Annotated publications and preprints list

Stefan Müller

Publications

Helicity of vector fields preserving a regular contact form and topologically conjugate smooth dynamical systems

With Peter Spaeth, to appear in *Ergod. Th. & Dynam. Sys.*, 35 pages, preprint at arXiv:1106.1968v2 [math.SG].

Suppose $\rho(X)$ is an invariant of a smooth vector field X that depends smoothly on X, and satisfies $\rho(\phi_* X) = \rho(X)$ for all diffeomorphisms ϕ. The invariant ρ can be considered as an invariant of the smooth isotopy corresponding to X, and ρ is invariant under conjugation by diffeomorphisms. A natural question is whether ρ can be *extended* to an invariant of an isotopy of *homeomorphisms*, and whether ρ is also invariant under *conjugation* by *homeomorphisms*, at least in the presence of an additional geometric structure (for example a volume, symplectic, or contact form) that is preserved by all vector fields, isotopies, and diffeomorphisms and homeomorphisms. These questions were asked by V. I. Arnold in the context of the *helicity* of a *divergence-free vector field* on the three-sphere with its standard volume form.

In this paper, we study Arnold's questions for closed three-manifolds equipped with a *regular contact form*; the isotopies and diffeomorphisms preserving this contact form are those that commute with the Reeb flow, and are precisely the *lifts* of *Hamiltonian isotopies* and *diffeomorphisms* of the quotient of the underlying contact manifold by the Reeb flow. Similarly, we consider *suspensions of surface isotopies* previously studied by G. G. Gambaudo and É. Ghys. In contrast to the case of a volume form considered by Arnold, it is *a priori* not clear what the correct *generalization* is to continuous Hamiltonian and contact isotopies, and homeomorphisms preserving a symplectic or contact form. Another natural question is whether there actually exist smooth vector fields that are topologically conjugate but not smoothly conjugate.

In both cases, we compute *explicit formulas* for the helicity in terms of the *generating Hamiltonian function* of the contact or Hamiltonian isotopy. It was previously shown by myself and Y.-G. Oh, and by A. Banyaga and P. Spaeth, respectively, that the concepts of Hamiltonian isotopies and isotopies preserving a contact form, admit *natural* and *nontrivial generalizations* to topological dynamical systems, and to *homeomorphisms* as *transformations preserving the underlying geometric structure*. Based on these results, we answer Arnold's questions in the *affirmative* in the present situations.

In the last part of the paper, we prove the *existence* of smooth (Hamiltonian and strictly contact) dynamical systems that are *topologically conjugate* but *not smoothly conjugate*. The proofs use the transformation laws and the uniqueness theorems of topological Hamiltonian and topological (strictly) contact dynamics. We also briefly discuss *higher-dimensional helicities*.

One of the key ingredients of the proofs is that our explicit formulas for the helicity are *invariant under conjugation*, and are *continuous* with respect to an appropriate choice of metric. Thus the ideas we present apply conceptually to other invariants ρ as well.

On properly essential classical conformal diffeomorphism groups

With Peter Spaeth, to appear in *Ann. of Global Anal. and Geom.*, 11 pages, online first DOI 10.1007/s10455-011-9304-y, November 2011.

In this paper, we consider geometrical structures that are defined as conformal classes σ of a tensor field τ on a smooth manifold. For example, the conformal structure $\sigma = [\tau]$ is an orientation if τ is a volume form, a conformal symplectic structure if τ is a symplectic form, and a contact structure if τ is a contact form; these three structures are the *classical conformal structures*.

The *conformal diffeomorphism group* of a conformal structure σ is by definition *properly essential* if there exists a conformal diffeomorphism of σ that does *not preserve any* of the tensor fields in the class σ. We prove this to be the case for *contact manifolds*, for *symplectic manifolds that are Liouville*, and for *oriented manifolds*. Our arguments are *local-to-global*, and rely on an obstruction in the form of a *cohomological equation*.

Moreover, we study the *orbit* of a given tensor field under the action of the conformal diffeomorphism group; the orbit of τ in the class σ can be identified with the quotient of the groups of diffeomorphisms that preserve σ by those that preserve τ. Among other results, we show that the orbit of a contact form α on a closed manifold is *not maximal*. That means there exists another contact form, defining the same contact structure, but not diffeomorphic to α. The method of proof is to show the Reeb flows of the two contact forms are not conjugated. Along the way, we demonstrate that every contact structure (not necessarily a given contact form) on a closed manifold admits a *closed Reeb orbit*.

We also relate our work to a *conformal invariant* defined by A. Banyaga and a theorem of W. H. Gottschalk and G. A. Hedlund, and define a *new contact invariant* (the *conformal length*) of a contact diffeomorphism.

The group of Hamiltonian homeomorphisms in the L^∞-norm

J. Korean Math. Soc. 45 (2008), no. 6, 1769-1784.

The notion of the *Hamiltonian metric* plays a central role in *topological Hamiltonian dynamics*. In this article, I reprove the results of my JSG paper with Y.-G. Oh in the case of the L^∞-*Hofer norm* in place of the $L^{(1,\infty)}$-*Hofer norm*, and compare the two choices of norm giving rise to different definitions of the Hamiltonian metric. See the description of the JSG paper below for details.

In the second part of the paper, I demonstrate the following result: every *topological Hamiltonian dynamical system* is *arbitrarily close* (with respect to the

$L^{(1,\infty)}$-Hamiltonian metric) to a *continuous Hamiltonian dynamical system* (i.e., one that is defined with respect to the stronger L^∞-Hamiltonian metric) with the same end point; moreover, the latter is smooth everywhere except possibly at time one. In particular, the two groups of *Hamiltonian homeomorphisms* (by definition the *time-one maps* of *topological* and *continuous Hamiltonian isotopies*) arising from the different choices of Hofer norm *coincide*.

The proof involves the observation by L. Polterovich that a generic smooth Hamiltonian isotopy is *regular* (in the sense that its tangent vector never vanishes), and an adaptation of a reparameterization procedure also due to Polterovich. A combination with further *approximation* and *reparameterization techniques* then completes the proof. Finally it is shown that the two a priori different *Hofer norms* on the group of Hamiltonian *homeomorphisms* are equal.

The group of Hamiltonian homeomorphisms and C^0-symplectic topology

With Yong-Geun Oh, *J. Symplectic Geom.* 5 (2007), no. 2, 167-219.

A *topological Hamiltonian dynamical system* is a tuple that consists of a *topological Hamiltonian isotopy* (a *continuous* isotopy of *homeomorphisms*), together with a possibly *non-smooth topological Hamiltonian function* on a symplectic manifold. By definition, this Hamiltonian function is the *limit* of a sequence of (normalized time-dependent) smooth Hamiltonian functions with respect to the usual *Hofer metric*; the corresponding sequence of smooth Hamiltonian isotopies *converges uniformly* to the above continuous isotopy. The key notion is the *Hamiltonian metric*, which is precisely the above combination of the *topological* C^0*-metric* with the *dynamical Hofer metric*.

A topological Hamiltonian isotopy is *determined uniquely* by its corresponding topological Hamiltonian function. In other words, given two topological Hamiltonian dynamical systems with the same topological Hamiltonian function, their topological Hamiltonian isotopies must coincide as well. This follows from the important *energy-capacity inequality* of F. Lalonde and D. McDuff, and can be formulated in three equivalent ways (cf. my joint papers on topological contact dynamics with P. Spaeth). The proof is thus already contained in the present paper (although somewhat in disguise); an earlier version for standard Euclidean space can be found in the monograph by H. Hofer and E. Zehnder.

As a consequence of the above uniqueness theorem, *composition* and *inversion* of topological Hamiltonian dynamical systems, topological Hamiltonian functions, and topological Hamiltonian isotopies and their time-one maps, can be defined as in the smooth case. Thus topological Hamiltonian dynamics is a *natural extension* of the smooth dynamics of a Hamiltonian vector field (or function) to topological dynamics. We show by example that a topological Hamiltonian isotopy need *not* even be Lipschitz *continuous* (in the space or time variable). In other words, the extension to topological dynamics is a *genuine extension* (on *any* symplectic manifold).

A *Hamiltonian homeomorphism* is by definition the *time-one map* of a topological Hamiltonian isotopy. Much attention in this article is focused on this group of Hamiltonian homeomorphisms and its topological properties. By continuity of the mass flow homomorphism (the topological dual to the volume flux homomorphism, which in turn is related to the symplectic flux homomorphism by multiplication in deRham cohomology), the *mass flow* of a topological Hamiltonian isotopy *vanishes*. This fact is related to the question of *(non-)simpleness* of the (kernel of the mass flow in the) *group of area-preserving homeomorphisms* of a surface, and thus yields an instance where symplectic methods enter topological dynamics.

A *symplectic homeomorphism* is by definition the C^0-*limit* of a sequence of symplectic diffeomorphisms. In other words, the group of symplectic homeomorphisms of a symplectic structure is the uniform closure of the group of symplectic diffeomorphisms in the group of homeomorphisms. This definition is motivated by Eliashberg-Gromov's celebrated C^0-*symplectic rigidity* theorem; a symplectic homeomorphism that is in addition smooth preserves the symplectic structure, and is thus a symplectic diffeomorphism in the usual sense. Moreover, the well known *transformation law* extends to topological Hamiltonian dynamical systems and symplectic homeomorphisms. Thus symplectic homeomorphisms can be considered as the *topological automorphisms* of the symplectic structure, and the group of Hamiltonian homeomorphisms forms a normal subgroup of the group of symplectic homeomorphisms. A symplectic homeomorphism preserves the measure obtained by integrating the canonical volume form induced by the symplectic form. As in the smooth case, the inclusion of symplectic homeomorphisms as a closed subgroup of the group of measure-preserving homeomorphisms is proper if and only if the dimension of the underlying manifold is greater than two. The properness can be deduced from Gromov's seminal non-squeezing theorem.

We repeatedly demonstrate that our choice of Hamiltonian metric gives the objects of topological Hamiltonian dynamics the correct *dynamical*, *topological*, and *algebraic* properties. For example, the set of topological Hamiltonian dynamical systems forms a *topological group* with respect to the *Hamiltonian topology* (the topology induced by the Hamiltonian metric), and smooth Hamiltonian dynamical systems form a topological subgroup. This article marks the beginning of the study of *topological Hamiltonian dynamics* in the sense explained here. The notation and language used to describe it have (in my opinion) improved significantly over the past few years. The present vocabulary is taken from my joint papers with P. Spaeth on topological contact dynamics.

Ph.d. thesis

The group of Hamiltonian homeomorphisms and C^0-symplectic topology

Ph.D. thesis, the University of Wisconsin - Madison, 2008, 104+v pages.

My thesis contains an improved presentation and elaborate development of *topological Hamiltonian dynamics*. Among other things, the results of my joint JSG paper with Y.-G. Oh and my JKMS paper are explained in greater detail. In particular, the case of open manifolds is developed rigorously, and the various choices in the definitions are discussed and justified at great length.

ArXiv preprints

Topological contact dynamics II: topological automorphisms, contact homeomorphisms, and non-smooth contact dynamical systems

With Peter Spaeth, 39 pages, arXiv:1203.4655v1 [math.SG].

This *sequel* to our previous paper continues the study of *topological contact dynamics* and applications to contact dynamics and topological dynamics. We provide further evidence that the *topological automorphism groups* of a *contact structure* and a *contact form* are the appropriate transformation groups of contact dynamical systems, and study the *topological properties* of the groups of *contact* and *strictly contact homeomorphisms*. On the latter we construct a *bi-invariant metric* that resembles the bi-invariant metric on the group of strictly contact *diffeomorphisms* studied in part I. Among other things, we show that a generic smooth contact isotopy is *regular* (in the sense that its tangent vector is never stationary). The proof is similar to the one given by L. Polterovich for smooth Hamiltonian isotopies. Generalizing the *approximation* and *reparameterization techniques* from my JKMS paper, we prove the following main lemma of part II: every *topological contact dynamical system* is *arbitrarily close* (with respect to the $L^{(1,\infty)}$-contact metric) to a *continuous contact dynamical system* (i.e., one defined with respect to the stronger L^∞-contact metric) with the same end point; moreover, the latter is smooth everywhere except possibly at time one. In particular, the two groups of *contact homeomorphisms* (by definition, the *time-one maps* of *topological* and *continuous contact isotopies*) arising from the different choices of a Hofer-like norm (on the space of Hamiltonian functions) *coincide*. The same holds for the two groups of *strictly contact homeomorphisms* (by definition, the *time-one maps* of *topological* and *continuous strictly contact isotopies*). Finally on every contact manifold we construct topological contact dynamical systems with time-one maps that *fail* to be *Lipschitz continuous*, and smooth contact vector fields whose flows are *topologically conjugate* but *not conjugate* by a contact C^1-*diffeomorphism*.

Topological contact dynamics I: symplectization and applications of the energy-capacity inequality

With Peter Spaeth, 44 pages, arXiv:1110.6705v2 [math.SG].

Some of the most prominent early results in symplectic topology include a profound *energy-capacity inequality* in Hamiltonian dynamics, the *non-degeneracy* of the remarkable *Hofer metric* on the group of Hamiltonian diffeomorphisms, and Eliashberg-Gromov's fundamental C^0-*rigidity* of symplectic diffeomorphisms. This article is the first part of a series of papers on *topological contact dynamics*. We derive an *energy-capacity inequality* for *contact diffeomorphisms*, which proves to be an equally powerful tool in contact dynamics. As an immediate consequence, we establish the *non-degeneracy* of a bi-invariant *Hofer-like metric* on the *group of diffeomorphisms preserving a contact form*.

A *topological contact dynamical system* consists of a *topological contact isotopy* (a *continuous* isotopy of *homeomorphisms*), together with a possibly *non-smooth topological Hamiltonian function* on the underlying contact manifold, and a *topological conformal factor* (a *continuous* function). By definition, this Hamiltonian function is the *limit* of a sequence of smooth (time-dependent) Hamiltonian functions with respect to a *Hofer-like metric*. Moreover, the corresponding sequence of smooth contact isotopies *converges uniformly* to the above continuous isotopy, and their smooth conformal factors *converge uniformly* to the continuous function associated to the limit isotopy. This definition is explained and justified in detail in this paper.

A topological contact isotopy is *not* generated by a vector field, and may *not* even be Lipschitz *continuous*; nonetheless, as a *consequence* of the *contact energy-capacity inequality*, it is *uniquely determined* by its associated topological Hamiltonian function. *Composition* and *inversion* of topological contact dynamical systems, topological Hamiltonian functions, and topological contact isotopies and their time-one maps, can therefore be defined as in the smooth case, and the usual *transformation law* continues to hold.

We show that the *topological automorphism groups* of a contact structure and a contact form exhibit surprising rigidity properties, including *uniqueness* of the *topological conformal factor*, and C^0-*rigidity* of *contact* and *strictly contact diffeomorphisms*, analogous to the above Eliashberg-Gromov rigidity in the case of a symplectic structure.

The upshot is a *natural* and *genuine extension* of the smooth dynamics of a contact vector field to topological dynamics. Consequences and *applications* to both contact dynamics and *topological dynamics*, such as the fact that a *topological automorphism* of a contact structure *conjugates* the corresponding *Reeb flows*, are discussed throughout the paper. The *uniqueness theorems* applied to *smooth* contact dynamical systems prove *rigidity* of contact isotopies and their conformal factors in the following sense: if a sequence of contact isotopies and their conformal factors are *uniformly Cauchy*, and the generating Hamiltonian functions *converge* (with respect to the Hofer-like metric) to another (continuously differentiable) Hamiltonian function (with uniquely integrable contact vector field), then the *limit* of the contact isotopies coincides with the contact isotopy *generated* by the limit Hamiltonian function, and likewise for the conformal factors.

Our general approach to relating topological contact dynamics to topological Hamiltonian dynamics is via *symplectization* of a contact manifold.

A note on the volume flux of smooth and continuous strictly contact isotopies

8 pages, arXiv:1107.4869v1 [math.SG].

This short note on the *flux homomorphism* for *strictly contact isotopies* complements my paper on the helicity written jointly with P. Spaeth. I compute the volume flux homomorphism restricted to symplectic and volume preserving contact isotopies and their C^0-*limits* for certain classes of symplectic and contact manifolds. A copious number of examples is given.

The above restrictions of the flux homomorphism may *fail* to be *surjective*. The flux homomorphism *vanishes* for an isotopy preserving a *regular* contact form, but can be *non-trivial* for *non-regular* contact forms. Applications are discussed in the article mentioned above. I also find an *obstruction* to *regularizing* a strictly contact isotopy that is *not* present for Hamiltonian isotopies or contact isotopies. This is related to the fact that (for non-regular contact forms) a *locally defined* strictly contact vector field may *not* extend to a *globally defined* strictly contact vector field. Conversely, if the flux of a given isotopy is *non-trivial*, its generating strictly contact vector field *cannot* be *fragmented* into a sum of strictly contact vector fields that are supported in Darboux charts.

Approximation of volume preserving homeomorphisms by volume preserving diffeomorphisms

8 pages, arXiv:0901.1002v3 [math.DS].

In the late '50s and early '60s, J. R. Munkres and M. W. Hirsch independently developed *obstruction theories* for when a given homeomorphism of a smooth manifold can be *approximated uniformly* by diffeomorphisms. In the past decade, Y.-G. Oh and J. C. Sikorav independently showed that if a *volume preserving homeomorphism* can be *approximated uniformly* by *diffeomorphisms*, it can also be *approximated uniformly* by *volume preserving diffeomorphisms*.

This note is an attempt to give a *necessary* and *sufficient condition* for when a *volume preserving homeomorphism* can be *approximated uniformly* by *(volume preserving) diffeomorphisms*. An update should appear shortly.

In preparation

Topological contact dynamics III

With Peter Spaeth.

The principal result of this third part of the series of papers on *topological contact dynamics* is the *converse* to the main uniqueness theorem(s) of part I: every topological contact isotopy *possesses* a *unique topological Hamiltonian function*. *Applications* are discussed as well.