Almost sure continuity along curves traversing the Mandelbrot set

Michael Benedicks (with Jacek Graczyk)

KTH Royal Institute of Technology

Everything is Complex
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The aim of this talk is to discuss the dimension of Julia sets and the harmonic measure on the Julia set J_c of $f_c(z) = z^2 + c$, where c is close to the boundary of the Mandelbrot set.
The idea is to use work of Graczyk-Swiatek and Smirnov which proves that for a.e point c_0 on the boundary of the Mandelbrot set $\mathcal{M}$ with respect to harmonic measure the function f_{c_0} satisfies the so called Collet-Eckmann condition.

$$|Df^n_{c_0}(c_0)| \geq Ce^{\kappa n} \qquad \forall n \geq 1. \tag{1}$$

Of course a.e. with respect to Harmonic measure is much related to Makarov's theorem.

The Collet-Eckmann condition together with another condition which we call *approach rate condition for the critical point* makes it possible to apply machinery based on Carleson-B to prove that a C^2-curve through c_0 has that point as a Lebesgue density point with respect to arclength for other CE-points.

After the parameter selection it is possible to apply results by Graczyk-Smirnov on the corresponding Julia sets and we get.

- Geometric measures associated to the Julia set and the corresponding dimension of the Julia set J_c.
- A Sinai-Ruelle-Bowen measure on J_c and its dimension.
- Continuity properties of measures, dimensions and Lyuapunov exponents.

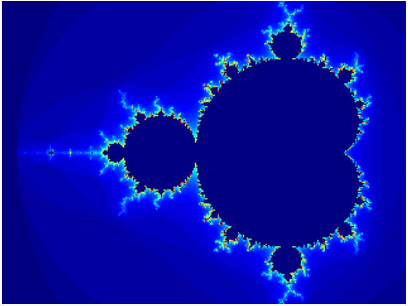

Figure: Mandelbrot set

Let us first start with the case c real.

Theorem (Jakobson (1978))

Consider the quadratic family $f_c : x \mapsto c + x^2$ for $c \in (-2, \frac{1}{4})$. There is a subset A of the parameter set $(-2, \frac{1}{2})$ of positive Lebesgue measure so that for a.e. $c \in A$, f_c has an absolutely continous invariant measure $d\mu_c(x) = \varphi_c(x)\,dx$.

As a consequence you get that for a.e. initial point x in the dynamical interval

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j x} \to d\mu_c \qquad \text{weak} - *$$

Later with Lennart Carleson we first gave another proof for Jakobson's theorem and also proved the following result

Theorem (Carleson - B.)

There are constants $C > 0$, $\kappa > 0$ and a set A_{CE} of positive Lebesgue measure so that for all $c \in A_{CE}$

$$|Df_c^n(c)| \geq Ce^{\kappa n} \qquad \text{for all } n \geq 0. \tag{2}$$

We say that c satisfies the Collet-Eckmann condition if (2) is satisfied In the inductive proof of (2) we also proved

$$|f_c^n(0)| \geq Ce^{-\alpha n} \qquad \forall n \geq 1. \tag{3}$$

The following famous result is called the Real Fatou Conjecture.

Theorem (Swiatek & Graczyk and Lyubich)

The set $B = \{c : f_c \text{has an attractive periodic orbit}\}$ *is open and dense in the parameter space* $(-2, \frac{1}{2})$.

This was further extended by Lyubich who proved

Theorem (Lyubich)

For the quadratic family the parameter space can be written as the disjoint union a.e. of the set A *of parameters with absolute continuous invariant measure and the set* B *of parameters with attractive periodic orbits*

$$A \cup B = (-2, \frac{1}{4}) \qquad a.e.$$

Finally Avila and Moreira proved

Theorem

For a.e. $c \in A$

(i) *The functions f_c satisfy the Collet-Eckmann condition $|Df_c^n(c)| \geq Ce^{\kappa n}$ for all $n \geq 0$.*

(ii) *There is $\alpha > 1$ and $C > 0$, so that $|f_c^j(0)| \geq Cj^{-\alpha}$ for all $j \geq 1$.*

In the original paper with Carleson and also in the paper on Hénon maps we did the perturbation from $x \mapsto 1 - 2x^2$ or equivalently from the map $-2 + x^2$ (the von Neumann-Ulam map) as the starting map, which have the initial expansion. (More about this later.) One can as well start to perturb from a map which satisfies the so called

Misiurewicz-Thurston condition:

The critical point is preperiodic and going to an unstable periodic orbit.

There are three important aspect on the unperturbed map to set up a proof of this type

- Expansion at the critical value (similar to the Collet-Eckmann condition).
- A transversality condition that gives comparasion between the phase and parmeter derivative.
- Mañé-Misiurewicz style lemma: Expansion outside a neighborhood of the critical point.

In the present work with Graczyk we replace the Misiurewicz-Thurston condition

$$|f_{c_0}^n(0)| \geq c_*, n \geq 1$$

by the initial approach rate condition

$$|f_{c_0}^n(0)| \geq K_0 e^{-\alpha_0 n}, \qquad n \geq 1$$

After the parameter selection for c in the selected set we will have

$$|f_c^n(0)| \geq K e^{-\alpha n}, \qquad n \geq 1.$$

A sufficient condition for transversality is that

$$T(c) = \sum_{j=1}^{\infty} \frac{1}{Df_c^{j-1}(c)} \neq 0 \tag{4}$$

and

$$T(c) = \sum_{j=1}^{\infty} \frac{1}{|Df_c^{j-1}(c)|} < \infty \tag{5}$$

Genadi Levin proved that (5) implies (4) for the quadratic family.

Mañé type lemma

We need the following property of the unperturbed map and it will be inherited by the perturbed maps
Let $f(z) = f_{c_0}(z)$ and fix $\delta > 0$.

- Suppose that there is an integer $m = m(\delta)$ so that if z, $f(z), \ldots, f^m(z) \notin B(0, \delta)$. Then

$$|Df^m(z)| \geq \lambda = e^{\kappa m} > 1$$

- There is $c_* > 0$ such that if z, $f(z), \ldots, f^{n-1}(z) \notin B(0, \delta)$ but $f^n(z) \in B(0, \delta)$ then

$$|Df^n(z)| \geq c_*. \tag{6}$$

As formulated (6) is used for $n < m$.

In principle the two conditions could be summarized into one There is $C > 0$ and $\kappa > 0$ such that if x, $f(z)$,...,$f^{n-1}(z) \notin B(0,\delta)$ but $f^n(z) \in B(0,\delta)$ then $|Df^n(z)| \geq Ce^{\kappa n}$.

The advantage of formulating the condition in two parts is that it is obviously perturbable. If f_{c_0} satisfies these two conditions then by continuity they are satisfied in an open neighborhood of c_0 in the parameter space.

The main result is

Theorem (Graczyk - B.)

For a.e. $c_0 \in \partial\mathcal{M}$ and every C^2 curve $\gamma : [-1,1] \mapsto \mathbb{C}$ we have for the set of Collet-Eckmann parameters E that

$$\frac{m(E \cap \gamma(-\varepsilon,\varepsilon))}{m(\gamma(-\varepsilon,\varepsilon))} \to 1$$

as $\varepsilon \to 0$ (m is arclength measure).

The starting point for our construction is

Theorem (Graczyk-Swiatek & Smirnov)

For almost every $c \in \partial\mathcal{M}$ with respect to the harmonic measure ω, the limit

$$\lim_{n\to\infty} \frac{1}{n} |\log (f_c^n)'(c)|$$

exists and is equal to $\log 2$.

The approach rate property

Proposition

For every $\eta > 0$ and almost every c in $\partial\mathcal{M}$ with respect to the harmonic measure, there exists n_0 such that for every $n \geq n_0$,

$$|f_c^n(c)| \geq 2^{-\eta n} .$$

Proof.

Let $\delta_0 > 0$. Choose a number n_0 such that for every $n \geq n_0$,

$$2^{(n+1)(1-\delta_0)} \leq |(f_c^{n+1})'(c)| \leq 2^{n(1+\delta_0)}\, 2|f_c^n(c)| \leq 2^{(n+1)(1+\delta_0)}\, |f_c^n(c)| .$$

Therefore,

$$|f_c^n(c)| \geq 2^{-2\delta_0(n+1)}$$

which completes the proof. $\square$

The transversality property

We introduce the transversality function

$$T(c) = \sum_{j=1}^{\infty} \frac{1}{(f_c^{j-1})'(c)}$$

which is needed to be different from 0.

Proposition

A.e. on $\partial\mathcal{M}$ with respect to Harmonic Measure the non-tangential boundary values $T(c) \neq 0$.

As mentioned before from a theorem by Levin it follows that if f_c satisfies the Collet-Eckmann property then $T(c) \neq 0$ but let us sketch another proof based on Privalov's theorem.

Sketch. By a result of Graczyk-Smirnov, $\mathrm{Lyap}(c) > 0$ for almost every $c \in \partial\mathcal{M}$ with respect to the harmonic measure. Therefore, by Abel's theorem, the transversality function $T(c)$ has angular limits at almost every c in the boundary of $\mathcal{M}$. The analytic function $T(c)$ is not equal identically 0 as $T(c) > 0$ for c large and positive. By Privalov's theorem, $T(c) \neq 0$ for almost every $c \in \mathcal{M}$ with respect to the harmonic measure.

Sketch of proof of Main Theorem (the CE-property)

Let us recall the statement:

Theorem (Graczyk - B.)

For a.e. $c_0 \in \partial\mathcal{M}$ and every C^2 curve $\gamma : [-1,1] \mapsto \mathbb{C}$ we have for the set of Collet-Eckmann parameters E so that

$$\frac{m(E \cap \gamma(-\varepsilon,\varepsilon))}{m(\ell(-\varepsilon,\varepsilon))} \to 1$$

as $\varepsilon \to 0$ (m is arclength measure) and for all $c \in E$

(i) $|Df_c^n(c)| \geq C_1 e^{\kappa n} \qquad \forall n \geq 1$

(ii) $|f_c^n(0)| \geq C_2 e^{-\alpha n} \qquad \forall n \geq 1$

Strategy of the proof

Definition

Let $\alpha > 1$ and $\beta > \alpha + 1 > 2$. We consider two parameters c_1 and c_2. We say that z is *bound* to 0 up to time p if p is maximal with the property

$$|f_{c_1}^j(0) - f_{c_2}^j(z)| \leq C_b e^{-\beta j} \qquad \text{for all } j \leq p.$$

In the application $z = f_{c_2}^n(0)$, where n is a return time to $B(0, \delta)$ of a partition curve segment ω containing c_1 and c_2.

An important lemma is the Bound Distorsion Lemma

$$C^{-1} \leq \frac{|Df_{c_1}^p(c_1)|}{|Df_{c_2}^p(f_{c_2}(z))|} \leq C$$

for some universal constant C which can be chosen close to 1. This is essentially due to the estimate

$$\frac{|Df_{c_1}^p|}{|Df_{c_2}^p(f_{c_2}(z))|} \leq \exp\{\sum_{n=1}^{p} e^{-(\beta-\alpha)n)}\}$$

Bound period derivative recovery.

Let the "arriving" derivative be D_n. After the bound period the bound relation reverses and we get the relation

$$|z|^2 D_{p+1} \geq \text{const.} e^{-\beta(p+1)}$$

This translates to

$$D_{n+p} \geq D_n \cdot |2z| \cdot D_p \geq \text{const.} D_n D_p^{1/2} e^{-\beta p/2}.$$

After the bound period we have a *free period L*, where we have exponential derivative increase until the return by the Mañé style lemma: The "outside derivative" $\geq \text{const.} e^{\kappa_0 L}$.

This argument does not let us fully recover the Lyapunov exponent. If $D_n \sim e^{\kappa n}$ we get roughly $D_{n+p} \sim e^{\kappa n + (\frac{\kappa}{2} - \beta)p}$ which is not quite good enough. Basically this argument is good enough to prove stretched exponential growth $D_n \sim e^{n^\gamma}$, $\gamma < 1$.

To improve the estimates we use a large deviation argument.
When an orbit circulates outside $B(0, \delta^2)$ we get by the Mañé style lemmas. If

$$z, f(z), \ldots, f^{n-1}(z) \notin B(0, \delta^2) \text{ and } f^n(z) \in B(0, \delta)$$

then $|Df^n(z)| \geq e^{\kappa_0 n}$.

For the selected parameters f_c satisfies an approach rate condition: there is an $\alpha > 0$ so that

$$|f_c^n(0)| \geq e^{-\alpha_0 n} \qquad \text{for all } n \geq 1.$$

This is obtained by deleting parameters c so the *Basic assumption*

$$|f_c^n(0)| \geq e^{-\alpha n} \qquad n \geq 1$$

is satisfied.

In this complex case if a curve segment γ_n arrives to $B(0, \delta)$ it is partitioned essentially according to which annualar region $A_{r\ell} = \{R_{r,\ell} \leq |z| \leq R_{r,(\ell+1)}\}$ it arrives, $R_{r,\ell} \sim e^{-r}$. In order to do so we must have a curvature control of γ_n

The partition.

We write the return ball as a union

$$B(0, \delta) = \bigcup_{r \geq r_\delta} A_r$$

Each A_r is subdivided into annuli $A_{r,\ell}$, $\ell = 0, 1, 2, \ldots, r^2 - 1$, which

$$A_{r,\ell} = \{z \in \mathbb{C} \,|\, R_{r,\ell} \leq |z| \leq R_{r,\ell}\},$$

where $R_{r,0} = e^{-r-1}$ and $R_{r,i+1} - R_{r,i} = (1 - e^{-1})\frac{1}{r^2} \cdot e^{-r}$. We also define $R_{r,r^2} = R_{r-1,0}$.

Some consequences using previous work by Graczyk and Smirnov

Theorem (Continuity of Hausdorff Dimension)

It follows that for the selected subset $\mathcal{E}_\gamma$ of the curve γ that

$$\lim_{\mathcal{E}_\gamma \ni c \to c_0} \mathrm{HDim}(\mathcal{J}_c) = \mathrm{HDim}(\mathcal{J}_{c_0})$$

and

$$\lim_{\mathcal{E}_\gamma \ni c \to c_0} \mathrm{Lyap}(f_c) = \mathrm{Lyap}(f_{c_0}) \; .$$

By Shishikura's theorem *(Shishikura, 1991)*, it is known that the Hausdorff dimension as a function of $c \in \mathbb{C} \setminus \mathcal{M}$ does not extend continuously to ∂M. Yet typically with respect to the harmonic measure of $\partial\mathcal{M}$ a continuous extension of $\mathrm{HDim}(\cdot)$ along restricted approaches is possible.

The geodesic landing at c_0 is called an external ray at c_0. Radial continuity of Hausdorff dimension for postcritically finite quadratic polynomials was established in *(McMullen 2000, Rivera 2001)*. The set of postcritically finite polynomials is of zero harmonic measure, *(Graczyk-Swiatek 2000, Smirnov 2000)*. This result was further extended in *(Rivera 2001)* for Misiurewicz polynomials.

Another consequence of (Graczyk-Swiatek 2000, Smirnov 2000) is a conformal analogue of Jakobson and Benedicks-Carleson's theorem.

Suppose that f_c has a geometric measure. We call a probabilistic measure μ, supported on the Julia set of f_c, a *Sinai-Ruelle-Bowen*, or SRB for short, measure if it is a weak accumulation point of measures μ_n equally distributed along the orbits $z, f_c(z), \ldots, f_c^n(z)$ for z in a positive geometric measure set.

A theorem by Graczyk-Smirnov from 1998 states that for almost all $c \in \partial\mathcal{M}$ with respect to the harmonic measure, there exists a unique geometric measure ν_c of f_c which is a weak limit of the normalized Hausdorff measures of $J_{c'}$ along external rays landing at c, ν_c is ergodic and non-atomic, $\mathrm{HDim}(\nu_c) = \mathrm{HDim}(J_c)$, f_c has an invariant SRB measure with a positive Lyapunov exponent which is equivalent to the geometric measure ν_c.

Theorem

For almost every parameter c_0 in the boundary of the Mandelbrot set $\mathcal{M}$ with respect of the harmonic measure and every C^2 curve $\gamma : [-1,1] \mapsto \mathbb{C}$, $c_0 = \gamma(0) \in \partial\mathcal{M}$, the point 0 is a Lebesque density point of the set $\mathcal{A}_\gamma \subset [-1,1]$ with the following properties:

1. *For every $x \in \mathcal{A}_\gamma$, there exists a unique geometric measure ν_x of $f_{\gamma(x)}$ which tends weakly to a unique geometric measure ν_0 of f_{c_0}.*
2. *For every $x \in \mathcal{A}_\gamma$, $f_{\gamma(x)}$ has an invariant and ergodic SRB measure μ_x with a positive Lyapunov exponent which tends weakly to μ_0. Every μ_x is equivalent to ν_x, $\mathrm{HDim}(\mu_x) = \mathrm{HDim}(\mathcal{J}_{\gamma(x)})$, and μ_x shows an exponential decay of correlation.*
3. $\lim_{x\in\mathcal{A}_\gamma\to 0} \mathrm{HDim}(\mathcal{J}_{\gamma(x)}) = \mathrm{HDim}(\mathcal{J}_{c_0})$.
4. $\lim_{x\in\mathcal{A}_\gamma\to 0} \mathrm{Lyap}(\gamma(x)) = \mathrm{Lyap}(c_0) = \log 2$.

The Mañé Lemma

Proposition

Let f_c, $c \in \partial\mathcal{M}$, be a Collet-Eckmann quadratic polynomial. There exist $C > 0$, $\lambda > 1$, and $\varepsilon > 0$ such that for every $\delta > 0$ and every z from ϵ-neigborhood of $\in \mathcal{J}_c$,

$$|(f_c^n)'(z)| \geq C\lambda^n$$

provided $z, f(z), \ldots, f^{n-1}(z)$ stay outside $B(0,\delta)$ and $f^n(z) \in B(0,\delta)$.
If we assume only that $z, f(z), \ldots, f^{n-1}(z)$ are outside $B(0,\delta)$ then

$$|(f_c^n)'(z)| \geq C\lambda^n\delta$$

The starting point is Proposition 7.2 and Remark 7.1 of (Graczyk-Smirnov). The claimes of that paper can be stated as follows: there exist $r > 0$, $C_1 > 0$ and $\lambda_1 > 1$ such that for every disk $D \ni 0$ of radius smaller than r, every n, and every branch of f_c^{-n} we have that

$$\mathrm{diam} f_c^{-n}(D) < C\lambda_1^{-n}\mathrm{diam}(D) .$$

Assume that $\delta < r/2$. Observe that the preimages of $D = B(f_c^n(z), \delta)$ by all branches f_c^{-k}, $1 \leq k \leq n$, do not contain 0. Indeed, suppose that $0 \in f_c^{-k}(D)$. By the well-known normality argument, k must be at least a constant multiple of $-\log \delta$.

Therefore,

$$\mathrm{diam} f_c^{-k}(D) < C\lambda_1^{C'\log\delta}\mathrm{diam} D \leq 2C\delta^{1+C'\log\lambda_1} \leq \frac{\delta}{10}$$

if only δ is small enough. This implies that $f^{n-k}(z) \in B(0,\delta)$, a contradiction. Hence, there exists a univalent branch of f^{-n} mapping D on some neighoborhood of z of the diameter smaller than $C\lambda_1^{-n}\delta$. The proposition follows by Schwarz lemma.

To prove the second claim of the Proposition under the weaker hypothesis that that $z, f(z), \ldots, f^{n-1}(z)$ are outside $B(0,\delta)$, choose the last moment $s \leq n$ such that $f_c^s(z) \in B(0,r)$. If s does not exist it means that the whole orbit $z, f(z), \ldots, f^{n-1}(z)$ stays outside $D(0,r)$. From a the normality argument of Graczyk-Smirnov, there exist $C_2 > 0$ and $\lambda_2 > 1$ which depend only on f such that

$$|(f_c^n)'(z)| \geq C_2 \lambda_2^n \; .$$

By Lemma 3.2 of Graczyk-Smirnov, there exist $C_3 > 0$ and $\lambda_3 > 1$ depending only on f such that

$$|(f_c^s)'(z)| \geq C_3 \lambda_3^s \, .$$

Observe that $|f_c'(f_c^s(z))| \geq 2\delta$ and the orbit $f_c^{s+1}(z), \ldots, f_c^n(z)$ stays away from $B(0, \delta)$.
Therefore,

$$|(f_c^n)'(z)| \geq C_3 \lambda_3^s \, \delta \, C_2 \lambda_2^{n-s-1} \geq C \lambda^n \delta \, .$$