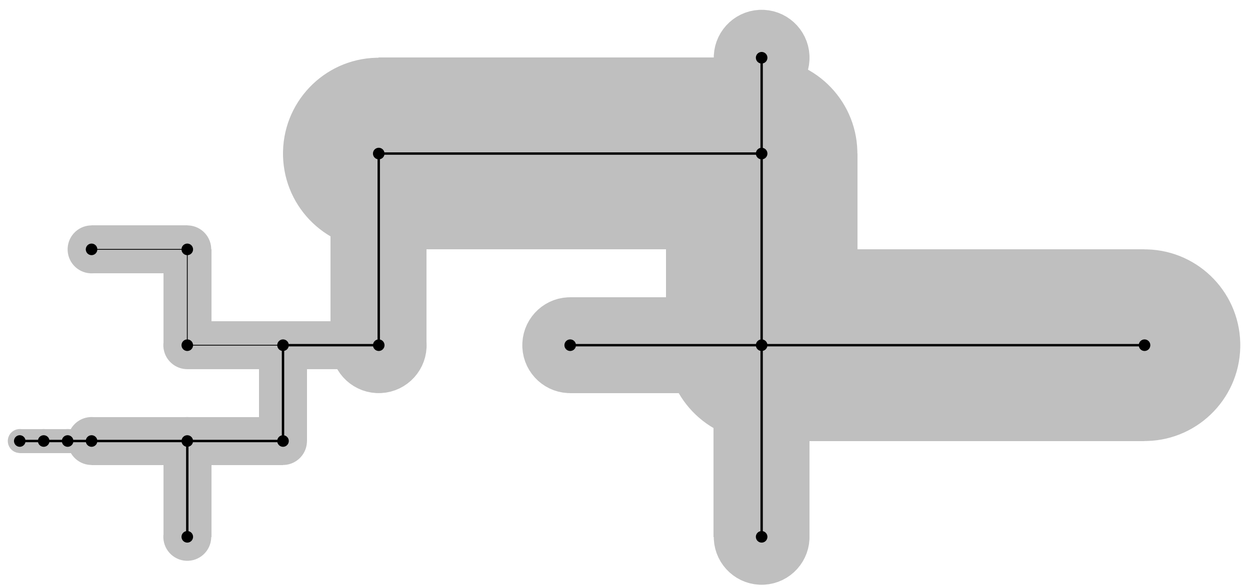
Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.

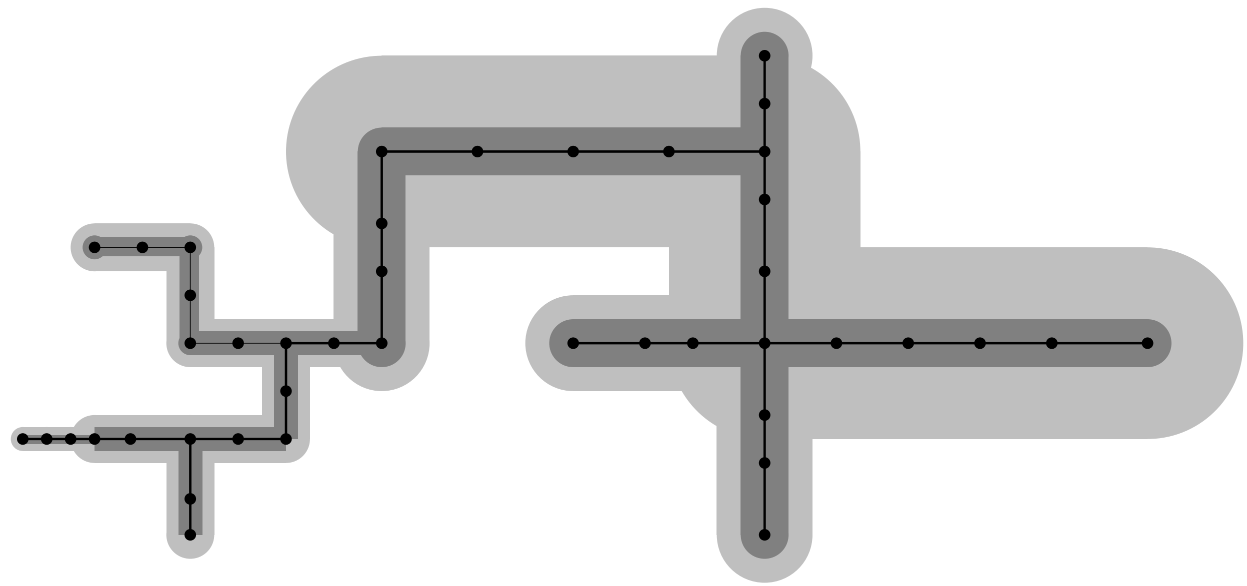


$T(r)$ for infinite tree replaces Hausdorff metric in finite case.

What is $T(r)$? If e is an edge of T and $r > 0$ let

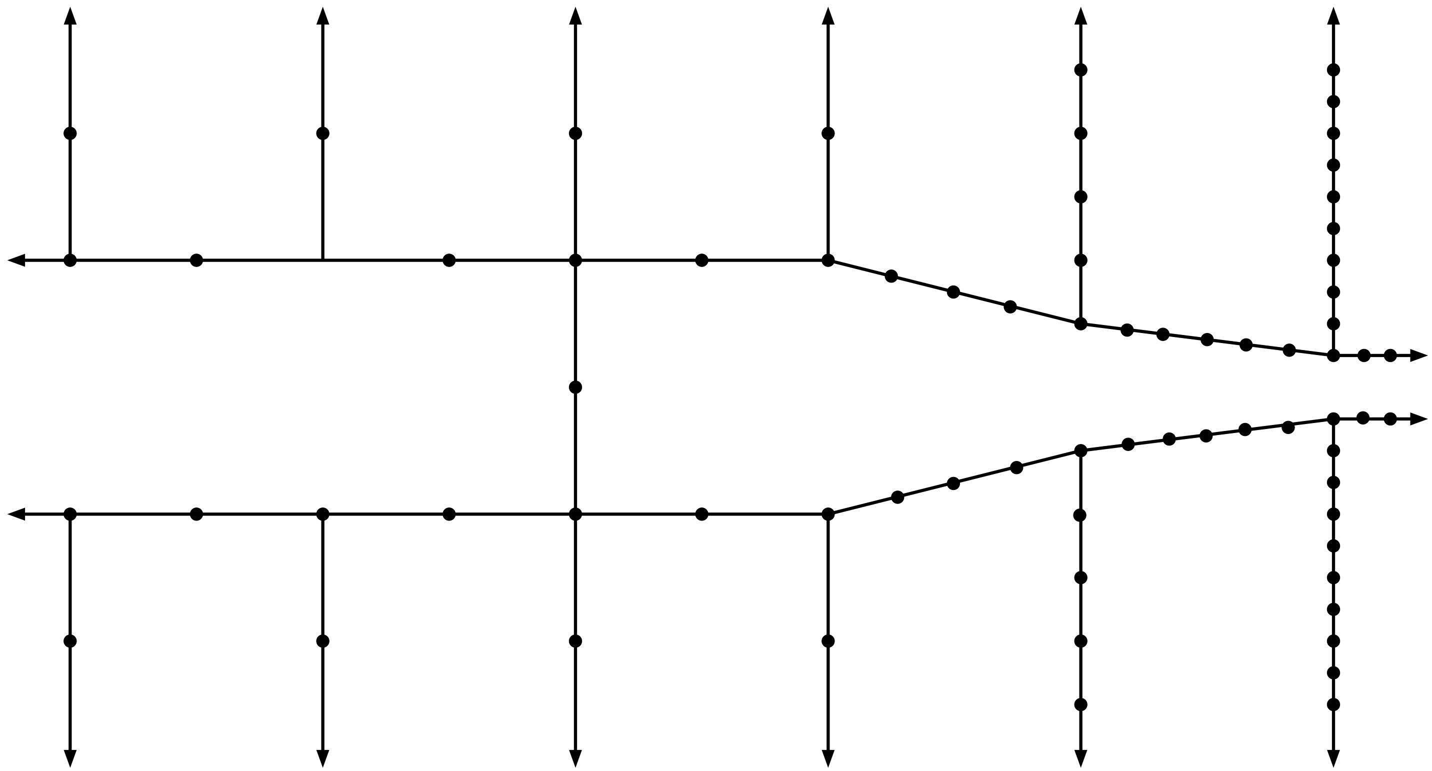
$$e(r) = \{z : \text{dist}(z, e) \leq r \cdot \text{diam}(e)\}$$

Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.



Adding vertices reduces size of $T(r)$.

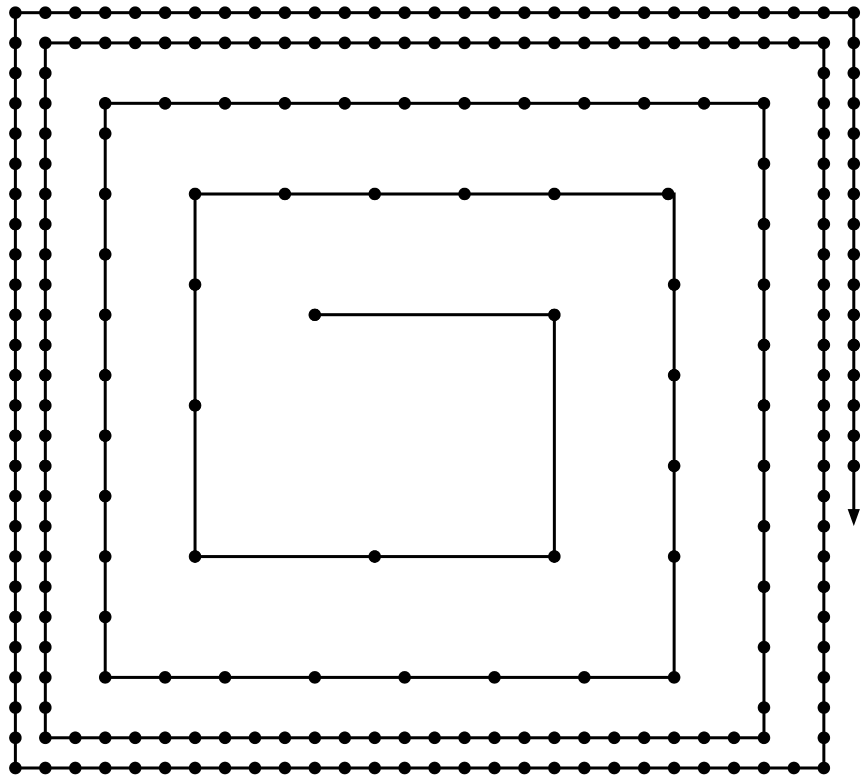
Rapid increase



f has two singular values, $f(x) \nearrow \infty$ as fast as we wish.

First such example due to Sergei Merenkov.

Fast spirals



Two singular values, the tract $\{|f| > 1\}$ spirals as fast as we wish.

Order of growth:

$$\rho(f) = \limsup_{|z|\to\infty} \frac{\log\log|f(z)|}{\log|z|}, \qquad \rho(e^{z^d}) = d.$$

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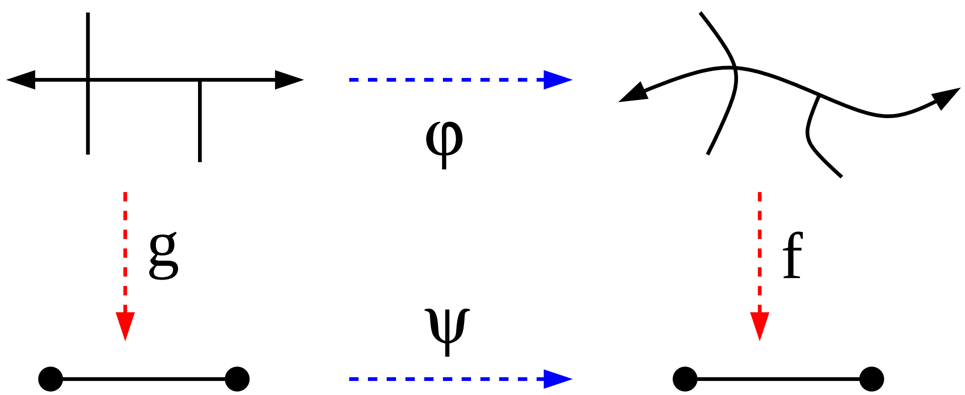
Order conjecture (A. Epstein): f, g QC-equivalent $\Rightarrow \rho(f) = \rho(g)$?

f, g are QC-equivalent
if $\exists$ QC ϕ, ψ such that
$f \circ \phi = \psi \circ g$.



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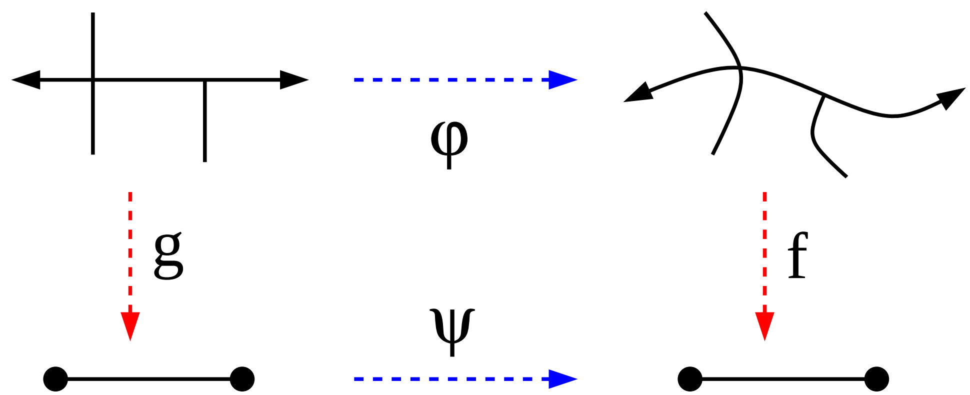
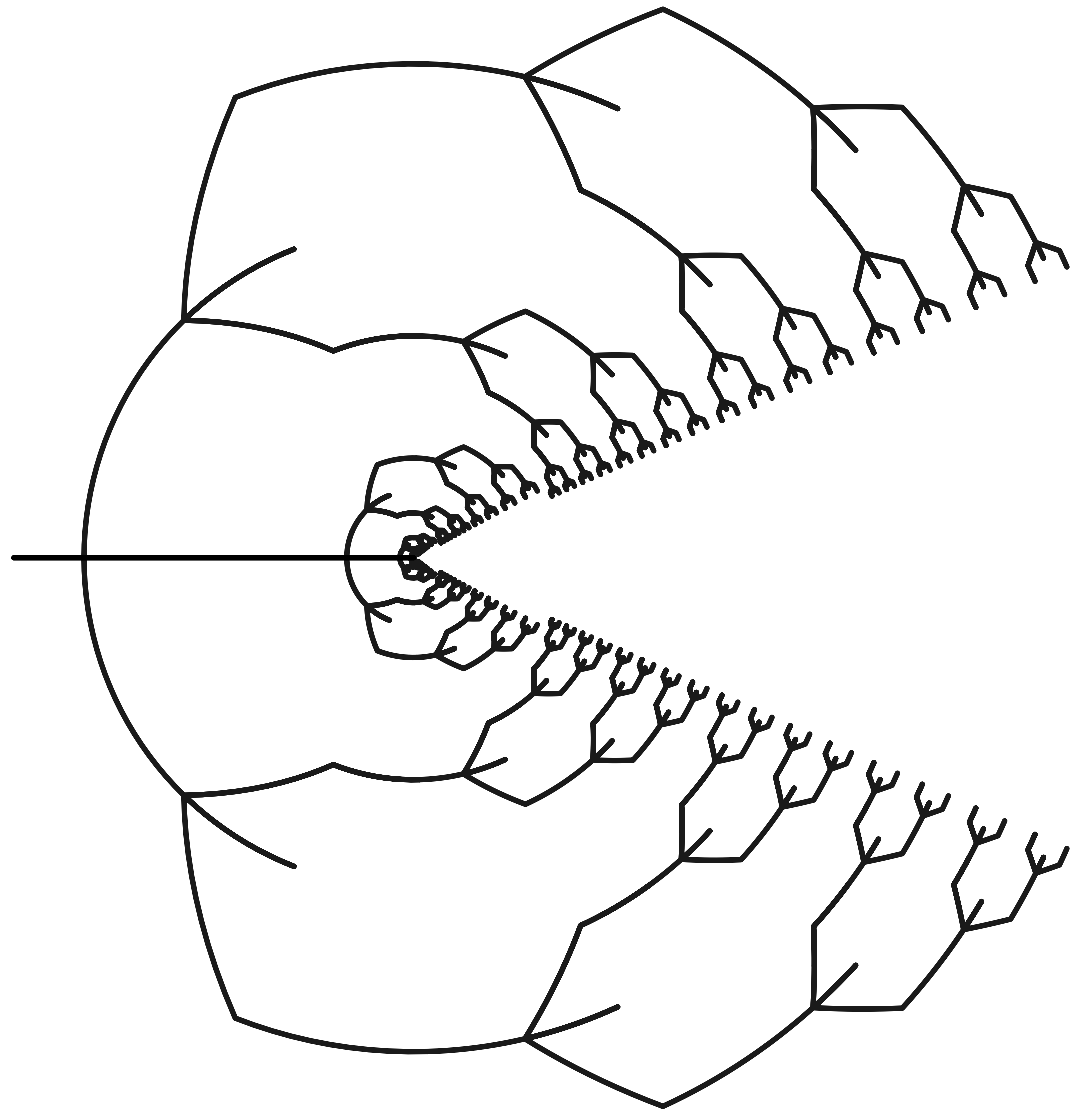


Conjecture true in some special cases.

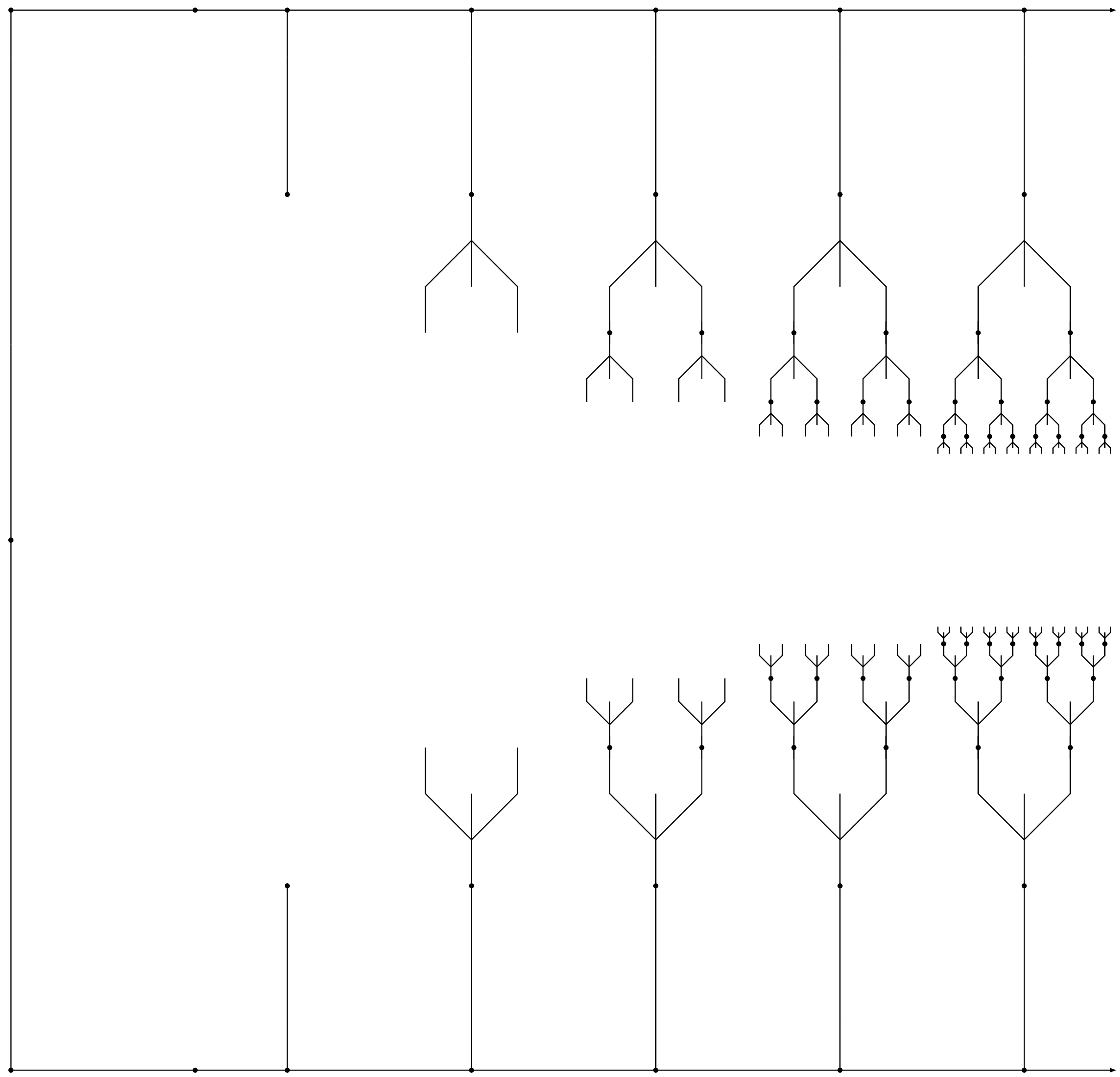
False in Eremenko-Lyubich class = bounded singular set (Epstein-Rempe)

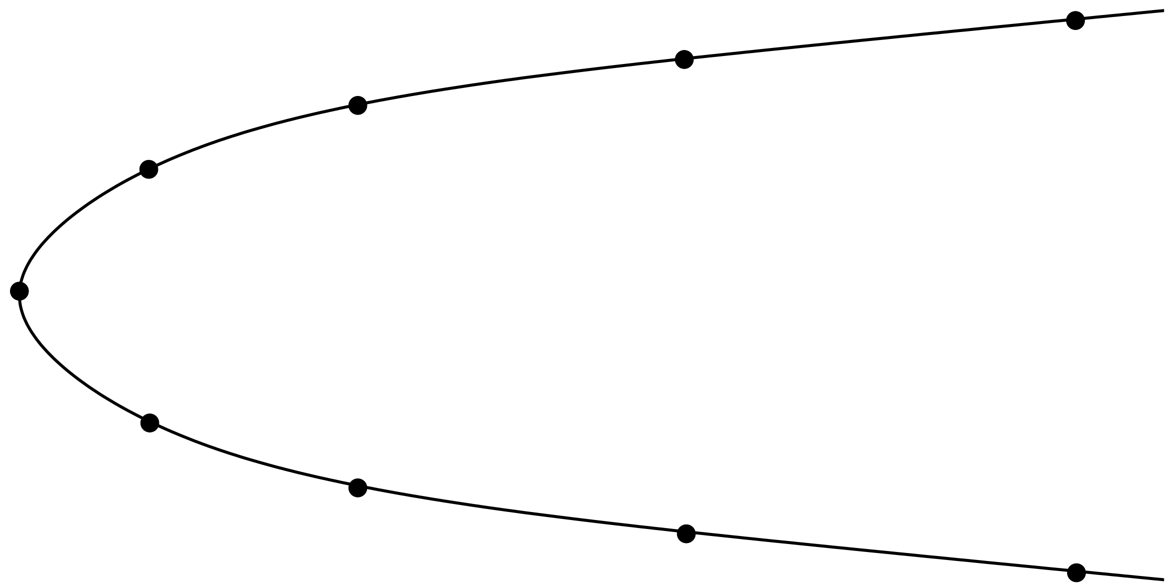
What about Speiser class = finite singular set? True for 2 singularities . . .

but counterexample with 3 singular values:

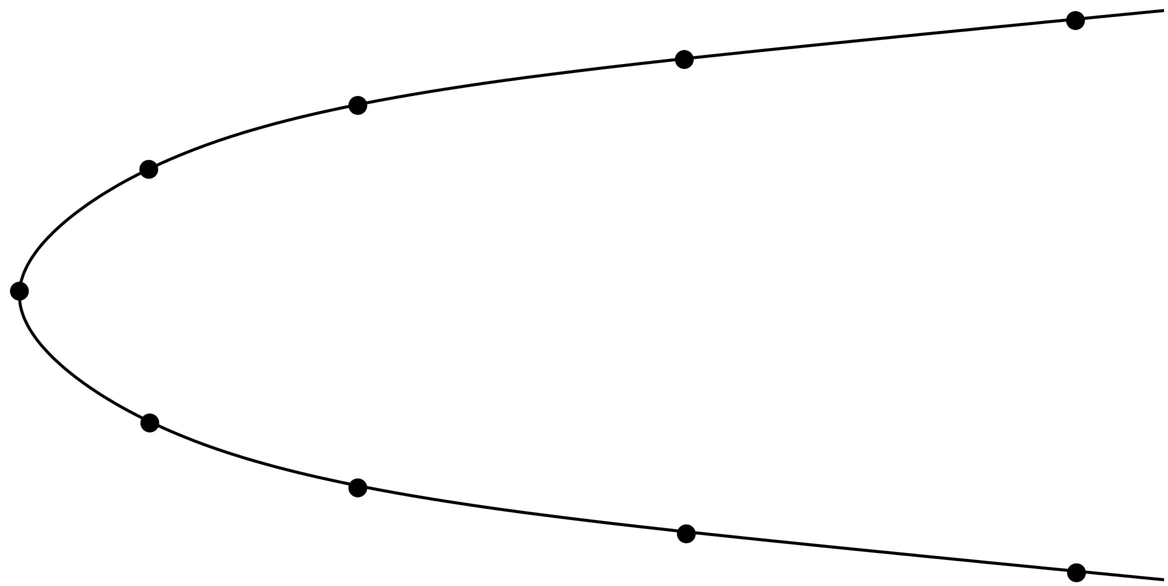


Same domain in logarithmic coordinates





Two singular values and $\dim(\mathcal{J}(f)) < 1 + \epsilon$?



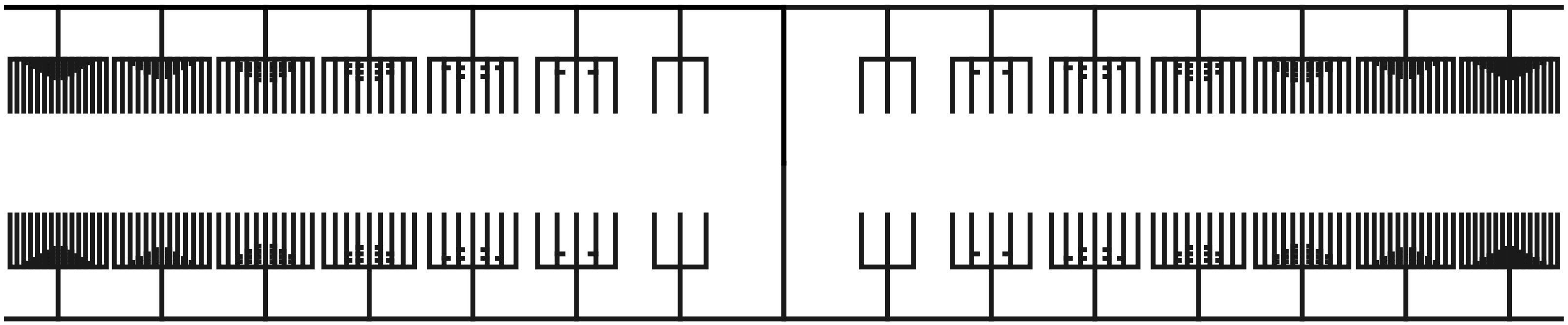
$\dim(\mathcal{J}) \geq 1$ for all transcendental entire functions (Baker).

All values in $[1, 2]$ occur (McMullen, Stallard, B).

$\dim(\mathcal{J}) > 1$ if singular set bounded; $1 + \epsilon$ occurs (Stallard).

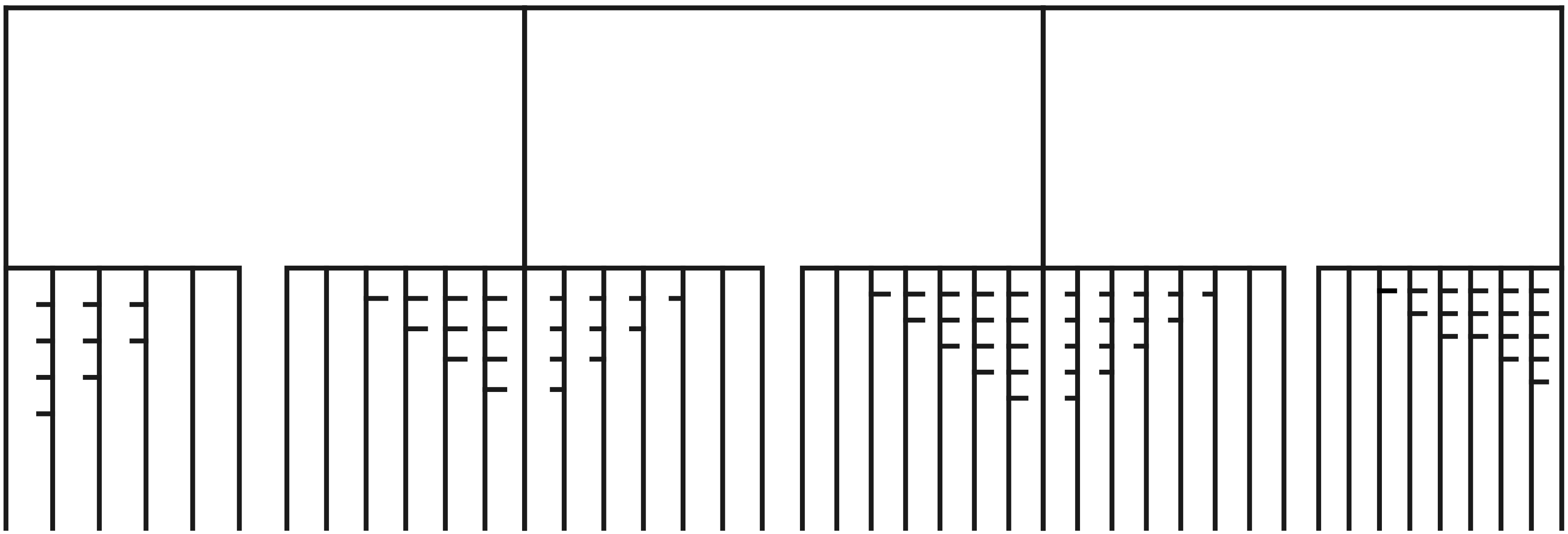
Can we get $\dim < 1 + \epsilon$ in Speiser class = finite singular set?

Hard to estimate QC correction map above. Need Lipschitz bound.

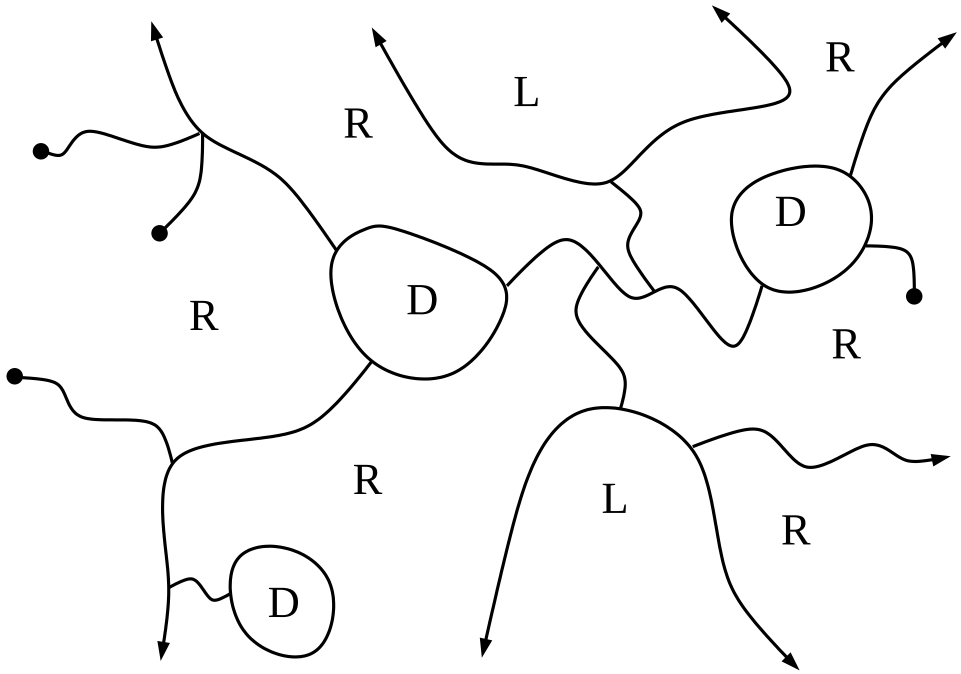


Two singular values and $\dim(\mathcal{J}(f)) < 1+\epsilon$? (B + Albrecht)

Complicated tree, but almost τ-balanced, so folding map is "simple".



Enlargement of portion of tree.



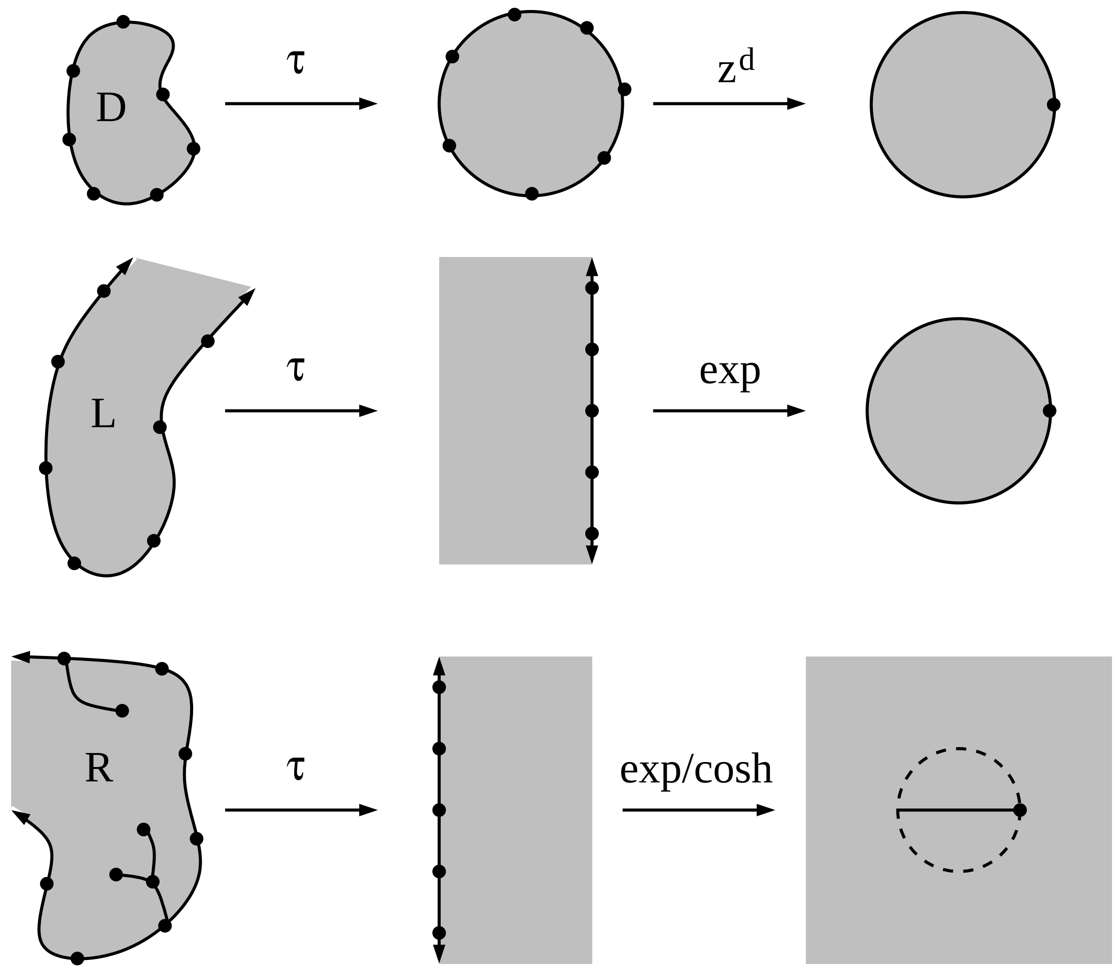
Folding theorem can be generalized from trees to graphs.

Graph faces labeled D,L,R.

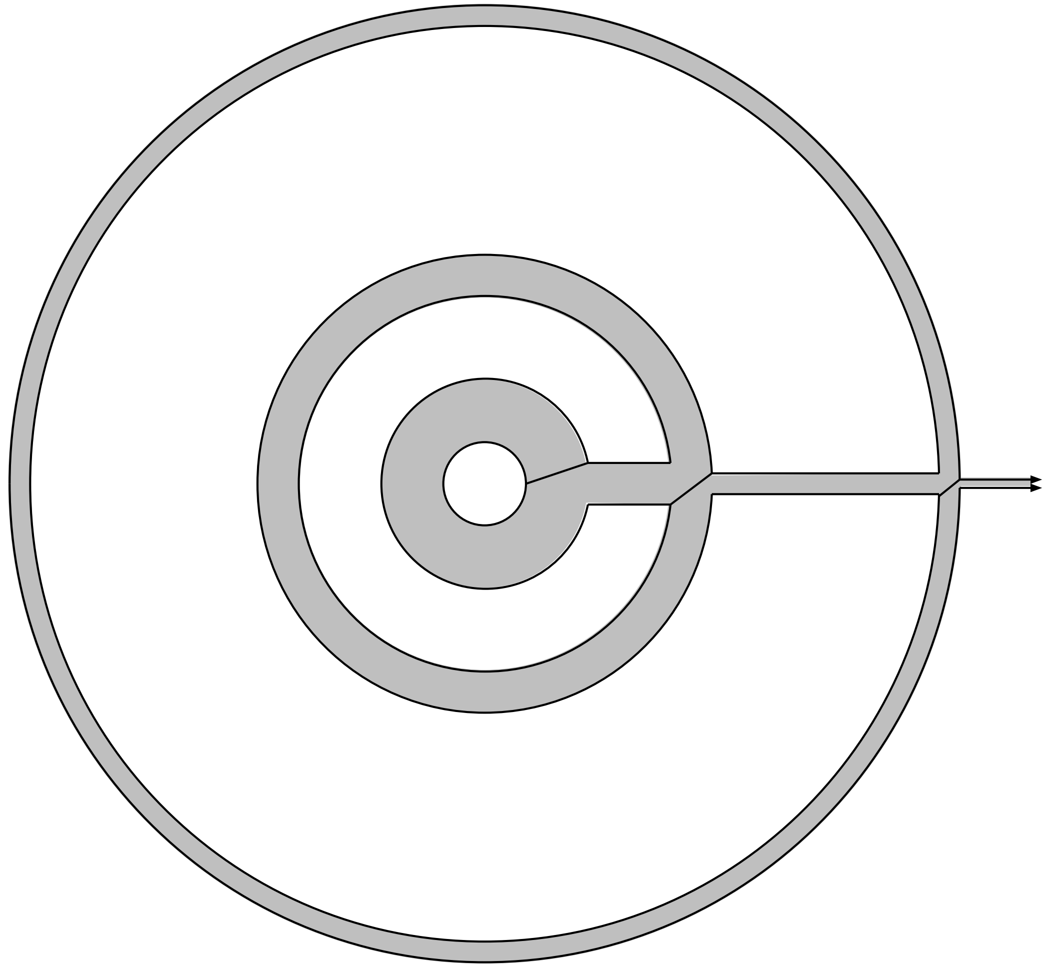
D = bounded Jordan domains (high degree critical points)

L = unbounded Jordan domains (asymptotic values)

D's and L's only share edges with R's.

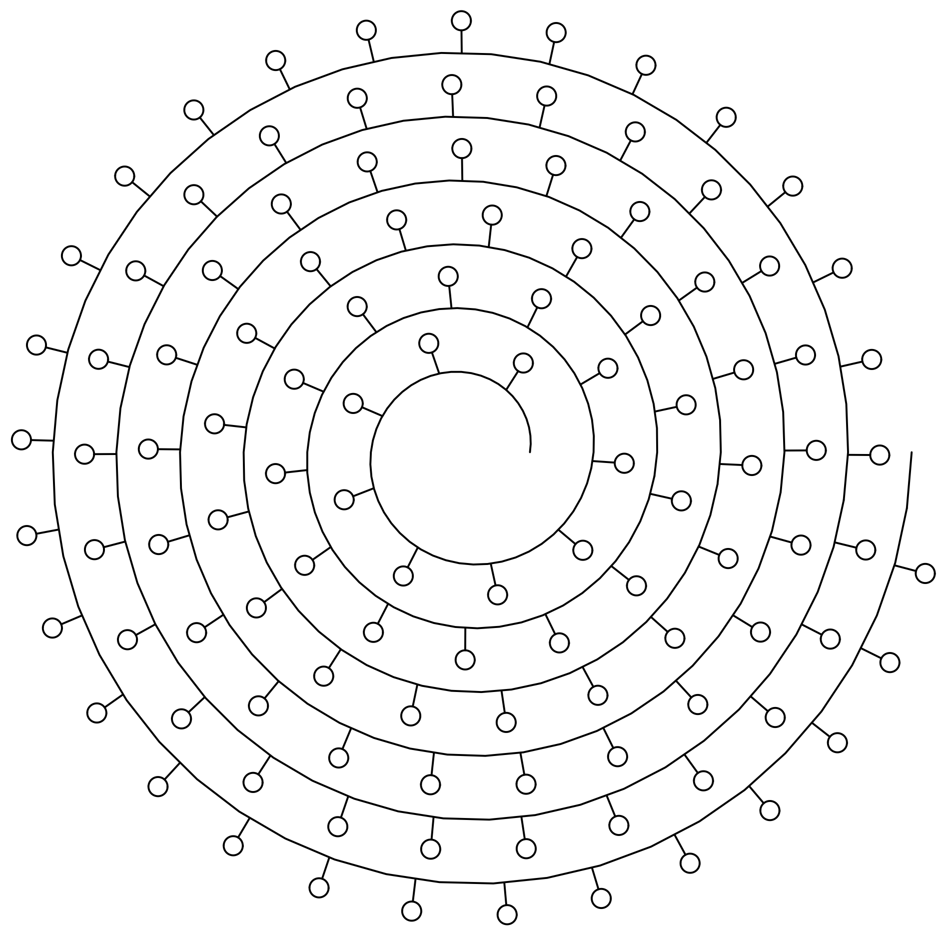


Can define holomorphic map on each component. Generalized folding modifies near graph to get quasiregular map. MRMT gives entire function.



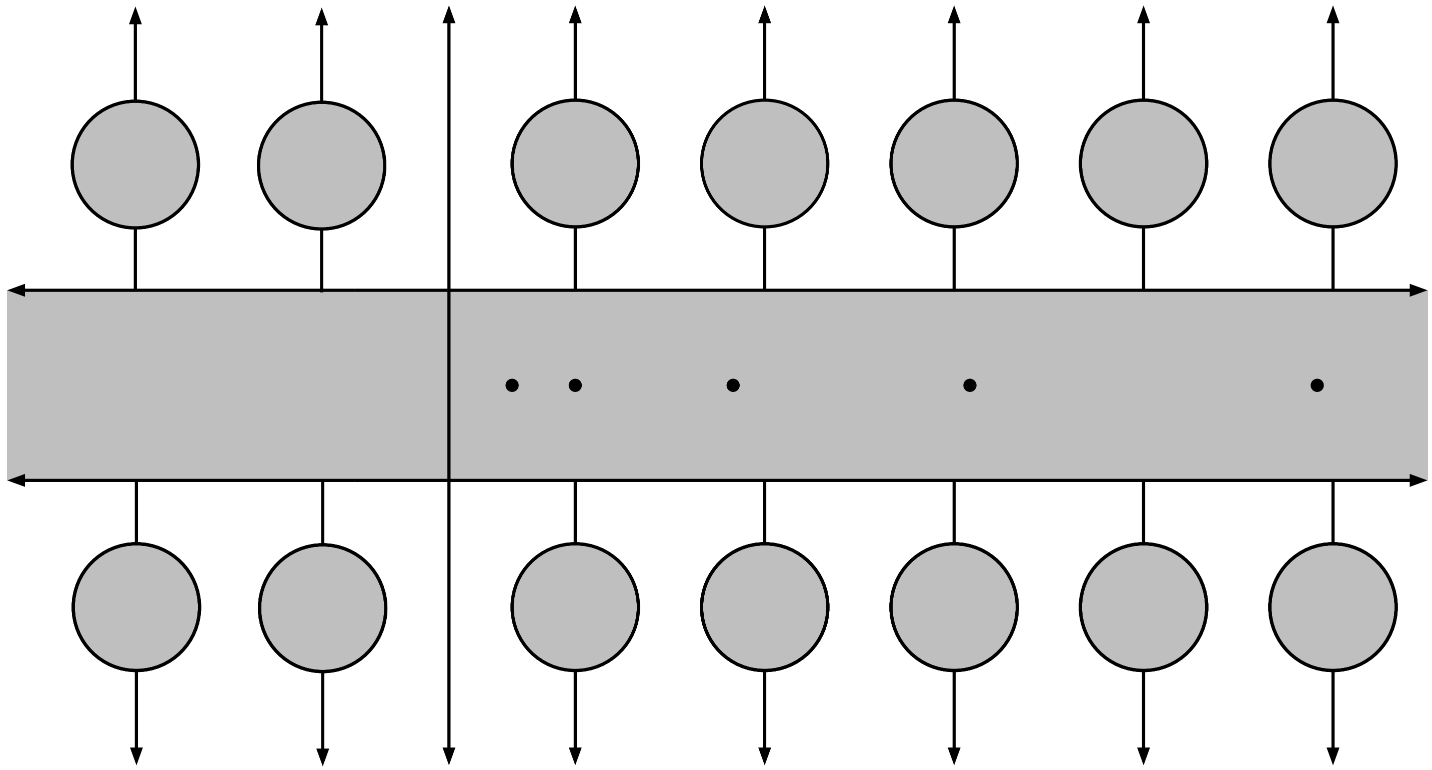
Counterexample to area conjecture (Eremenko-Lyubich)

Three critical values, $\text{area}(\{|f| > \epsilon\}) < \infty$ for all $\epsilon > 0$.



$$3 \text{ critical values and } \limsup_{r\to\infty} \frac{\log m(r,f)}{\log M(r,f)} = -\infty.$$

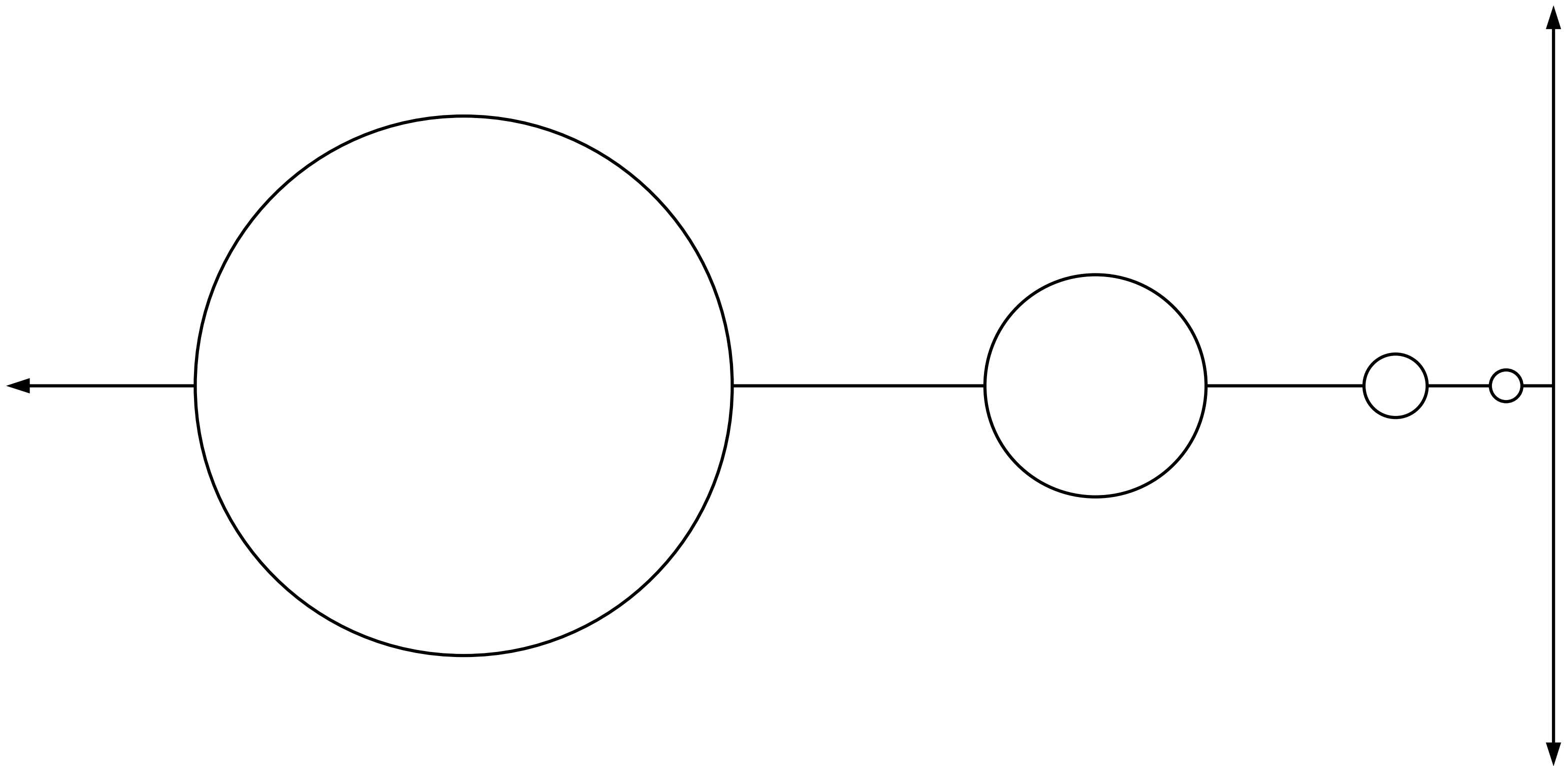
Wiman conjectured ≥ -1 (consider e^z). Disproved by Hayman.



Wandering domain in Eremenko-Lyubich class = bounded singular set

No wandering in Speiser class = finite singular set (Sullivan, Eremenko-Lyubich, Goldberg-Keen).

First wandering domains for entire functions due to Baker.



Wandering domain in Eremenko-Lyubich class with finite order of growth?

Thanks for listening. Questions?

With apologies to Rodgers and Hammerstein, “Nick’s favorite things”

Maps both conformal and slightly distorting,
Logs that repeat, but never are boring,
Bounding derivatives that make wild swings,
These are a few of Nick’s favorite things.

Gauss fields with freedom and metrics most random,
Harmonic measure and integral spectrum,
Brennan’s conjecture, the questions it brings,
These are a few of Nick’s favorite things.

Growth by diffusion and fractals dynamic,
Loewner’s equation, when data’s erratic,
Curves that are random, containing no rings,
These are a few of Nick’s favorite things.

When the proof fails, and the truth stings,
When I’m feeling dumb,
I simply remember Nick’s favorite things,
And then I don’t feel so glum.