Research statement and research plan

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My main interests and areas of research are *symplectic* and *contact geometry* and *topology*, and their deeper underlying topological structure. In joint work I have developed *topological Hamiltonian* and *contact dynamics* (with Yong-Geun Oh and Peter Spaeth, respectively), which are *natural* and *genuine extensions* of smooth Hamiltonian and contact dynamics to topological dynamics, and of symplectic and contact diffeomorphisms to homeomorphisms as transformations preserving the additional geometric structure. These two theories have numerous applications to their smooth counterparts, as well as to other areas of mathematics, such as *topological dynamics*, in particular in *low dimensions*, and to *geodesic flows* on Riemannian manifolds. I also have an interest in *action selectors* in *Hamiltonian Floer theory* and related subjects.

Topological Hamiltonian and contact dynamics

Several well-known topological phenomena in symplectic and contact topology, as well as more recent developments, have revealed the need for a generalization of Hamiltonian and contact dynamical systems to topological dynamical systems with non-smooth Hamiltonian functions and transformations. By definition, such a dynamical system is the *limit* of a sequence of smooth Hamiltonian or contact dynamical systems with respect to a metric that combines the *topological* C^0-*metric* with the *dynamical Hofer metric*. Similarly, topological automorphisms of a *symplectic* or *contact structure* or *form* are C^0-limits of symplectic or contact diffeomorphisms (together with their conformal factors in the contact case).

The objects of these theories are non-smooth in general. Nevertheless, ample evidence asserts they comprise the correct topological analogs of Hamiltonian and contact dynamics. For instance, a topological Hamiltonian or contact isotopy is *not* generated by a vector field, and may *not* even be Lipschitz *continuous*; nonetheless, it is *uniquely determined* by its associated topological Hamiltonian function, and in turn, in the contact case it *determines uniquely* its topological conformal factor. *Composition* and *inversion* can therefore be defined as in the smooth case, and the usual *transformation law* continues to hold. Conversely, every topological Hamiltonian or contact isotopy *possesses* a *unique* topological Hamiltonian function.

Moreover, the *topological automorphism groups* of a contact structure and a contact form exhibit surprising rigidity properties analogous to the well-known Eliashberg-Gromov rigidity in the case of a symplectic structure. In a similar vain, suppose a sequence of Hamiltonian or contact isotopies (and their conformal factors) are C^0-*Cauchy*, and the generating Hamiltonian functions *converge* (with respect to the Hofer or contact metric) to another Hamiltonian function that is at least continuously differentiable with uniquely integrable Hamiltonian or contact vector field. Then the *limit* of the Hamiltonian or contact isotopies coincides with the Hamiltonian or contact isotopy *generated* by the limit Hamiltonian function (and likewise for the conformal factors). A *converse* rigidity result also holds for Cauchy sequences of Hamiltonian functions.

In addition, topological Hamiltonian and contact dynamics have *applications* to *topological dynamics*, by extending a priori smooth invariants to topological Hamiltonian and contact dynamical systems or their time-one maps (notably in dimensions two and three), and via construction of special homeomorphisms that are not C^1 but otherwise behave like Hamiltonian and contact diffeomorphisms. For example, on every symplectic or contact manifold there exist pairs of smooth Hamiltonian or contact vector fields that are topologically conjugate but are not conjugate by symplectic or contact C^1-diffeomorphisms. Explicitly, topological Hamiltonian and contact dynamics can be used in the study of *area preserving homeomorphism groups* of surfaces, and one can answer important questions of Arnold in dimension three, regarding *continuity* and behavior under *conjugation* by volume preserving *homeomorphisms* of the *helicity* of a *volume preserving isotopy*, provided the isotopy and the transformation can be described as *lifts* from a surface. Applications are also expected to billiard dynamics.

The relation between topological Hamiltonian and contact dynamics is the same as in the smooth case. In particular, topological Hamiltonian dynamics of an integral symplectic manifold is intimately related to *topological strictly contact dynamics* (introduced by Banyaga and Spaeth) of the total space of the associated Boothby-Wang *prequantization bundle*. In turn, topological contact dynamics of a contact manifold corresponds to (admissible) topological Hamiltonian dynamics of its *symplectizations*. More generally, topological Hamiltonian dynamics can be defined on other types of non-compact symplectic manifolds that appear for example in the context of symplectic field theories (e.g. symplectic manifolds with *cylindrical ends*).

In another direction, one can replace the Hofer metric in the constructions outlined above by other suitable metrics, for example Viterbo-type distances, or study (abstract) completions with respect to the Hofer or Viterbo metric alone.

Research plan

I am most interested in exploring other *applications* to symplectic and contact geometry and topology, Hamiltonian and contact dynamics, and other areas of mathematics. To this end, I am interested in expanding results on non-compactly supported Hamiltonians to for instance *quadratic forms* on *cotangent bundles*, such as *magnetic Hamiltonians* in Riemannian and sub-Riemannian geometry. In a related direction, I intend to adapt the present theory to *generating functions*, which play an important role in symplectic topology.

Another important goal is to develop *generalizations* of other key objects of symplectic and contact geometry and topology, such as topological versions of symplectic and contact structures. Several particularly promising directions to be explored further can be described as follows.

By Darboux's theorem, a symplectic structure is equivalent to an *atlas* whose transition charts are symplectic diffeomorphisms of subsets of standard Euclidean space. By Banyaga's contributions to Klein's Erlanger program, its automorphism group determines a symplectic structure up to rescaling by a non-zero constant. More generally, one can consider manifolds that admit an atlas whose transition charts are symplectic *homeomorphisms* of subsets of Euclidean space, and using the transformation law, define topological Hamiltonian dynamics. Symplectic manifolds in the usual sense clearly define such objects. A crucial question is if this gives rise to a genuine extension of the notion of symplectic manifold, or more specifically, if it is possible to define such an atlas on an *even-dimensional sphere*. On the other hand, there are *topological obstructions* that prohibit the existence of actual symplectic structures on spheres in dimension at least four.

A symplectic structure is also determined by the one-to-one correspondence between Hamiltonian isotopies and smooth functions on the manifold, together with the induced Poisson bracket on the latter. Therefore one can use the ideas of topological Hamiltonian dynamics to define generalized symplectic structures that again extend the usual definition.

In another direction, it is feasible to define a topological symplectic structure as a limit of smooth symplectic structures in a sense still to be made precise. Such an approach will most likely lead into the realm of deRahm currents. In fact, it is worthwhile to further investigate the ideas in Sullivan's Inventiones paper involving *compact convex cones of currents* with compact convex sub-cones of cycles (closed currents). Notably, there exists a theory of approximation by closed differential forms. Such cycles have already proved useful in the study of *foliations*, and are related to other geometric structures such as complex, symplectic, and contact structures.

I am looking forward to completing a pending joint project with Peter Spaeth concerning the *continuity* of action selectors in Hamiltonian Floer theory with respect to *deformations* of the *symplectic structure*. We have already solved the problem of *exact deformations* of the symplectic structure. Following arguments of Lê-Ono, it is possible to reduce the general problem to situations in which the set of *generators* of the Floer complex is *fixed* throughout the deformation of the symplectic structure. Moreover, the relevant *energy estimates* for J-*holomorphic cylinders* with appropriate prescribed asymptotic conditions that allow to define *chain homomorphisms* between different Floer complexes have been established.

Appendix: background and motivation

Symplectic geometry has its origins in the study of *classical mechanical systems* such as the planetary system or the harmonic oscillator. The geodesic flow on a Riemannian manifold provides another example. The discovery of symplectic structures allows the study of such *Hamiltonian dynamical systems* in a coordinate-free way, and thus on *manifolds* (of *even dimensions*) other than classical phase space. By Darboux's theorem, symplectic structures have no *local invariants*, and the study of the *global features* of symplectic geometry is known as *symplectic topology*.

Contact geometry is an *odd-dimensional* analog to symplectic geometry, and also has its origins in classical mechanics, such as time-dependent mechanics and Huygens' work on geometric optics. Again Darboux's theorem dictates that contact structures and forms have *no local invariants*. Modern *contact topology* is concerned with *global* and *topological features* of contact geometry.

The two theories are *linked* in many ways, either via prequantization bundles over integral symplectic manifolds, with total space a *regular* contact manifold (such circle bundles are of great interest in mechanics), or by *contactization* of an exact symplectic manifold, and conversely, and perhaps more prominently, via symplectization or *symplectic filling* of a contact manifold.

A *symplectic structure* is a *closed* and *non-degenerate two-form*, and therefore provides a means of identifying smooth vector fields with differential one-forms. As a consequence, every smooth (time-dependent) function on the underlying manifold gives rises to a *unique smooth* isotopy of *diffeomorphisms* that *preserve* the symplectic structure, and conversely, the *Hamiltonian function generating* such a *Hamiltonian isotopy* is unique (up to normalization). Thus invariants of Hamiltonian isotopies (and sometimes their time-one maps) can be defined via (invariants of) their generating Hamiltonian functions. The automorphisms of a symplectic structure are *symplectic diffeomorphisms* (diffeomorphism preserving the symplectic structure). A *Hamiltonian dynamical system transforms* under such a symplectic change of coordinates in the expected manner.

A *contact structure* is a coorientable nowhere integrable field of hyperplanes in the tangent bundle of the contact manifold; it can be described globally as the kernel of a *contact form*, that is, a one-form α such that the top-dimensional differential form $\alpha \wedge (d\alpha)^n$ never vanishes. Again a smooth function defines (together with a contact form) a *unique smooth* isotopy of *diffeomorphisms* that *preserve* the contact structure (not necessarily the contact form), and vice versa. Invariants of contact isotopies again coincide with invariants of their generating Hamiltonian functions, and a *transformation law* for *contact dynamical systems* and *contact diffeomorphisms* (diffeomorphisms preserving the contact structure, or conformally rescaling the contact form) holds. The contact structure is also *determined completely* by its automorphism group.

It is natural from a physics point of view to consider dynamical systems (and transformations) with *low order of regularity*. But *a priori*, both symplectic and contact geometry are *smooth* theories. For example, when the regularity of a Hamiltonian function is less than differentiable with locally Lipschitz derivative, methods of *ordinary differential equations* are no longer available. In this situation topological Hamiltonian and contact dynamics come into play.

Striking phenomena in symplectic topology and Hamiltonian dynamics that are *topological in nature* include the *non-degeneracy* of *Hofer's metric*, and the (existence and) behavior of *symplectic capacities*. The important *action selectors* (or *spectral invariants*) in *Hamiltonian Floer theory* depend *continuously* on the generating Hamiltonian. Yet another fascinating example is that the property of being a *Hamiltonian loop* (up to homotopy in the group of all diffeomorphisms) is preserved under *small perturbations* of the symplectic structure, or the fact that the corresponding Seidel element depends only on the homotopy class (again in the group of all diffeomorphisms) of a Hamiltonian loop. Moreover, if a symplectic homeomorphism is smooth, it is a symplectic diffeomorphism in the usual sense. Due to the very close relationship between the symplectic and contact worlds, the above remarks apply almost verbatim to contact structures as well. These phenomena indicate toward the *topological features* underlying Hamiltonian and contact dynamics. The study of various C^0-phenomena in symplectic topology, such as the C^0-*robustness* of the *Poisson bracket* induced by a symplectic form, has attracted much recent attention by many mathematicians.