THE GROUP OF HAMILTONIAN HOMEOMORPHISMS AND C^0-SYMPLECTIC TOPOLOGY

By

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Abstract

The purpose of this dissertation is to carry out a foundational study of C^0-Hamiltonian geometry and C^0-symplectic topology. Chapter 1 begins with a brief review of some important results in symplectic topology that motivate the study of C^0-symplectic topology, accompanied by some more recent related developments. We then outline our general approach to a C^0-complement to Hamiltonian geometry and symplectic topology from the point of view adopted in this work. We introduce the basic concepts and recapitulate the definitions of symplectic and Hamiltonian diffeomorphisms, Hamiltonian paths, and the Hofer and C^0-topologies, and some of their important properties. We discuss in detail two different norms, the $L^{(1,\infty)}$ and L^∞-norms, on the space of Hamiltonian functions and the space of Hamiltonian paths. We take a close look at the 'closeness' of reparameterizations of Hamiltonian paths, and derive some estimates for the aforementioned norms, which will be of fundamental importance in later chapters.

In Chapter 2 we define the group $\mathrm{Sympeo}(M, \omega)$ of symplectic homeomorphisms, and study its relation to measure-preserving homeomorphisms. We also compare this definition to other notions of symplectic homeomorphisms, which have previously appeared in the literature. We then define the Hamiltonian topology on the space of Hamiltonian paths, and on the group of Hamiltonian diffeomorphisms, and compare it to other well-known topologies. Subsequently, we define the completion of the space of Hamiltonian paths with respect to the Hamiltonian metric, resulting in the definitions of the space of topological Hamiltonian paths, the space of topological Hamiltonian functions, and the group of Hamiltonian homeomorphisms, denoted by $\mathrm{Hameo}(M, \omega)$. From our earlier estimates in Chapter 1, we derive some fundamental estimates concerning the two norms above when considering the completion of the space of Hamiltonian

paths, resulting in the proof of the fact that this completion forms a topological group. This in turn has immediate implications for the time-one evaluation map and the topological properties of the spaces of Hamiltonian diffeomorphisms and homeomorphisms in the Hamiltonian topology. We prove that $\mathrm{Hameo}(M, \omega)$ is a normal subgroup of $\mathrm{Sympeo}(M, \omega)$, and contains all the time-one maps of Hamiltonian vector fields of $C^{1,1}$-functions, and that $\mathrm{Hameo}(M, \omega)$ is path connected and so contained in the identity component $\mathrm{Sympeo}_0(M, \omega)$ of $\mathrm{Sympeo}(M, \omega)$. We show that the spaces of topological Hamiltonian paths and functions contain non smooth elements, and that $\mathrm{Ham}(M, \omega) \subsetneq \mathrm{Hameo}(M, \omega) \subset \mathrm{Sympeo}_0(M, \omega)$. We consider many cases in which $\mathrm{Hameo}(M, \omega)$ is a proper subgroup of $\mathrm{Sympeo}_0(M, \omega)$. For this purpose, we review the mass flow homomorphism for measure-preserving homeomorphisms and the flux homomorphisms for symplectic and volume-preserving diffeomorphisms. We 'extend' Hamilton's equation to a larger class of Hamiltonians and corresponding flows. As a consequence of an argument due to Polterovich, we show that the two groups of Hamiltonian homeomorphisms arising from the two different norms coincide. The chapter ends with a brief discussion of Hofer norms for Hamiltonian homeomorphisms.

In Chapter 3 we discuss the case of noncompact manifolds and manifolds with nonempty boundary. Finally, we consider and compare some variations of the notion of Hamiltonian topology adopted in this work.

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Contents

Chapter 1

Introduction

C^0-symplectic topology is the topological complement to symplectic topology. Latter is a priori a smooth theory. For example, a symplectic diffeomorphism must be at least C^1 for its very definition to make sense, and a Hamiltonian function on a symplectic manifold should be at least $C^{1,1}$, so that the existence, uniqueness, and regularity results from the theory of ordinary differential equations can be applied to its associated Hamiltonian vector field.

On the other hand, there are many important results in symplectic topology that are C^0 in nature, such as the existence of symplectic capacities and its consequences (Hofer et al), for instance the Hofer-Zehnder capacity (see [HZ94]), which is defined in terms of the oscillation of a special class of Hamiltonians, the symplectic rigidity theorem [Eli87, Gro86], and the nondegeneracy of Hofer's metric [Hof90, Vit92, Pol93, LM95a]. Another striking example is the topological rigidity of Hamiltonian loops [LMP99]. Such phenomena are the starting point of C^0-symplectic topology. Other related works include [Hof93, Bat94], and more recently [Vit06b, Vit06a, CV07, EPZ07, EP07, Hum07, Rou07, Ban08a, Ban08b].

Our general approach to a C^0-counterpart to symplectic topology is to consider completions with respect to a suitable metric on the space of Hamiltonian paths, which we call the Hamiltonian metric. This leads to the definition of the group $\mathrm{Hameo}(M,\omega)$ of Hamiltonian homeomorphisms, such that

$$\mathrm{Ham}(M,\omega) \subsetneq \mathrm{Hameo}(M,\omega) \subset \mathrm{Sympeo}(M,\omega),$$

where $\mathrm{Sympeo}(M,\omega)$ is the group of symplectic homeomorphisms. In fact, the notion of

Hamiltonian topology has been vaguely present in the literature, without much emphasis on its significance, see e.g. [Vit92, Hof93, HZ94, Oh02] for some theorems related to this topology. However, all of the previous works fell short of constructing a 'group' of Hamiltonian homeomorphisms. A precise formulation of the topology will be essential in our study of C^0-symplectic analogs corresponding to various C^∞-objects or invariants. We provide many evidences for our thesis that the Hamiltonian topology is the 'correct' topology for the study of topological Hamiltonian geometry. We prove C^0-analogs to some well-known facts concerning the groups of symplectic and Hamiltonian diffeomorphisms. In particular in dimension two, a symplectic form is just an area form, and often the 'correct' generalization of area-preserving diffeomorphisms in higher dimensions is to symplectic or Hamiltonian diffeomorphisms. We will show that symplectic and Hamiltonian homeomorphisms generalize measure-preserving homeomorphisms in a nontrivial way. We discuss some of the fundamental questions concerning C^0-symplectic topology, and in particular the structure of the group $\mathrm{Hameo}(M, \omega)$ of Hamiltonian homeomorphisms.

In Chapter 1 we introduce the basic concepts and review the definitions of symplectic and Hamiltonian diffeomorphisms, Hamiltonian paths, and the Hofer and C^0-topologies, together with some of their important properties. We discuss in detail two different norms, the $L^{(1,\infty)}$ and L^∞-norms, on the space of Hamiltonian functions and the space of Hamiltonian paths. We take a close look at the 'closeness' of reparameterizations of Hamiltonian paths, and derive some estimates for the aforementioned norms, which will be of fundamental importance in later chapters.

In Chapter 2 we define the group of symplectic homeomorphisms, and study its relation to measure-preserving homeomorphisms. We also compare this definition to other notions of symplectic homeomorphisms, which have previously appeared in the literature. We then define the Hamiltonian topology on the space of Hamiltonian paths, and on the group of Hamiltonian diffeomorphisms, and compare it to other well-known topologies. Subsequently, we define the completion of the space of Hamiltonian paths with respect to the Hamiltonian metric, resulting in the definitions of the space of topological Hamiltonian paths, the space of topological Hamiltonian functions, and the group of Hamiltonian homeomorphisms. From our earlier estimates in Chapter 1, we derive some fundamental estimates concerning the two norms above when considering the completion of the space of Hamiltonian paths, resulting in the proof of the fact that this completion forms a topological group. This in turn has immediate implications for the time-one evaluation map and the topological properties of the spaces of Hamiltonian diffeomorphisms and homeomorphisms in the Hamiltonian topology. We prove that $\mathrm{Hameo}(M, \omega)$ is a normal subgroup of $\mathrm{Sympeo}(M, \omega)$, and that $\mathrm{Hameo}(M, \omega)$ is path connected and so contained in the identity component $\mathrm{Sympeo}_0(M, \omega)$ of $\mathrm{Sympeo}(M, \omega)$. In addition, we will see that the set $\mathrm{Hameo}(M, \omega)$ equipped with the Hamiltonian topology is locally path connected. We show that the spaces of topological Hamiltonian paths and functions contain non smooth elements, and that $\mathrm{Ham}(M, \omega) \subsetneq \mathrm{Hameo}(M, \omega) \subset \mathrm{Sympeo}_0(M, \omega)$. We consider many cases in which $\mathrm{Hameo}(M, \omega)$ is a proper subgroup of $\mathrm{Sympeo}_0(M, \omega)$. For this purpose, we review the mass flow homomorphism (or mean rotation vector) for measure-preserving homeomorphisms and the flux homomorphisms for symplectic and volume-preserving diffeomorphisms, and show that the mass flow with respect to the Liouville measure of ω vanishes on $\mathrm{Hameo}(M, \omega)$. If $M \neq S^2$ is a closed orientable surface, this implies $\mathrm{Hameo}(M, \omega) \subsetneq \mathrm{Sympeo}_0(M, \omega)$. Under some additional hypothesis on the (co-)homology groups of the manifold, this result extends to higher dimensions. In particular, this holds for all Kähler manifolds with $H^1(M) \neq 0$. We prove that $\mathrm{Hameo}(M, \omega)$ contains all the time-one maps of Hamiltonian vector fields of $C^{1,1}$-functions, and thus for all our purposes, we could equally well work with $C^{1,1}$ (rather than C^∞) Hamiltonians in the development of the theory. We 'extend' Hamilton's equation to a larger class of Hamiltonians and corresponding flows. As a consequence of a reparameterization procedure due to Polterovich [Pol01], we show that the two groups of Hamiltonian homeomorphisms arising from the two different norms coincide. The chapter ends with a brief discussion of Hofer norms for Hamiltonian homeomorphisms.

In Chapter 3 we discuss the case of noncompact manifolds and manifolds with nonempty boundary. Finally, we consider some variations of the notion of Hamiltonian topology adopted in this work, and compare them in terms of the essential features of the Hamiltonian topology used throughout this work.

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1.1 Preliminaries

In the remainder of this introduction, we assemble some notations and background material, as well as some other results that will be used in later chapters.

Let (M, ω) denote a connected symplectic manifold of dimension $2n$. That is, M is a smooth manifold, equipped with a closed nondegenerate 2-form $\omega \in \Omega^2(M)$. Nondegeneracy means that on each tangent space T_xM, the bilinear form ω_x is nondegenerate. In other words, the map $\mathcal{X}(M) \to \Omega^1(M)$, defined by $X \mapsto \iota(X)\omega$, is an isomorphism between the vector spaces of smooth vector fields and 1-forms on M, where $\iota(X)$ denotes interior multiplication by the vector field X. Equivalently, nondegeneracy of ω can be expressed as requiring that its top power $\omega^n = \omega \wedge \ldots \wedge \omega > 0$ is a volume form on M. The distinguished volume form

$$\Omega = \frac{\omega^n}{n!} \tag{1.1}$$

on M is called the *Liouville volume form* induced by ω. The normalization factor in (1.1) is chosen so that the standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

on $\mathbb{R}^{2n}$ (or the torus T^{2n}) induces the standard volume form $dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$. By Darboux's theorem, every symplectic manifold (M, ω) is locally diffeomorphic to standard $(\mathbb{R}^{2n}, \omega_0)$. We refer for example to [MS98] for more details on basic terminology and an introduction to symplectic manifolds. Unless explicit mention is made to the contrary, M will be closed, i.e. compact and without boundary. The open case will be discussed in Section 3.1.

1.1.1 Symplectic and Hamiltonian diffeomorphisms

Denote by $\mathrm{Diff}(M)$ the group of smooth diffeomorphisms of M, equipped with the C^∞-topology. In this work, smooth will always mean of class C^∞, and unless we explicitly mention otherwise, all diffeomorphisms of M and all functions on M are assumed to be smooth. Denote by $\mathrm{Symp}(M,\omega)$ the group of symplectic diffeomorphisms, i.e. the subgroup of $\mathrm{Diff}(M)$ consisting of all diffeomorphisms ψ of M that preserve the symplectic form ω, that is, $\psi^*\omega = \omega$. $\mathrm{Symp}(M,\omega)$ with the induced C^∞-topology forms a closed topological subgroup of $\mathrm{Diff}(M)$. We denote by $\mathrm{Symp}_0(M,\omega)$ the identity component in $\mathrm{Symp}(M,\omega)$, with the subspace topology. By Weinstein's theorem [Wei71], this coincides with the path component of the identity in $\mathrm{Symp}(M,\omega)$, i.e. the group of all symplectic diffeomorphisms that are smoothly isotopic to the identity through symplectic diffeomorphisms. Denote by $\mathcal{P}\mathrm{Diff}(M)$ the set of smooth paths $\lambda\colon [0,1] \to \mathrm{Diff}(M)$, with $\lambda(0) = \mathrm{id}$. Each $\lambda \in \mathcal{P}\mathrm{Diff}(M)$ defines a smooth map $\Lambda\colon [0,1]\times M \to M$, by $\Lambda(t,\cdot) = \lambda(t)$, and we give $\mathcal{P}\mathrm{Diff}(M)$ the C^∞-topology as a subspace of $C^\infty([0,1]\times M, M)$.

Denote by $C^\infty([0,1]\times M)$ the vector space of smooth functions $H\colon [0,1]\times M \to \mathbb{R}$. Often we view H as a family of functions $H_t\colon M \to \mathbb{R}$, and think of $t \in [0,1]$ as the time variable. As is customary, we will refer to such a function H as a *Hamiltonian function*, or simply a *Hamiltonian*. We define the (time-dependent) Hamiltonian vector field X_H associated to H by

$$\iota(X_H(t,x))\omega = dH_t(x), \qquad \text{for all } t \in [0,1], x \in M. \tag{1.2}$$

Each Hamiltonian $H \in C^\infty([0,1]\times M)$ generates a family of diffeomorphisms ϕ_H^t of M, $0 \le t \le 1$, with $\phi_H^0 = \mathrm{id}$, by integrating the Hamiltonian vector field X_H. In other words, $[0,1]\times M \to M$, $(t,x) \mapsto \phi_H^t(x)$, is the flow of the Hamiltonian differential equation $\dot{x}(t) = X_H(t,x(t))$, or

$$\frac{d}{dt}\phi_H^t \circ \left(\phi_H^t\right)^{-1}(x) = X_H(t,x), \tag{1.3}$$

for all $0 \le t \le 1$, and $x \in M$. We will always denote by ϕ_H the corresponding path $t \mapsto \phi_H^t \in \mathrm{Symp}(M,\omega)$, $t \in [0,1]$, and call it the *Hamiltonian path* generated by H, and by $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ the set of all Hamiltonian paths. Conversely, the function H is called the *generating Hamiltonian* of a Hamiltonian path ϕ_H. We can give $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ the C^∞-topology as a subspace of $\mathcal{P}\mathrm{Diff}(M)$. However, we will mostly use a different (smaller) topology in this work.

A diffeomorphism $\phi \in \mathrm{Diff}(M)$ is called *Hamiltonian* if $\phi = \phi_H^1$ for some Hamiltonian function H, where ϕ_H^1 denotes the time-one map of the path ϕ_H. We will write $H \mapsto \phi$ when $\phi = \phi_H^1$, and say the diffeomorphism ϕ is generated by the Hamiltonian H. We denote the set of Hamiltonian diffeomorphisms by $\mathrm{Ham}(M,\omega)$, and recall that $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}_0(M,\omega)$. There is the time-one evaluation map

$$\mathrm{ev}_1 \colon \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \longrightarrow \mathrm{Symp}_0(M), \qquad \phi_H \longmapsto \phi_H^1, \tag{1.4}$$

and by definition, $\mathrm{Ham}(M,\omega)$ is precisely the image of the map ev_1. We equip $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}(M,\omega)$ with the subspace topology, i.e. the C^∞-topology. Hamiltonian diffeomorphisms play a prominent role in many problems in the development of symplectic topology, starting implicitly from Hamiltonian mechanics, and more conspicuously from the Arnold conjecture. One of the purposes of the present work is to give a precise definition of the C^0-counterpart of $\mathrm{Ham}(M,\omega)$.

By standard existence, uniqueness, and regularity results from the theory of ordinary differential equations, each Hamiltonian defines a unique Hamiltonian path ϕ_H. Conversely, the Hamiltonian path ϕ_H determines H up to an additive constant. We call a Hamiltonian H *normalized* if

$$\int_M H_t \omega^n = 0, \qquad \text{for all } t \in [0,1], \tag{1.5}$$

where we recall that ω^n defines a volume form on M. We denote by $C_m^\infty([0,1] \times M)$ the vector space of normalized Hamiltonian functions, where m stands for 'mean zero'. Hence there is a one-to-one correspondence between Hamiltonian paths and their generating normalized Hamiltonian functions. From now on, we will always assume that all Hamiltonians are normalized.

Of course, whenever we define a Hamiltonian in one way or the other, we need to prove that it satisfies the above normalization condition (1.5). This is often trivial and will then be omitted.

We recall that for two Hamiltonian functions H and K, the 'product' Hamiltonian $H\#K$ is given by the formula

$$(H\#K)_t = H_t + K_t \circ (\phi_H^t)^{-1}, \tag{1.6}$$

and generates the path $\phi_H \circ \phi_K : t \mapsto \phi_H^t \circ \phi_K^t$. And the 'inverse' Hamiltonian $\overline{H}$, corresponding to the inverse path $(\phi_H)^{-1} : t \mapsto (\phi_H^t)^{-1}$, is defined by

$$(\overline{H})_t = -H_t \circ \phi_H^t. \tag{1.7}$$

Note that $(\phi_H)^{-1}$ denotes the path of inverse diffeomorphisms, not the time-reversed path $t \mapsto \phi_H^{1-t}$. We also recall that the pulled-back Hamiltonian $\psi^* H$,

$$(\psi^* H)_t = H_t \circ \psi, \tag{1.8}$$

generates the path $\psi^{-1} \circ \phi_H \circ \psi : t \mapsto \psi^{-1} \circ \phi_H^t \circ \psi$, for any $\psi \in \mathrm{Symp}(M, \omega)$. These formulas can easily be verified directly from the definitions (1.2) and (1.3). The product and inverse operations define a (nonstandard) group structure on $C_m^\infty([0,1] \times M)$, which will be used mostly in this work. We will be mainly interested in paths of the form $\phi_H^{-1} \circ \phi_K$. This path is generated by the Hamiltonian $\overline{H}\#K$, where

$$(\overline{H}\#K)_t = -H_t \circ \phi_H^t + K_t \circ \phi_H^t = (K_t - H_t) \circ \phi_H^t. \tag{1.9}$$

The above considerations imply that $\mathrm{Ham}(M, \omega)$ is a normal subgroup of $\mathrm{Symp}(M, \omega)$. Banyaga [Ban78] proved that this group is simple, and by the C^∞-Flux Conjecture, now a theorem [Ono06], $\mathrm{Ham}(M, \omega)$ is a closed subgroup of $\mathrm{Symp}_0(M, \omega)$, and locally contractible in the C^∞-topology.

As remarked above, there is a one-to-one correspondence between the set of Hamiltonian paths and the set of (normalized) Hamiltonians in the smooth category. In fact, this one-to-one correspondence continues to hold even if H is only $C^{1,1}$, although ϕ_H will then no longer be smooth in general, but only C^1 in the time variable and continuous in the space variable. However, this correspondence gets murkier when the regularity of the Hamiltonian is less than $C^{1,1}$. Because of this, we introduce the following terminology for our later discussions.

Definition 1.1.1. We define two maps

$$\begin{aligned} \mathrm{Dev}, \mathrm{Tan} \colon \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) &\longrightarrow C^\infty_m([0,1]\times M), \\ \mathrm{Dev}(\phi_H)(t,x) &= H(t,x), \qquad (1.10) \\ \mathrm{Tan}(\phi_H)(t,x) &= H\left(t, (\phi_H^t)(x)\right), \qquad (1.11) \end{aligned}$$

and call them the *developing map* and the *tangent map*, respectively.

In other words, the developing map assigns to a path ϕ_H its normalized generating Hamiltonian. With the usual identification of the Lie algebra of $\mathrm{Ham}(M,\omega)$ with the space $C^\infty_m(M)$ of normalized functions on M, the tangent map corresponds to assigning to a path ϕ_H the vector field of tangent vectors $H_t \circ \phi_H^t$ to ϕ_H. Assigning the generating Hamiltonian H to a Hamiltonian path corresponds to the developing map in Lie group theory: one can 'develop' any differentiable path in a Lie group to a path in its Lie algebra using the tangent map and then by right translations.

By a slight abuse of notation, we will often write

$$(H \circ \phi_H)(t,x) = H\left(t, \phi_H^t(x)\right),$$

and thus denote the tangent map by $\mathrm{Tan}(\phi_H) = H \circ \phi_H$. From the definitions, we immediately get the useful identity

$$\mathrm{Tan}(\phi_H) = -\mathrm{Dev}\left(\phi_H^{-1}\right) = -\overline{H}. \qquad (1.12)$$

Note that this identity does not make sense in general even for C^1-functions H, because their Hamiltonian vector field X_H would be only C^0, and so their flow $\phi_H^t(x)$ may not exist. Understanding what is going on in such a case touches the heart of C^0-Hamiltonian geometry and dynamics.

1.1.2 The Hofer topology

Let $H \in C^\infty([0,1] \times M)$. Denote by

$$\operatorname{osc}(H_t) = \max_{x \in M} H_t(x) - \min_{x \in M} H_t(x),$$

for $t \in [0,1]$, the *oscillation*, or total variation, of H_t. For later reference, note that for a normalized Hamiltonian H, and for every $t \in [0,1]$, both $\max H_t(x)$ and $-\min H_t(x)$ are always nonnegative. Then define by

$$\|H\|_{(1,\infty)} = \int_0^1 \operatorname{osc}(H_t) dt \tag{1.13}$$

the *mean oscillation* and by

$$\|H\|_\infty = \max_{t \in [0,1]} \operatorname{osc}(H_t) \tag{1.14}$$

the *maximum oscillation* of the Hamiltonian H on the interval $[0,1]$. We call $\| \cdot \|_{(1,\infty)}$ and $\| \cdot \|_\infty$ the $L^{(1,\infty)}$*-norm* and the L^∞*-norm* on the space of (time-dependent) Hamiltonians, respectively. Strictly speaking, these are norms only when restricted to the space $C_m^\infty([0,1] \times M)$ of normalized Hamiltonian functions, since both (1.13) and (1.14) are invariant under adding functions that depend only on t. In particular, normalizing a Hamiltonian does not change either one of them. There are also the norms

$$\|H\|_{C^0} = \max_{t \in [0,1]} \|H_t\|_{C^0} = \max_{t \in [0,1]} \max_{x \in M} |H_t(x)|,$$

and

$$\max_{(t,x) \in [0,1] \times M} H(t,x) - \min_{(t,x) \in [0,1] \times M} H(t,x).$$

However, these are not invariant under adding functions of t. And for normalized Hamiltonians, both these norms are equivalent to the L^∞-norm, so we will have little use for them in this work.

Clearly $\| \cdot \|_{(1,\infty)} \leq \| \cdot \|_\infty$, but the two norms are not equivalent. Indeed, it is easy to see that there are sequences H_i of Hamiltonians such that $\|H_i\|_{(1,\infty)} = 1$ for all i, but

$\|H_i\|_\infty \to \infty$ as $i \to \infty$. In fact, one can take any time-independent Hamiltonian H with $\|H\|_{(1,\infty)} = \|H\|_\infty = 1$, and define a sequence of time-reparameterizations (see below) of H so that the constructed sequence has this property. However, for a generic Hamiltonian, after suitable reparameterization we have $\|H\|_\infty \approx \|H\|_{(1,\infty)}$. This is explained in greater detail in Section 2.6 below.

We will consistently use the subscripts (or superscripts) $(1,\infty)$ and ∞ to distinguish the two cases, and use them to denote any object defined using the one or the other norm. We will omit them and write for example $\|\cdot\|$ to denote either one of the two cases. That is, when we omit the subscripts (or superscripts), the particular statement is true in both the $L^{(1,\infty)}$-case and the L^∞-case.

Due to the above one-to-one correspondence between Hamiltonian functions and Hamiltonian paths, we frequently consider these norms as norms on the space $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ of Hamiltonian paths. In that case, we also refer to them as the *Hofer norms*. They induce metrics, called the *Hofer metrics*, on $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ in the usual way by

$$d_H \colon \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \times \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \longrightarrow \mathbb{R}, \qquad d_H(\phi_H, \phi_K) = \left\|\overline{H}\#K\right\|.$$

We call the induced topology on $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ the *Hofer topology*. By a slight abuse of notation, we often denote this metric by $\|\cdot\|$ instead of by d_H.

The above Hofer norm $\|\cdot\|_{(1,\infty)}$ on $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ can be identified with the Finsler length

$$\mathrm{leng}(\phi_H) = \int_0^1 \Big(\max_{x\in M} H\left(t, (\phi_H^t)(x)\right) - \min_{x\in M} H\left(t, (\phi_H^t)(x)\right)\Big) dt$$

of the path $\phi_H \colon t \mapsto \phi_H^t$, where the norm on $T_{\mathrm{id}}\mathrm{Ham}(M,\omega) \cong C_m^\infty(M) \cong C^\infty(M)/\mathbb{R}$ is of course defined by

$$\|H\| = \mathrm{osc}(H) = \max_{x\in M} H(x) - \min_{x\in M} H(x)$$

for a normalized Hamiltonian function $H \colon M \to \mathbb{R}$. From the definitions, we have $\|H\|_{(1,\infty)} =$

$\operatorname{leng}(\phi_H)$, and in particular,

$$\operatorname{leng}\left(\phi_H^{-1} \circ \phi_K\right) = \left\|\overline{H} \# K\right\|_{(1,\infty)}.$$

The following simple identities and inequalities, which will be useful later, follow immediately from the definitions, so their proofs are omitted. For H, K, $L\colon [0,1] \times M \to \mathbb{R}$, we have

$$\left\|\overline{H} \# K\right\| = \|H - K\|.$$

Moreover, the norm is symmetric,

$$\left\|\overline{H}\right\| = \|H\|, \qquad \left\|\overline{H} \# K\right\| = \left\|\overline{K} \# H\right\|,$$

and satisfies the triangle inequality,

$$\|H \# K\| \leq \|H\| + \|K\|.$$

It is easy to see that $\|\cdot\|$ is left (but in general not right) invariant, i.e. we have the identity $\|(H \# K) - (H \# L)\| = \|K - L\|$. The identity (1.12) immediately implies

$$\|\operatorname{Tan}(\phi_H) - \operatorname{Tan}(\phi_K)\| = \left\|H \# \overline{K}\right\|. \tag{1.15}$$

The norms $\|\cdot\|$ induce metrics on the space $C_m^\infty([0,1] \times M)$ in the usual way. By another slight abuse of notation, we denote these metrics by $\|\cdot\|$ as well. The completion of $C_m^\infty([0,1] \times M)$ with respect to $\|\cdot\|_\infty$ is the space $C_m^0([0,1] \times M)$ of normalized (uniformly) continuous functions on $[0,1] \times M$. On the other hand, we denote the completion of $C_m^\infty([0,1] \times M)$ with respect to $\|\cdot\|_{(1,\infty)}$ by $L_m^{(1,\infty)}([0,1] \times M)$. Note that a typical element $H \in L_m^{(1,\infty)}([0,1] \times M)$ is not a continuous function. However, by standard arguments from measure theory (see e.g. [Fol99]), H_t is defined for a.e. $t \in [0,1]$, and is a continuous function of the space variable for each such t. Moreover, the normalization condition does make sense a.e., and is satisfied for all such t, which justifies usage of the subscript m in the notation.

Strictly speaking, elements of $L_m^{(1,\infty)}([0,1] \times M)$ are equivalence classes of functions, where two functions are considered equivalent if and only if they agree for a.e. $t \in [0,1]$, but as is customary in measure theory, we will mostly disregard this subtlety in our treatment. The uniform convergence (to a uniformly continuous limit function) is an obvious advantage for many arguments in the L^∞-case. One might be tempted to work exclusively with the L^∞-norm. However, the $L^{(1,\infty)}$-norm has its own merits, which will be pointed out in detail below. For brevity, when making statements about both spaces simultaneously, we denote the completion of $C_m^\infty([0,1] \times M)$ with respect to $\|\cdot\|$ by $H([0,1] \times M)$ (the normalization condition being understood).

The remarkable *Hofer norm* of a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M, \omega)$ introduced in [Hof90, Hof93] is defined by

$$\|\phi\| = \inf_{H \mapsto \phi} \|H\|. \tag{1.16}$$

While it is easy to see that (1.16) defines a pseudo-norm on $\mathrm{Ham}(M, \omega)$, it is a highly non-trivial fact that this indeed gives a genuine norm, which was first proved by Hofer [Hof90] for $\mathbb{C}^n$ or $\mathbb{R}^{2n}$, by Polterovich [Pol93] for rational symplectic manifolds, and by Lalonde and McDuff [LM95a] in complete generality.

The Hofer norm (1.16) induces a (bi-invariant) metric on $\mathrm{Ham}(M, \omega)$ by $\rho(\phi, \psi) = \|\phi^{-1} \circ \psi\|$, called the *Hofer metric*, and we define the *Hofer topology* on $\mathrm{Ham}(M, \omega)$ to be the induced metric topology. It is immediate to check that the Hofer topology is locally path connected. Polterovich proved the following interesting lemma.

Lemma 1.1.2 ([Pol01],Lemma 5.1.C)**.** $\|\phi\|_{(1,\infty)} = \|\phi\|_\infty$ *for each* $\phi \in \mathrm{Ham}(M, \omega)$.

Since for any Hamiltonian H, we have $\|H\|_{(1,\infty)} \leq \|H\|_\infty$, the L^∞-Hofer norm on $\mathrm{Ham}(M, \omega)$ is a priori larger (or stronger) than the $L^{(1,\infty)}$-Hofer norm. Lemma 1.1.2 is thus sometimes expressed as the fact that the 'coarse' Hofer norm coincides with the 'fine' Hofer norm on

$\mathrm{Ham}(M, \omega)$ [Pol01]. Polterovich's proof is constructive, and since we will make use of his arguments in Section 2.6, the proof will be given there.

1.1.3 The C^0-topology

We denote by $\mathrm{Homeo}(M)$ the group of homeomorphisms of M, equipped with the compact-open topology, and by $\mathrm{Homeo}_0(M)$ the path component of the identity in $\mathrm{Homeo}(M)$, endowed with the subspace topology. Denote by $\mathcal{P}\mathrm{Homeo}(M)$ the set of continuous paths $\lambda\colon [0,1] \to \mathrm{Homeo}(M)$, with $\lambda(0) = \mathrm{id}$. Recall that this is the same as the set of continuous maps $\Lambda\colon [0,1] \times M \to M$, such that each map $\lambda_t = \Lambda(t, \cdot)\colon M \to M$, $t \in [0,1]$, is a homeomorphism, and $\lambda_0 = \mathrm{id}$, and equip $\mathcal{P}\mathrm{Homeo}(M)$ with the compact-open topology. Both $\mathrm{Homeo}(M)$ and $\mathcal{P}\mathrm{Homeo}(M)$ are topological groups under composition (see below), and as above there is a time-one evaluation map

$$\mathrm{ev}_1\colon \mathcal{P}\mathrm{Homeo}(M) \longrightarrow \mathrm{Homeo}_0(M), \qquad \lambda \longmapsto \lambda_1 = \lambda(1),$$

which is a surjective continuous homomorphism.

Fix any Riemannian metric on M, and denote by d the induced distance function on M. Consider the metrics $\hat{d}$ and $\overline{d}$ on $\mathrm{Homeo}(M)$ defined by

$$\hat{d}(h, g) = \max_{x \in M} \left(d(h(x), g(x)) \right)$$

and

$$\overline{d}(h, g) = \max \left(\hat{d}(h, g), \hat{d}(h^{-1}, g^{-1}) \right). \tag{1.17}$$

Since M is compact, both metrics induce the compact-open topology on $\mathrm{Homeo}(M)$. In particular, the compact-open topology is metrizable, and is independent of the choice of Riemannian metric on M. It is preferable to work with the metric $\overline{d}$, since it is easily seen to be a complete metric, while the metric $\hat{d}$ can never be complete. We refer to the metric $\overline{d}$ as the C^0-*metric*, and therefore also call the induced (compact-open) topology the C^0-*topology*, on Homeo(M). The compact-open topology on $\mathcal{P}$Homeo(M) is also induced by a complete metric, given by

$$\overline{d}(\lambda, \mu) = \max_{t \in [0,1]} \overline{d}(\lambda(t), \mu(t)). \tag{1.18}$$

If we define

$$\hat{d}(\lambda, \mu) = \max_{t \in [0,1]} \hat{d}(\lambda(t), \mu(t)),$$

then we can also write $\overline{d}(\lambda, \mu) = \max\left(\hat{d}(\lambda, \mu), \hat{d}(\lambda^{-1}, \mu^{-1})\right)$. Here $\lambda^{-1} \colon [0,1] \to \mathrm{Homeo}(M)$ denotes the path $t \mapsto (\lambda(t))^{-1}$ (compare to the earlier remark about Hamiltonian paths). We call the metric $\overline{d}$ the C^0-*metric*, and the induced (compact-open) topology the C^0-*topology*, on $\mathcal{P}$Homeo(M). Note that we use the notations $\hat{d}$ and $\overline{d}$ both for the distance of maps as well as the distance of paths. Since it is clear from the context which one is meant, this should not lead to any confusion.

If h_i is a sequence of homeomorphisms, converging in the C^0-metric to a homeomorphism $h \in \mathrm{Homeo}(M)$, we will simply write $\lim_{C^0} h_i = h$, and similarly $\lambda = \lim_{C^0} \lambda_i$, if a sequence λ_i of continuous paths converges in the C^0-metric to a continuous path $\lambda \in \mathcal{P}\mathrm{Homeo}(M)$. One readily checks that for any given sequences h_i and $g_i \in \mathrm{Homeo}(M)$, with $\lim_{C^0} h_i = h$ and $\lim_{C^0} g_i = g$, we have $\lim_{C^0} h_i \circ g_i = h \circ g$, and $\lim_{C^0} h_i^{-1} = h^{-1}$. Similarly, if λ_i and $\mu_i \in \mathcal{P}\mathrm{Homeo}(M)$ converge in the C^0-metric to continuous paths λ and μ, respectively, then $\lim_{C^0} \lambda_i \circ \mu_i = \lambda \circ \mu$, and $\lim_{C^0} \lambda_i^{-1} = \lambda^{-1}$. In other words, the spaces Homeo(M) and $\mathcal{P}$Homeo(M) form topological groups. Moreover, the metrics $\hat{d}$ are right (but not left) invariant. We will use these simple facts, and completeness of $\overline{d}$, frequently in this work.

1.2 Reparameterization of Hamiltonian paths

It is immediate to check from the definition the following well-known formula for the Hamiltonian generating a reparameterized Hamiltonian path. For a given Hamiltonian $H\colon [0,1]\times M \to \mathbb{R}$, generating the Hamiltonian path $\phi_H\colon t \mapsto \phi_H^t$, and any smooth function $\zeta\colon [0,1] \to [0,1]$, the reparameterized path $\phi_{H^\zeta}\colon t \mapsto \phi_H^{\zeta(t)}$, is generated by the Hamiltonian function H^ζ, defined by the formula

$$H^\zeta(t,x) = \zeta'(t)H(\zeta(t),x). \tag{1.19}$$

Here and in the following ζ' denotes the derivative of the function ζ. If H is normalized, then so is H^ζ. If $\zeta(0) = 0$, $\zeta(1) = 1$, and ζ is monotone, so that the reparameterized path traverses the same 'image' as the original path at different speed, we refer to the function ζ as a *reparameterization function*. In the special case $\zeta(t) = st$, for some $s \in [0,1]$, we also write $H^\zeta = H^s$.

Very often in the study of the geometry of Hamiltonian paths, one needs to reparameterize a given Hamiltonian path in a way that the reparameterization is close enough to the given parameterization, e.g. in the smoothing process of the concatenation of two paths. We will discuss this 'closeness' now.

Lemma 1.2.1. *Let H be a normalized Hamiltonian, and ζ_1, $\zeta_2\colon [0,1] \to [0,1]$ be two smooth functions. Then for every $t \in [0,1]$, we have*

$$0 \leq \operatorname{osc}\left(H_t^{\zeta_1} - H_t^{\zeta_2}\right) \leq 2L \cdot |\zeta_1'(t)| \cdot |\zeta_1(t) - \zeta_2(t)| + |\zeta_1'(t) - \zeta_2'(t)| \cdot \operatorname{osc}(H_{\zeta_2(t)}),$$

where the constant $L < \infty$ depends only on H.

Proof. Fix $t \in [0,1]$. We compute

$$\begin{aligned}
0 &\leq \max_{x\in M}\Big(H^{\zeta_1}(t,x) - H^{\zeta_2}(t,x)\Big) \\
&= \max_{x\in M}\big(\zeta_1'(t)H(\zeta_1(t),x) - \zeta_2'(t)H(\zeta_2(t),x)\big) \\
&\leq \max_{x\in M}\big(\zeta_1'(t)(H(\zeta_1(t),x) - H(\zeta_2(t),x))\big) + \max_{x\in M}\big((\zeta_1'(t) - \zeta_2'(t))H(\zeta_2(t),x)\big),
\end{aligned}$$

and similarly for $-\min$, so that

$$\begin{aligned}
0 &\leq \operatorname{osc}\Big(H_t^{\zeta_1} - H_t^{\zeta_2}\Big) \\
&\leq |\zeta_1'(t)|\Big[\max_{x\in M}\big(H(\zeta_1(t),x) - H(\zeta_2(t),x)\big) - \min_{x\in M}\big(H(\zeta_1(t),x) - H(\zeta_2(t),x)\big)\Big] \\
&\qquad + |\zeta_1'(t) - \zeta_2'(t)| \cdot \operatorname{osc}(H_{\zeta_2(t)}) \\
&\leq 2L \cdot |\zeta_1'(t)| \cdot |\zeta_1(t) - \zeta_2(t)| + |\zeta_1'(t) - \zeta_2'(t)| \cdot \operatorname{osc}(H_{\zeta_2(t)}),
\end{aligned}$$

where L is a Lipschitz constant that depends only on the function H. $\square$

In the situation of Lemma 1.2.1, if in addition ζ_1 is monotone, then

$$\Big\|H^{\zeta_1} - H^{\zeta_2}\Big\|_{(1,\infty)} \leq 2L \max_{t\in[0,1]} |\zeta_1(t) - \zeta_2(t)| + \|H\|_\infty \int_0^1 |\zeta_1'(t) - \zeta_2'(t)|dt.$$

That motivates the following

Definition 1.2.2. For a function $\zeta\colon [0,1] \to [0,1]$, we define its *hamiltonian norm* by

$$\|\zeta\|_{\mathrm{ham}} = \|\zeta\|_{C^0} + \|\zeta'\|_{L^1} = \max_{t\in[0,1]} |\zeta(t)| + \int_0^1 |\zeta'(t)|dt.$$

We say that two functions are *hamiltonian close* to each other, if they are close in the metric induced by the hamiltonian norm.

We have proved

Lemma 1.2.3. *Let $H\colon [0,1] \times M \to \mathbb{R}$ be a normalized Hamiltonian, and let ζ_1, $\zeta_2\colon [0,1] \to [0,1]$ be two smooth functions. Assume in addition that ζ_1 is monotone. Then*

$$\Big\|H^{\zeta_1} - H^{\zeta_2}\Big\|_{(1,\infty)} \leq C\|\zeta_1 - \zeta_2\|_{\mathrm{ham}},$$

for some constant $C < \infty$ that depends only on the function H.

We consider the special case $\zeta_1 = \mathrm{id}$, and write $\zeta_2 = \zeta$. The following definition will be useful.

Definition 1.2.4. We call a Hamiltonian path $\phi_H \colon [0,1] \to \mathrm{Symp}(M, \omega)$ boundary flat near $t = 0$ (or $t = 1$) if ϕ_H is constant near $t = 0$ (or $t = 1$), and we call the path *boundary flat* if it is constant near both end points $t = 0$ and $t = 1$. In terms of the normalized generating Hamiltonian H of ϕ_H, this means $H_t \equiv 0$ near the end points.

We would like to point out that the set of boundary flat Hamiltonians (or Hamiltonians flat near $t = 0$ or $t = 1$) is closed under the operations of the product $(H, K) \mapsto H \# K$, and taking the inverse $H \mapsto \overline{H}$. Given any Hamiltonian path ϕ_H, by choosing ζ so that $\zeta \equiv 0$ near $t = 0$ and $\zeta \equiv 1$ near $t = 1$, the reparameterized path H^ζ becomes boundary flat. It is easy to see that one can find such a reparameterization function ζ, with the property that $\|\zeta - \mathrm{id}\|_{\mathrm{ham}}$ is arbitrarily small (see Figure 1). That proves the following result.

Lemma 1.2.5. *Let $H \colon [0,1] \times M \to \mathbb{R}$ be a normalized Hamiltonian, and $\epsilon > 0$. Then there exists a reparameterization function $\zeta \colon [0,1] \to [0,1]$, such that H^ζ is boundary flat, and*

$$\left\| H - H^\zeta \right\|_{(1,\infty)} < \epsilon.$$

In other words, any Hamiltonian path can be approximated arbitrarily closely in the Hofer metric $\| \cdot \|_{(1,\infty)}$ by a boundary flat Hamiltonian path. By considering the estimate in Lemma 1.2.1 separately on the three intervals $0 \leq t \leq \delta$, $\delta \leq t \leq 1 - \delta$, and $1 - \delta \leq t \leq 1$, for some sufficiently small $\delta > 0$ (depending only on H), we obtain the following estimate in the L^∞-case.

Lemma 1.2.6. *Let $H \colon [0,1] \times M \to \mathbb{R}$ be a normalized Hamiltonian, and $\epsilon > 0$. Then there exists a reparameterization function $\zeta \colon [0,1] \to [0,1]$, such that H^ζ is boundary flat, and*

$$\left\| H - H^\zeta \right\|_{\infty} < \epsilon + \max(\mathrm{osc}(H_0), \mathrm{osc}(H_1)).$$

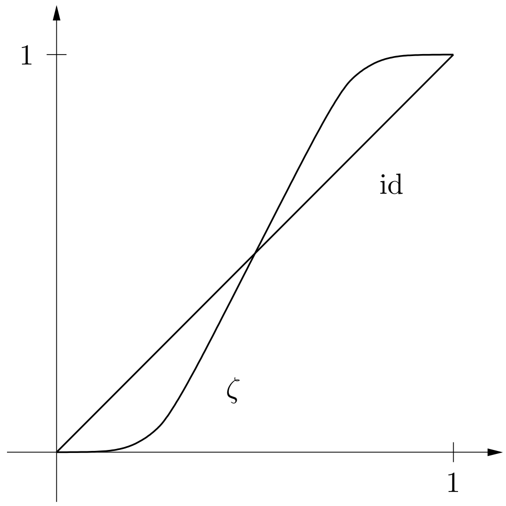

Figure 1: A boundary flat reparameterization function hamiltonian close to the identity

We state and prove the following approximation lemma, which is a variation of Lemma 5.2 in [Oh02], and Lemma A.1 in [OM07], adapted to our setting.

Lemma 1.2.7 (Approximation lemma)**.** *Let $H\colon [0,1]\times M \to \mathbb{R}$ be a normalized Hamiltonian. Then we can reparameterize the Hamiltonian path ϕ_H in time, so that the Hamiltonian $F = H^\zeta$, generating the reparameterized path, satisfies the following properties:*

(i) $\phi_F^0 = \phi_H^0 = \mathrm{id}$ *and* $\phi_F^1 = \phi_H^1$*,*

(ii) $F_t \equiv 0$ *near* $t = 0$ *and* $t = 1$*, and in particular,* F *can be extended to a time-periodic function on* $\mathbb{R}\times M$*, and the path* ϕ_F *is boundary flat,*

(iii) there is a canonical one-to-one correspondence between the periods $\mathrm{Per}(H)$ *and* $\mathrm{Per}(F)$*, and the critical points of the action functionals* $\mathrm{Crit}\mathcal{A}_H$ *and* $\mathrm{Crit}\mathcal{A}_F$*, with their actions fixed,*

(iv) the C^0*-distance* $\overline{\mathrm{d}}(\phi_H,\phi_F)$ *can be made as small as we want. In fact,* $\overline{\mathrm{d}}(\phi_H,\phi_F) < L\|\zeta - \mathrm{id}\|_{C^0} \le L\|\zeta - \mathrm{id}\|_{\mathrm{ham}}$*, where the constant* $L < \infty$ *depends only on* H*, and*

(v) the Hofer distance $\mathrm{leng}(\phi_H, \phi_F) = \|H - F\|_{(1,\infty)}$ *can be made as small as we want. In fact,* $\|H - F\|_{(1,\infty)} < C\|\zeta - \mathrm{id}\|_{\mathrm{ham}}$*, where the constant* $C < \infty$ *depends only on* H*, and we can choose* ζ *arbitrarily hamiltonian close to the identity.*

(vi) If in addition $H_0 \equiv 0 \equiv H_1$*, then the Hofer distance* $\|H - F\|_\infty$ *can be made as small as we want.*

Proof. For any $\epsilon > 0$, we can find $\delta > 0$, and a reparameterization function $\zeta\colon [0,1] \to [0,1]$, such that $\zeta(t) = 0$ for all $t < \delta$, $\zeta(t) = 1$ for all $t > 1 - \delta$, and $\|\zeta - \mathrm{id}\|_{\mathrm{ham}} < \epsilon$. Then $F = H^\zeta$ clearly satisfies (i) and (ii), and by Lemma 1.2.3, we have $\|H - F\|_{(1,\infty)} < C\|\zeta - \mathrm{id}\|_{\mathrm{ham}} < C\epsilon$, where C is a constant that depends only on H. And since $(t, x) \mapsto \phi_H^t(x)$ and $(t, x) \mapsto (\phi_H^t)^{-1}(x)$ are Lipschitz continuous, we find $\overline{d}(\phi_H, \phi_{H^\zeta}) < L\|\zeta - \mathrm{id}\|_{C^0} \le L\|\zeta - \mathrm{id}\|_{\mathrm{ham}}$, for some constant L that depends only on H. That proves (iv) and (v). And (vi) follows similarly from Lemma 1.2.6. Statement (iii) follows from simple comparison of the corresponding actions of periodic orbits [Oh02]. $\square$

Note that part (v) of this lemma is false in general in the L^∞-case, because the derivative of the cut-off function ζ could blow up in the above approximation. In fact, if $\zeta \equiv 0$ near $t = 0$, and $\zeta \equiv 1$ near $t = 1$, then $\|H - H^\zeta\|_\infty \ge \max(\mathrm{osc}(H_0), \mathrm{osc}(H_1))$, so that $\|H - H^\zeta\|_\infty$ is always bounded away from 0 unless $H_t \equiv 0$ at time $t = 0$ and $t = 1$. We remark that the continuity of this *boundary flattening procedure* is one of the main advantages of the $L^{(1,\infty)}$-norm, and is one of the primary reasons to study the $L^{(1,\infty)}$-case, not just the (seemingly easier) L^∞-case. We would also like to point out that the $L^{(1,\infty)}$-norm seems more natural in the context of Floer theory, see [OM07, Oh07a], and for the notion of the length of a Hamiltonian path.

For later reference, we wish to consider another way of reparameterizing a Hamiltonian H, which will be used to concatenate boundary flat Hamiltonian paths. Given $0 \le a < b \le 1$, and a smooth Hamiltonian function H defined on $[0,1] \times M$, we denote by $\zeta_{a,b}\colon [a,b] \to [0,1]$

the unique linear function with $\zeta(a) = 0$ and $\zeta(b) = 1$, and by $H^{\zeta_{a,b}}$ the reparameterized Hamiltonian defined on $[a,b] \times M$. Of course, if H is normalized, then so is $H^{\zeta_{a,b}}$, and if H is boundary flat, then again so is $H^{\zeta_{a,b}}$. Obviously, $\|H^{\zeta_{a,b}}\|_{(1,\infty)} = \|H\|_{(1,\infty)}$, and $\|H^{\zeta_{a,b}}\|_\infty = \frac{1}{b-a}\|H\|_\infty$.

To conclude this section, we derive two immediate consequences of the above estimates for Cauchy sequences of Hamiltonians. These will play an important role in the next chapter.

Lemma 1.2.8. *Suppose $H_i \colon [0,1] \times M \to \mathbb{R}$ is a Cauchy sequence in the $L^{(1,\infty)}$-metric, i.e. $\|H_i - H_j\|_{(1,\infty)} \to 0$ as i, $j \to \infty$, and ζ_1, $\zeta_2 \colon [0,1] \to [0,1]$ are two monotone smooth functions. Given $\epsilon > 0$, there exist constants $\delta = \delta(\{H_i\}) > 0$, and $i_0 = i_0(\{H_i\}) > 0$, such that: if ζ_1, ζ_2 satisfy $\|\zeta_1 - \zeta_2\|_{\mathrm{ham}} < \delta$, then*

$$\left\| H_i^{\zeta_1} - H_i^{\zeta_2} \right\|_{(1,\infty)} < \epsilon,$$

for all $i \geq i_0$.

Proof. We can find i_0 sufficiently large, such that $\|H_i - H_{i_0}\|_{(1,\infty)} < \epsilon/3$, for all $i \geq i_0$. Choose $\delta = \epsilon/3C$, where the constant C is given by applying Lemma 1.2.3 to the Hamiltonian H_{i_0}. Then

$$\left\| H_{i_0}^{\zeta_1} - H_{i_0}^{\zeta_2} \right\|_{(1,\infty)} < \frac{\epsilon}{3},$$

provided $\|\zeta_1 - \zeta_2\|_{\mathrm{ham}} < \delta$. Therefore,

$$\begin{aligned} \left\| H_i^{\zeta_1} - H_i^{\zeta_2} \right\|_{(1,\infty)} &\leq \left\| H_i^{\zeta_1} - H_{i_0}^{\zeta_1} \right\|_{(1,\infty)} + \left\| H_{i_0}^{\zeta_1} - H_{i_0}^{\zeta_2} \right\|_{(1,\infty)} + \left\| H_{i_0}^{\zeta_2} - H_i^{\zeta_2} \right\|_{(1,\infty)} \\ &\leq \|H_i - H_{i_0}\|_{(1,\infty)} + \left\| H_{i_0}^{\zeta_1} - H_{i_0}^{\zeta_2} \right\|_{(1,\infty)} + \|H_{i_0} - H_i\|_{(1,\infty)} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

as long as $\|\zeta_1 - \zeta_2\|_{\mathrm{ham}} < \delta$, and $i \geq i_0$. ∎

Note that H_i converges to an $L^{(1,\infty)}$-function H, but that we cannot replace H_{i_0} by H in the above proof, since H is not even continuous in general.

Lemma 1.2.9. *Suppose* $H_i\colon [0,1] \times M \to \mathbb{R}$ *is a Cauchy sequence in the metric* $\|\cdot\|$, *i.e.* $\|H_i - H_j\| \to 0$ *as* $i, j \to \infty$, *and* $\lambda, \mu \in \mathcal{P}\mathrm{Homeo}(M)$ *are two continuous paths. Given* $\epsilon > 0$, *there exist constants* $\delta = \delta(\{H_i\}) > 0$, *and* $i_0 = i_0(\{H_i\}) > 0$, *such that: if* λ, μ *satisfy* $\hat{d}(\lambda, \mu) < \delta$, *then*

$$\|H_i \circ \lambda - H_i \circ \mu\| < \epsilon,$$

for all $i \geq i_0$.

Proof. Choose i_0 sufficiently large as in the proof of Lemma 1.2.8. By uniform continuity of H_{i_0}, there exists $\delta > 0$ such that

$$\|H_{i_0} \circ \lambda - H_{i_0} \circ \mu\|_\infty < \frac{\epsilon}{3},$$

provided that $\hat{d}(\lambda, \mu) < \delta$. Since $\|\cdot\|_{(1,\infty)} \leq \|\cdot\|_\infty$, this implies

$$\|H_{i_0} \circ \lambda - H_{i_0} \circ \mu\|_{(1,\infty)} < \frac{\epsilon}{3},$$

when $\hat{d}(\lambda, \mu) < \delta$. In both cases, apply the triangle inequality as in the proof of Lemma 1.2.8. $\square$

Chapter 2

C^0-Hamiltonian geometry and C^0-symplectic topology

We begin our study of C^0-symplectic topology by defining the C^0-counterpart to symplectic diffeomorphisms.

2.1 The symplectic homeomorphism group

Consider the group $\mathrm{Symp}(M, \omega)$ of symplectic diffeomorphisms as a subspace of the group $\mathrm{Homeo}(M)$ of homeomorphisms of M, with the induced subspace topology, i.e. the C^0-topology.

Definition 2.1.1. We define the group of *symplectic homeomorphisms* of (M, ω) as the closure

$$\mathrm{Sympeo}(M, \omega) = \overline{\mathrm{Symp}(M, \omega)} \subset \mathrm{Homeo}(M)$$

of the group of symplectic diffeomorphisms inside the group $\mathrm{Homeo}(M)$ of homeomorphisms of M, with respect to the C^0-topology, or in other words, as the completion of $\mathrm{Symp}(M, \omega)$ in $\mathrm{Homeo}(M)$ with respect to the C^0-metric. We equip $\mathrm{Sympeo}(M, \omega)$ with the C^0-topology.

This closure obviously forms a closed topological subgroup of $\mathrm{Homeo}(M)$ with respect to the induced C^0-topology. The above definition is partly motivated by Gromov-Eliashberg's C^0-symplectic rigidity theorem.

Theorem 2.1.2 ([Eli87, Gro86])**.** $\mathrm{Symp}(M, \omega) \subset \mathrm{Diff}(M)$ *is closed in the* C^0*-topology.*

A symplectic homeomorphism that is also a diffeomorphism is therefore a symplectic diffeomorphism. In fact, using the proof of rigidity based on symplectic capacities (see for example Section 12.2 in [MS98]), we can prove

Proposition 2.1.3. *The derivative of a symplectic homeomorphism is symplectic wherever it exists, and in particular, any smooth symplectic homeomorphism is automatically a symplectic diffeomorphism.*

Proof. The second statement follows immediately from the first and the inverse function theorem. The first statement is a local statement, so by Darboux's theorem, we may without loss of generality assume $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$, and consider the derivative $A = d\psi(0)$ of a symplectic homeomorphism ψ at the origin. Suppose $\psi = \lim_{C^0} \psi_i$, where each ψ_i is a symplectic diffeomorphism. Let c be a symplectic capacity on $(\mathbb{R}^{2n}, \omega_0)$ (see for example [HZ94, MS98] for the definition). Then by the (relative) monotonicity axiom for a capacity, each symplectic diffeomorphism ψ_i preserves the capacity of ellipsoids. By [MS98, Lemma 12.11], the limit ψ also preserves the capacity of ellipsoids. By the conformality axiom, the maps $\frac{1}{t}\psi(tz)$ also preserve the capacity of ellipsoids, and thus so does the limit A as $t \to 0$ again by [MS98, Lemma 12.11]. By [MS98, Proposition 12.10], the linear map A is either symplectic or anti-symplectic. Applying the same argument to $\psi \times \mathrm{id}$ on $(\mathbb{R}^{2n+2n}, \omega_0 \times \omega_0)$, we see that $A = d\psi(0)$ must be symplectic. □

It is easy to see that any symplectic homeomorphism preserves the Liouville measure induced by the Liouville volume form (1.1), which is an easy consequence of Fatou's lemma in measure theory. More generally, recall that we obtain a measure on M by integrating any volume form Ω. Denote by $\mathrm{Homeo}^\Omega(M) \subset \mathrm{Homeo}(M)$ the group of measure-preserving homeomorphisms of M, equipped with the subspace topology, i.e. the C^0-topology, and by $\mathrm{Homeo}^\Omega_0(M)$ the path component of the identity in $\mathrm{Homeo}^\Omega(M)$, with the subspace topology. By [Fat80, Corollary 1.6], $\mathrm{Homeo}^\Omega(M)$ is a closed topological subgroup of $\mathrm{Homeo}(M)$

in the compact-open topology. Since every symplectic diffeomorphism preserves the Liouville volume form, and hence the induced Liouville measure, and $\mathrm{Sympeo}(M,\omega)$ is the closure of $\mathrm{Symp}(M,\omega) \subset \mathrm{Homeo}(M)$ in the C^0-topology, we obtain the following.

Proposition 2.1.4. *Any symplectic homeomorphism $h \in \mathrm{Sympeo}(M,\omega)$ preserves the Liouville measure. Consequently, $\mathrm{Sympeo}(M,\omega)$ forms a closed subgroup of $\mathrm{Homeo}^\Omega(M)$, where $\mathrm{Homeo}^\Omega(M)$ denotes the group of homeomorphisms of M that preserve the Liouville measure.*

It is easy to derive from the nonsqueezing theorem (for $(\mathbb{R}^{2n},\omega_0)$, combined with Darboux's theorem) and Eliashberg's rigidity theorem, together with the general fact that a measure-preserving diffeomorphism preserves the corresponding volume form, the properness of the subgroup $\mathrm{Sympeo}(M,\omega) \subsetneq \mathrm{Homeo}^\Omega(M)$, when $\dim M \geq 4$:

$$\begin{array}{ccc} \mathrm{Sympeo}(M,\omega) \cap \mathrm{Diff}(M) = \mathrm{Symp}(M,\omega) & \subsetneq & \mathrm{Diff}^\Omega(M) = \mathrm{Homeo}^\Omega(M) \cap \mathrm{Diff}(M) \\ \cap & & \cap \\ \mathrm{Sympeo}(M,\omega) & \subsetneq & \mathrm{Homeo}^\Omega(M), \end{array}$$

where $\mathrm{Diff}^\Omega(M)$ denotes the group of diffeomorphisms of M preserving the volume form Ω. Regarding the case of dimension two, Oh and Sikorav independently proved the following theorem.

Theorem 2.1.5 ([Oh06, Sik07])**.** *Let M be a smooth closed n-manifold, equipped with a measure induced by some volume form Ω on M. If a measure-preserving homeomorphism h can be C^0-approximated by diffeomorphisms (e.g. if $n \leq 3$), then it can be C^0-approximated by volume-preserving diffeomorphisms.*

As remarked in [Oh06, Sik07], this theorem is well-known among many experts in the dynamical systems community, but so far, there did not seem to exist any published proofs. The fact that approximation of homeomorphisms by diffeomorphisms can be done in dimension $n \leq 3$ is well-known as well, however, this need not be true in dimension 4 or higher, see

[Oh06] and the references therein. We give the details of (a slight variation of) Sikorav's proof here. Oh's proof is more sophisticated since he also proves a parametrized version of the statement above. In the language of Section 2.5 below, Oh shows that the C^0-closure of the space $\mathcal{P}\mathrm{Symp}(M,\omega)$ of symplectic isotopies coincides with the space $\mathcal{P}\mathrm{Homeo}^{\Omega}(M)$ of measure-preserving isotopies in dimension two [Oh06, Theorem I']. The result by Sikorav is all that is needed here.

Proof. As in Section 1.1, fix any Riemannian metric on M, and denote by d the induced distance function on M. Let $\epsilon > 0$. Let K be a smooth triangulation of M such that $\mathrm{diam}(\sigma)$ and $\mathrm{diam}(h(\sigma)) < \epsilon$ for every n-simplex $\sigma \in K^{(n)}$. By hypothesis, for any $\delta > 0$ there exists $\vartheta \in \mathrm{Diff}(M)$ such that $\overline{d}(h, \vartheta) < \delta$, and thus

$$|\mathrm{vol}(\vartheta(\sigma)) - \mathrm{vol}(\sigma)| = |\mathrm{vol}(\vartheta(\sigma)) - \mathrm{vol}(h(\sigma))| = O(\delta),$$

where vol denotes the volume, or measure, with respect to Ω. Using an idea of E. Giroux, we choose some maximal tree T in the dual triangulation, and write T as a union of almost-disjoint simplicial paths $P_1, \ldots, P_k$. The vertices of these paths all lie in distinct n-simplices σ, and each simplex contains exactly one such vertex. We can modify ϑ along the paths $P_1, \ldots, P_k$ to obtain $\varphi \in \mathrm{Diff}(M)$ such that $\mathrm{vol}(\varphi(\sigma)) = \mathrm{vol}(\sigma)$ for every σ. If δ is sufficiently small, this can be done so that $\overline{d}(h, \varphi) < \epsilon$, and $\mathrm{diam}(\varphi(\sigma)) < \epsilon$, for every $\sigma \in K^{(n)}$. By construction, the n-form $\alpha = \varphi^*\Omega - \Omega$ satisfies $\int_\sigma \alpha = 0$ for every n-simplex σ. Therefore it has a primitive β' such that $\int_\tau \beta' = 0$ for every $\tau \in K^{(n-1)}$. That means β' is exact on $K^{(n-1)}$, and there exists $\gamma \in \Omega^{n-2}(M)$ such that $d\gamma = \beta'$ on $K^{(n-1)}$. The primitive $\beta = \beta' - d\gamma$ of α then vanishes on $K^{(n-1)}$. Applying Moser's isotopy method [Mos65] to the smooth family of volume forms $\Omega_t = \Omega + t d\beta$ yields a (time-dependent) vector field X_t such that $\iota(X_t)\Omega = -\beta$. Since M is closed, X_t integrates to an isotopy $\phi_t \in \mathrm{Diff}(M)$ with $\phi_0 = \mathrm{id}$, and $\phi_t^*\Omega_t = \Omega$ for all $0 \le t \le 1$. Then $\psi = \varphi \circ \phi$ preserves the volume form Ω, where $\phi = \phi_1$. Since $\beta|_{K^{(n-1)}} = 0$, we have

$\phi_t = \mathrm{id}$ on $K^{(n-1)}$. Therefore

$$\hat{d}(h,\psi) = \hat{d}(h,\varphi\circ\phi) \le \hat{d}(h,\varphi) + \hat{d}(\varphi,\varphi\circ\phi) \le \overline{d}(h,\varphi) + \max_{\sigma\in K^{(n)}} \operatorname{diam}(\varphi(\sigma)) < 2\epsilon,$$

and

$$\begin{aligned}\hat{d}(h^{-1},\psi^{-1}) &= \hat{d}(h^{-1},\phi^{-1}\circ\varphi^{-1}) = \hat{d}(h^{-1}\circ\varphi,\phi^{-1}) \le \hat{d}(h^{-1}\circ\varphi,\mathrm{id}) + \hat{d}(\mathrm{id},\phi^{-1})\\ &= \hat{d}(h^{-1},\varphi^{-1}) + \hat{d}(\phi,\mathrm{id}) \le \overline{d}(h,\varphi) + \max_{\sigma\in K^{(n)}} \operatorname{diam}(\sigma) < 2\epsilon.\end{aligned}$$

That completes the proof. □

Corollary 2.1.6. *Let M be an orientable surface, and $\omega = \Omega$ be any area form on M, then*

$$\mathrm{Sympeo}(M,\omega) = \mathrm{Homeo}^{\Omega}(M), \qquad \mathrm{Sympeo}_0(M,\omega) = \mathrm{Homeo}_0^{\Omega}(M).$$

These relations are the precise analogs to the relation between $\mathrm{Symp}(M,\omega)$ and $\mathrm{Diff}^{\Omega}(M)$ in dimension two, and higher dimensions, respectively. In this sense, the symplectic homeomorphism group is a ‘good’ *symplectic* generalization of the group of area-preserving homeomorphisms. Examples of non smooth symplectic homeomorphisms, on any general symplectic manifold, which are in addition isotopic to the identity in $\mathrm{Sympeo}(M,\omega)$, will be given in Example 2.4.5 below. Therefore we have the proper inclusion relations $\mathrm{Symp}(M,\omega) \subsetneq \mathrm{Sympeo}(M,\omega)$, and $\mathrm{Symp}_0(M,\omega) \subsetneq \mathrm{Sympeo}_0(M,\omega)$.

Remark 2.1.7. The above definition of a symplectic homeomorphism differs from the ones previously given in the literature, for example in [Bat94, HZ94, MS98], where symplectic homeomorphisms of the (noncompact) symplectic manifold $(\mathbb{R}^{2n},\omega_0)$ are defined in terms of symplectic capacities on $(\mathbb{R}^{2n},\omega_0)$ and $(\mathbb{R}^{2n+2},\omega_0)$. As a consequence of the definition in [Bat94], the derivative of a so-called c-symplectic homeomorphism, where c is some capacity, is symplectic wherever it exists. The groups of c-symplectic homeomorphisms and symplectic homeomorphisms (homeomorphisms that are c-symplectic for all capacities) are closed in the group of all homeomorphisms of $\mathbb{R}^{2n}$ in the C^0-topology. With the definition given in [HZ94], any smooth symplectic homeomorphism is automatically a symplectic or anti-symplectic diffeomorphism. However, it is not clear whether their group is closed under locally uniform limits, except under some additional hypothesis. It is not known whether a homeomorphism preserving some given capacity also preserve Lebesque measure. However, if it preserves all capacities, it is Lebesque measure-preserving. Following [MS98], the group of symplectic homeomorphisms is closed in the group of all homeomorphisms with respect to the C^0-topology, and a smooth symplectic homeomorphism is again a symplectic diffeomorphism. In this case it is also not known whether these homeomorphisms are measure-preserving.

Definition 2.1.1 and the proof of rigidity imply that our notion of symplectic homeomorphism (see Section 3.1 for the precise definition of symplectic homeomorphisms of noncompact manifolds) is stronger than all of the above, that is, a symplectic homeomorphism in our sense satisfies the definitions given in [Bat94, HZ94, MS98]. These notions generalize to general symplectic manifolds, by restricting to capacities of 'small' sets, and using Darboux's theorem. However, it is unclear whether these maps preserve the capacities of 'large' sets as well. See the cited references for more details. It does not seem to be worked out yet what would be the 'correct' definition of a symplectic homeomorphism in terms of capacities. From our point of view, the measure-preserving property is crucial, in particular in the important special case of dimension two.

2.2 Definition and properties of the Hamiltonian topology

In this section, we introduce the Hamiltonian topology on the set of Hamiltonian paths, and construct the group of Hamiltonian homeomorphisms. The definition of the Hamiltonian topology is in part motivated by the following theorem. This is a reformulation of Theorem 6, Chapter 5 in [HZ94], in our general context, which Hofer and Zehnder proved for compactly supported Hamiltonian diffeomorphisms of $(\mathbb{R}^{2n}, \omega_0)$. In the presence of the general energy-capacity inequality [LM95a], their proof can easily be adapted to our general context.

Theorem 2.2.1. *Let $\phi_{H_i} \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M, \omega)$ be a sequence of Hamiltonian paths, ϕ_H be another Hamiltonian path, and $\phi\colon M \to M$ be a function, such that*

(i) $\|\overline{H} \# H_i\| \to 0$, *and*

(ii) $\phi_{H_i}^1 \to \phi$ *uniformly,*

as $i \to \infty$. Then we must have $\phi = \phi_H^1$.

Proof. We first note that ϕ must be continuous since it is the uniform limit of continuous maps $\phi_{H_i}^1$. Suppose the contrary that $\phi \neq \phi_H^1$, or equivalently, $(\phi_H^1)^{-1} \circ \phi \neq \mathrm{id}$. Then we can find a (small) compact ball $B \subset M$, such that

$$B \cap \left((\phi_H^1)^{-1} \circ \phi\right)(B) = \emptyset.$$

Since B, and thus $\left((\phi_H^1)^{-1} \circ \phi\right)(B)$, is compact, and $\phi_{H_i}^1 \to \phi$ uniformly, we have

$$B \cap \left((\phi_H^1)^{-1} \circ \phi_{H_i}^1\right)(B) = \emptyset,$$

for all sufficiently large i. By definition of the Hofer displacement energy e (see [Hof90, LM95a] for the definition), we have $e(B) \leq \|(\phi_H^1)^{-1} \circ \phi_{H_i}^1\|$. Now by the energy-capacity inequality from [LM95a], we know $e(B) > 0$, and hence

$$0 < e(B) \leq \left\|(\phi_H^1)^{-1} \circ \phi_{H_i}^1\right\|,$$

for all sufficiently large i. On the other hand, by hypothesis we have

$$\left\|(\phi_H^1)^{-1} \circ \phi_{H_i}^1\right\| \leq \left\|\overline{H} \# H_i\right\| \to 0.$$

The last two inequalities certainly contradict each other. That completes the proof. $\square$

Hofer and Zehnder's result on $\mathbb{R}^{2n}$ can also be obtained via the methods of [Vit92]. In fact, it is an immediate corollary to Corollary 4.19 in that paper.

Corollary 2.2.2. *Let* $\phi_{H_i} \in \mathcal{P}^{\text{ham}}\text{Symp}(M, \omega)$ *be a sequence of Hamiltonian paths,* ϕ_H *be another Hamiltonian path, and* $\lambda \colon t \mapsto \phi^t$ *be a paths of functions* $\phi^t \colon M \to M$*, such that*

(i) $\left\|\overline{H} \# H_i\right\| \to 0$*, and*

(ii) $\phi_{H_i} \to \lambda$ *uniformly on* $[0,1] \times M$*,*

as $i \to \infty$*. Then we must have* $\lambda = \phi_H$*.*

Proof. Suppose the contrary that $\lambda \neq \phi_H$, i.e. there exists $s \in (0,1]$, such that $\phi^s \neq \phi_H^s$. Then the sequence $t \mapsto \phi_{H_i}^{st}$ of Hamiltonian paths contradicts Theorem 2.2.1. $\square$

What this corollary indicates for the practical purpose is that it is consistent to simultaneously impose both $H_i \to H$ in the metric induced by $\| \cdot \|$, and $\phi_{H_i} \to \phi_H$ in the C^0-metric.

Remark 2.2.3. To put Theorem 2.2.1 into further prospective, remark that the evaluation map ev_1 (1.4) is *not* continuous if we equip $\mathcal{P}^{\text{ham}}\text{Symp}(M, \omega)$ with the Hofer topology, and $\text{Ham}(M, \omega)$ with the C^0-topology. If it were, for every sequence H_i, such that $\|H_i\| \to 0$, we would have $\phi_{H_i}^1 \to \text{id}$. But for any pair of points x, $y \in M$, there is such a sequence with $\phi_{H_i}^1(x) = y$, for all i. In other words, the 'transport energy' of a point from one place to any other place is always zero, that is,

$$\inf\left\{\|H\| \mid \phi_H^1(x) = y\right\} = \inf\left\{\|\phi\| \mid \phi(x) = y\right\} = 0.$$

This fact is expressed very nicely in [Rou07] by saying that the norm $\|\phi\|$ is 'small', or the diffeomorphism ϕ is 'close' to the identity with respect to Hofer's distance, as long as points can be moved smoothly to the position prescribed by ϕ, in a way that at each time of the move, there is no 'big' region of points moving 'too fast' in the same direction.

In particular, the assumption of uniform convergence in Theorem 2.2.1 and its corollary is necessary.

2.2.1 The Hamiltonian topology

We will now define the Hamiltonian topology. The corresponding definition of the Hamiltonian topology in the open case is given in Section 3.1.

Definition 2.2.4. We define the *Hamiltonian topology* on the set $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ of Hamiltonian paths by the one having the following collection of subsets as a subbasis: for $\phi_H \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, and ϵ_1, $\epsilon_2 > 0$, define

$$\mathcal{U}(\phi_H, \epsilon_1, \epsilon_2) = \left\{ \phi_K \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \,\middle|\, \|\overline{H} \# K\| < \epsilon_1,\, \overline{d}(\phi_H, \phi_K) < \epsilon_2 \right\}. \tag{2.1}$$

We define the *Hamiltonian topology* on $\mathrm{Ham}(M,\omega)$ to be the topology induced by the evaluation map (1.4), i.e. the largest (or strongest) topology such that the evaluation map ev_1 is continuous. We denote the resulting topological space by $\mathcal{H}\mathrm{am}(M,\omega)$.

Remark 2.2.5. The sets (2.1) form, in fact, a basis of the Hamiltonian topology. And for fixed $\phi_H \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, the open sets (2.1) form a neighborhood basis of the Hamiltonian topology at ϕ_H. Consider the inclusion

$$\iota_{\mathrm{ham}} \colon \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \longrightarrow \mathcal{P}\mathrm{Homeo}(M),$$

and recall the map

$$\mathrm{Dev} \colon \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \longrightarrow C^\infty_m([0,1] \times M).$$

The Hamiltonian topology on $\mathcal{P}^{\text{ham}}\text{Symp}(M,\omega)$ is nothing but the smallest (or weakest) topology for which the map

$$\iota_{\text{ham}} \times \text{Dev}\colon \mathcal{P}^{\text{ham}}\text{Symp}(M,\omega) \longrightarrow \mathcal{P}\text{Homeo}(M) \times C^{\infty}_{m}([0,1] \ \times \ M)$$

is continuous. Here the target carries the product topology induced by the C^0-topology and the Hofer topology.

It turns out that the Hamiltonian topology on $\mathcal{P}^{\text{ham}}\text{Symp}(M,\omega)$ is metrizable.

Definition 2.2.6. We define a metric d_{ham} on $\mathcal{P}^{\text{ham}}\text{Symp}(M,\omega)$, called the *Hamiltonian metric*, by

$$d_{\text{ham}}(\phi_H,\phi_K) = \left\|\overline{H}\#K\right\| + \overline{d}(\phi_H,\phi_K) = \|H-K\| + \overline{d}(\phi_H,\phi_K).$$

The second equality in the previous line follows from the identity (1.9) above, where we recall that $\overline{H}\#K$ is the unique normalized Hamiltonian that generates the path $\phi_H^{-1}\circ\phi_K$.

Proposition 2.2.7. *The Hamiltonian topology on* $\mathcal{P}^{\text{ham}}\text{Symp}(M,\omega)$ *is equivalent to the metric topology induced by* d_{ham}.

Proof. Let $\phi_H \in \mathcal{P}^{\text{ham}}\text{Symp}(M,\omega)$. As remarked above, a neighborhood basis of the Hamiltonian topology at ϕ_H is given by the sets (2.1), and a neighborhood basis of the metric topology at ϕ_H is given by the metric balls

$$\mathcal{U}(\phi_H,\epsilon) = \left\{\phi_K \in \mathcal{P}^{\text{ham}}\text{Symp}(M,\omega) \mid d_{\text{ham}}(\phi_H,\phi_K) < \epsilon\right\}$$

centered at ϕ_H. But $\mathcal{U}(\phi_H,\frac{\epsilon}{2},\frac{\epsilon}{2}) \subset \mathcal{U}(\phi_H,\epsilon)$, and conversely, if we set $\epsilon = \min(\epsilon_1,\epsilon_2)$, then $\mathcal{U}(\phi_H,\epsilon) \subset \mathcal{U}(\phi_H,\epsilon_1,\epsilon_2)$. $\square$

The way how we define a topology on $\text{Ham}(M,\omega)$, starting from one on the path space $\mathcal{P}^{\text{ham}}\text{Symp}(M,\omega)$, seems quite natural, since the group $\text{Ham}(M,\omega)$ itself is defined in terms of

Hamiltonian paths connecting its elements to the identity. We will repeatedly use this strategy in this work.

Observe for later reference that for a (smooth) Hamiltonian path ϕ_H, the assignment $[0,1] \to \mathcal{H}\mathrm{am}(M,\omega)$, $t \mapsto \phi_H^t$, is continuous with respect to the Hamiltonian topology: this map factors through

$$[0,1] \longrightarrow \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \longrightarrow \mathcal{H}\mathrm{am}(M,\omega), \qquad s \longmapsto \phi_{H^s} \longmapsto \phi_H^s,$$

where the latter map is the evaluation map ev_1. Thus, by definition of the Hamiltonian topology on the set $\mathrm{Ham}(M,\omega)$, it suffices to show that the first map is continuous with respect to the metric d_{ham}. But

$$d_{\mathrm{ham}}(\phi_H^r, \phi_H^s) < C|r-s|,$$

for some constant $C < \infty$ that depends only on H, since the maps $(t,x) \mapsto H(t,x)$, $\phi_H^t(x)$, and $(\phi_H^t)^{-1}(x)$, are Lipschitz continuous. In fact, the next result, due to Banyaga [Ban78], implies that any smooth path in $\mathrm{Ham}(M,\omega)$ is continuous with respect to the Hamiltonian topology, or in short, that smoothness implies continuity with respect to the Hamiltonian topology.

Proposition 2.2.8 ([Ban78],Proposition II.3.3)**.** *Let* $\lambda\colon [0,1] \to \mathrm{Symp}(M,\omega)$ *be a smooth path (or more generally, a* C^1*-path), such that* $\lambda(t) \in \mathrm{Ham}(M,\omega) \subset \mathrm{Symp}(M,\omega)$*, for all* $0 \le t \le 1$*, and denote by* X_t *the vector field*

$$X_t = \frac{d}{dt}\lambda(t) \circ (\lambda(t))^{-1}.$$

Then the closed one-form $\iota(X_t)\omega$ *is exact for all* $t \in [0,1]$*.*

In other words, any smooth path in $\mathrm{Symp}(M,\omega)$, whose image lies in $\mathrm{Ham}(M,\omega)$, is in fact a Hamiltonian path. Note that this statement does not make sense if the path is not at least C^1 in t, i.e. when we consider a continuous path in $\mathrm{Homeo}(M)$, whose image lies in $\mathrm{Ham}(M,\omega)$. As far as I know, it is not known whether one can always approximate a continuous path $\lambda\colon [0,1] \to \mathrm{Ham}(M,\omega) \subset \mathrm{Homeo}(M)$ by a sequence of smooth Hamiltonian paths. More precisely, it is not known in general whether there is a sequence of smooth Hamiltonian functions $H_i\colon [0,1]\times M \to \mathbb{R}$, such that the Hamiltonian paths $t \mapsto \phi^t_{H_i}$ uniformly converge to λ.

Remark 2.2.9. The relation between the Hofer topology and the C^0-topology on $\mathrm{Ham}(M,\omega)$ is rather delicate. It is known however that they are *not* equivalent in general, and in fact, neither one is larger (or stronger) than the other in general. There are many compact manifolds, including all closed orientable surfaces, with infinite Hofer diameter [Pol01, Pol98, LM95b, LM96, Sch00]. Of course the C^0-diameter of these manifolds is finite, so that the C^0-metric is not larger (or stronger) than the Hofer metric on these manifolds. In fact, Polterovich [Pol01] shows that on any closed orientable surface, there exists a sequence of Hamiltonian diffeomorphisms that converges to the identity in the C^0-metric, but diverges in Hofer's metric. (In fact, the corresponding Hamiltonian isotopies in his example also converge to the identity in the C^0-metric.) For noncompact manifolds, a very nice explicit example on $\mathbb{R}^{2n}$, with the standard symplectic form, was constructed by Hofer and Zehnder [HZ94]. Thus the C^0-metric is not larger than the Hofer metric in general. In particular, the Hofer norm function $\phi \mapsto \|\phi\|$ on $\mathrm{Ham}(M,\omega)$ is *not* continuous with respect to the C^0-topology in general. On the other hand, by Remark 2.2.3, there are sequences of Hamiltonian diffeomorphisms, whose Hofer distance to the identity becomes arbitrarily small, while their C^0-distance to the identity remains bounded from below (by the distance of the two points x, y), so that the Hofer topology on $\mathrm{Ham}(M,\omega)$ is not larger than the C^0-topology.

The following remark further clarifies our definition of the Hamiltonian topology.

Remark 2.2.10. It is not too hard to see that the C^∞-topology, or more generally, the C^1-topology on $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, is larger (or stronger) than the Hamiltonian topology defined above. Indeed, fix $t \in [0,1]$, and let x, $y \in M$. Let γ be a geodesic from y to x (recall M is compact and connected), such that $\text{leng}(\gamma) = d(x,y)$, where $\text{leng}(\gamma)$ denotes the length of the path γ and $d(x,y)$ denotes the distance of the two points, both with respect to the Riemannian metric on M induced by some compatible almost complex structure J. Then

$$\begin{aligned} |H_t(x) - H_t(y)| &= \left| \int_0^1 \frac{d}{ds} H_t(\gamma(s)) ds \right| \\ &= \left| \int_0^1 \langle dH_t(\gamma(s)), (\dot{\gamma}(s)) \rangle \, ds \right| \\ &\leq \|\nabla H_t\|_{C^0} \cdot \text{leng}(\gamma) \\ &\leq \|X_H\|_{C^0} \cdot \text{diam}(M). \end{aligned}$$

Taking the maximum on the left hand side over all x, y shows that the oscillation of H_t is bounded by a constant (independent of t) times the C^1-norm of the path ϕ_H. In addition, the C^1-topology is larger (or stronger) than the C^0-topology. But the C^1-topology is too large for the purpose of studying C^0-phenomena in symplectic topology, such as the ones discussed in the introduction.

On the other hand, the Hamiltonian topology on $\mathcal{P}^{\text{ham}}\text{Symp}(M,\omega)$ is larger, and by the previous remark often strictly larger, than the Hofer or C^0-topology alone. The combination of the Hofer topology and the C^0-topology in (2.1) or Definition 2.2.6 will be essential in our study of C^0-analogs to various objects in Hamiltonian geometry and symplectic topology in this work. Such a phenomenon was first indicated by Eliashberg [Eli87], and partly demonstrated by Hofer [Hof90, Hof93] and Viterbo [Vit92]. While the Hofer metric is included in our definition of the Hamiltonian topology for its relevance in Hamiltonian dynamics, the inclusion of the C^0-metric in the definition of the Hamiltonian metric d_{ham} will guarantee that a limit of Hamiltonian paths in this metric lies in $\mathcal{P}\text{Homeo}(M)$.

Other possible 'Hamiltonian topologies' suitable for our purposes are discussed in Section 3.2, where we also hint toward why the C^0-metric alone is not appropriate to study topological Hamiltonian dynamics.

2.3 Topological Hamiltonian paths, topological Hamiltonian functions, and Hamiltonian homeomorphisms

We are now in a position to define the notions of topological Hamiltonian path, topological Hamiltonian function, and Hamiltonian homeomorphism. Consider a sequence H_i of Hamiltonians, generating the Hamiltonian paths $\phi_{H_i} \colon t \mapsto \phi_{H_i}^t$. Denote by $\phi_i = \phi_{H_i}^1 \in \mathrm{Ham}(M,\omega)$ the time-one maps. Suppose the sequence ϕ_{H_i} is Cauchy in the Hamiltonian metric, i.e. $\overline{d}\left(\phi_{H_i}, \phi_{H_j}\right) \to 0$, and $\|H_i - H_j\| \to 0$, as i, $j \to \infty$. Then the sequence ϕ_{H_i} converges in the C^0-metric to a continuous path $\lambda \in \mathcal{P}\mathrm{Homeo}(M)$, and the time-one maps ϕ_i converge in the C^0-metric to a homeomorphism $h = \lambda(1) \in \mathrm{Homeo}(M)$. The functions H_i converge as well, in the L^∞-case to a continuous function $H \in C^0_m([0,1] \times M)$, and in the $L^{(1,\infty)}$-case to an $L^{(1,\infty)}$-function $H \in L^{(1,\infty)}_m([0,1] \times M)$. Recall that for brevity we often denote both $C^0_m([0,1] \times M)$ and $L^{(1,\infty)}_m([0,1] \times M)$ by $H([0,1] \times M)$. We call the continuous path λ a *topological Hamiltonian path*, the function H a *topological Hamiltonian function*, or simply a *topological Hamiltonian*, and the map h a *Hamiltonian homeomorphism.*

More precisely, recall the map

$$\iota_{\mathrm{ham}} \times \mathrm{Dev} \colon \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \longrightarrow \mathcal{P}\mathrm{Homeo}(M) \times H([0,1] \times M), \tag{2.2}$$

defined by $\lambda = \phi_H \mapsto (\lambda, H)$. This is an isometric embedding of $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, with the Hamiltonian metric, into the product $\mathcal{P}\mathrm{Homeo}(M) \times H([0,1] \times M)$, with the product metric $\overline{d} + \|\cdot\|$. We identify the space $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ with its image, and refer to the (subspace) topology as the Hamiltonian topology. We denote the metric $\overline{d} + \|\cdot\|$ when restricted to $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ by d_{ham}, and call it the Hamiltonian metric. To emphasize this identification, we often write an element $\phi_H \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ as a pair

$$(\phi_H, H) \in \mathcal{P}\mathrm{Symp}(M,\omega) \times C^\infty_m([0,1] \times M) \subset \mathcal{P}\mathrm{Homeo}(M) \times H([0,1] \times M).$$

(The path $\lambda = \phi_H$ 'forgets' that it is a Hamiltonian path or even a smooth path, and H that it is a smooth function.) The crucial point is that $\mathcal{P}\mathrm{Homeo}(M) \times H([0,1] \times M)$, with the product metric, is a *complete* metric space.

By Definition 2.2.6, both ι_{ham} and Dev are Lipschitz continuous with respect to d_{ham} on $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, and the C^0-metric $\overline{d}$ on $\mathcal{P}\mathrm{Homeo}(M)$, respectively the metric induced by $\|\cdot\|$ on $H([0,1] \times M)$, with Lipschitz constants $L = 1$. These maps induce natural (Lipschitz continuous) projections from $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \subset \mathcal{P}\mathrm{Homeo}(M) \times H([0,1] \times M)$ onto the first and second factor, still denoted by ι_{ham} and Dev, respectively. The time-one evaluation map

$$\mathrm{ev}_1 \colon \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \longrightarrow \mathrm{Ham}(M,\omega) \subset \mathrm{Homeo}(M),$$

is also Lipschitz continuous, with respect to d_{ham} on $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, and the C^0-metric on $\mathrm{Ham}(M,\omega) \subset \mathrm{Homeo}(M)$, with Lipschitz constant $L = 1$ as well.

We denote by

$$\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \subset \mathcal{P}\mathrm{Homeo}(M) \times H([0,1] \times M) \tag{2.3}$$

the closure (or completion) of $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, with respect to the above product topology (or metric), equipped with the subspace topology, i.e. the metric topology. That is, a pair (λ, H), where $\lambda \colon [0,1] \to \mathrm{Homeo}(M)$ is a continuous path, with $\lambda(0) = \mathrm{id}$, and $H \colon [0,1] \times M \to \mathbb{R}$ is an ($L^{(1,\infty)}$ or continuous) function, lies in $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$, if and only if there is a sequence (ϕ_{H_i}, H_i), where the H_i are normalized Hamiltonians, and the ϕ_{H_i} are the corresponding Hamiltonian paths, such that the sequence H_i converges to H in the norm $\|\cdot\|$, and the sequence ϕ_{H_i} converges to λ in the C^0-metric. We will denote the metric restricted to $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$ by d_{ham} as well. By Lipschitz continuity, all of the above maps extend continuously to the completion. Note that

$$\begin{aligned}\|\mathrm{Tan}(\phi_H) - \mathrm{Tan}(\phi_K)\| &= \|H \circ \phi_H - K \circ \phi_K\| \\ &\leq \|H \circ \phi_H - H \circ \phi_K\| + \|H \circ \phi_K - K \circ \phi_K\| \\ &\leq 2L \cdot \hat{d}(\phi_H, \phi_K) + \|H - K\|,\end{aligned}$$

where $L < \infty$ is a Lipschitz constant that depends on H. So the map Tan is continuous as well. However, since the constant L depends on H, Tan is not Lipschitz continuous in general, and hence it is not at all obvious that it also extends continuously to $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M, \omega)$. We will prove this fact later.

By the preceding discussion, the following definitions are well-defined.

Definition 2.3.1. We denote by

$$\overline{\mathrm{Dev}} \colon \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M, \omega) \longrightarrow H([0,1] \times M), \qquad (\lambda, H) \longmapsto H,$$

the (Lipschitz) continuous extension of Dev, and also call this map the *developing map*. We denote the image of $\overline{\mathrm{Dev}}$ by

$$\mathcal{H}([0,1] \times M) \subset H([0,1] \times M),$$

equipped with the subspace topology, and call an element of $\mathcal{H}([0,1] \times M)$ a *topological Hamiltonian function*, or simply a *topological Hamiltonian*. Then a function $H \in H([0,1] \times M)$ is a topological Hamiltonian, if and only if in $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M, \omega)$ there exists a Cauchy sequence (ϕ_{H_i}, H_i), with respect to the Hamiltonian metric, such that $\|H - H_i\| \to 0$.

Obviously we have

$$C^\infty_m([0,1] \times M) \subset \mathcal{H}^\infty([0,1] \times M) \subset \mathcal{H}^{(1,\infty)}([0,1] \times M).$$

Proposition 2.3.2. *The map $\overline{\mathrm{Dev}} \colon \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M, \omega) \to H([0,1] \times M)$ is injective. That is, if (λ, H) and $(\mu, H) \in \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M, \omega)$, then we have $\lambda = \mu$.*

Proof. By definition of $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$, there exist sequences (ϕ_{H_i}, H_i) and (ϕ_{K_i}, K_i), such that $\phi_{H_i} \to \lambda$, $\phi_{K_i} \to \mu$, in the C^0-metric, and H_i, $K_i \to H$ in the metric $\|\cdot\|$. Applying Corollary 2.2.2 to the sequence $\phi_{H_i}^{-1} \circ \phi_{K_i}$, and the zero Hamiltonian, proves the proposition. □

By this proposition, we may extend the group structure $(\#, \overline{H})$ on $C^\infty_m([0,1] \times M)$ to the space $\mathcal{H}([0,1] \times M)$ of topological Hamiltonians in a natural way: for $H \in \mathcal{H}([0,1] \times M)$, let λ be the unique continuous path such that $(\lambda, H) \in \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$, and for $K \in \mathcal{H}([0,1] \times M)$ another topological Hamiltonian, define the functions $H\#K$ and $\overline{H}$ by

$$(H\#K)_t = H_t + K_t \circ (\lambda_t)^{-1}, \tag{2.4}$$

and

$$(\overline{H})_t = -H_t \circ \lambda_t. \tag{2.5}$$

These functions are well-defined by Proposition 2.3.2, and $C^\infty_m([0,1] \times M)$ becomes a subgroup of $\mathcal{H}([0,1] \times M)$. We have thus proved

Proposition 2.3.3. *The operation $\#$ defines a group structure on $\mathcal{H}([0,1] \times M)$, making $C^\infty_m([0,1] \times M)$ a subgroup. The above extension* $\overline{\mathrm{Dev}}$ *of the isomorphism* Dev *(1.10), is an injective homomorphism, and hence an isomorphism onto its image $\mathcal{H}([0,1] \times M)$.*

Note that the proofs of the two preceding propositions rest on Theorem 2.2.1, and thus on some of the deep 'C^0-type' results cited in the introduction, namely the concepts of the energy of a Hamiltonian diffeomorphism and capacity, and the energy-capacity inequality.

Definition 2.3.4. We denote by

$$\overline{\iota_{\mathrm{ham}}} \colon \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \longrightarrow \mathcal{P}\mathrm{Homeo}(M), \qquad (\lambda, H) \longmapsto \lambda,$$

the (Lipschitz) continuous extension of the map ι_{ham}. By definition of $\mathrm{Sympeo}(M,\omega)$, it follows that the image of $\overline{\iota_{\mathrm{ham}}}$ is contained in the space $\mathcal{P}\mathrm{Sympeo}(M,\omega) \subset \mathcal{P}\mathrm{Homeo}(M)$ of continuous paths $\lambda\colon [0,1] \to \mathrm{Sympeo}(M,\omega)$, with $\lambda(0) = \mathrm{id}$. We denote this image by

$$\mathcal{P}^{\mathrm{ham}}\mathrm{Sympeo}(M,\omega) \subset \mathcal{P}\mathrm{Sympeo}(M,\omega) \subset \mathcal{P}\mathrm{Homeo}(M),$$

equipped with the subspace topology, i.e. the C^0-topology. An element λ of $\mathcal{P}^{\mathrm{ham}}\mathrm{Sympeo}(M,\omega)$ is called a *topological Hamiltonian path*. In other words, a continuous path $\lambda \in \mathcal{P}\mathrm{Homeo}(M)$ is a topological Hamiltonian path, if and only if there exists a Cauchy sequence $(\phi_{H_i}, H_i) \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, with respect to the Hamiltonian metric, such that $\lim_{C^0} \phi_{H_i} = \lambda$.

Clearly

$$\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \subset \mathcal{P}^{\mathrm{ham}}_{\infty}\mathrm{Sympeo}(M,\omega) \subset \mathcal{P}^{\mathrm{ham}}_{(1,\infty)}\mathrm{Sympeo}(M,\omega).$$

Proposition 2.3.2 justifies calling the topological Hamiltonian associated to a topological Hamiltonian path its 'generating Hamiltonian'. There is the following uniqueness theorem, proved by Viterbo [Vit06b, Vit06a] in the L^∞-case, which is the analog to Proposition 2.3.2, concerning the injectivity of the map $\overline{\iota_{\mathrm{ham}}}$.

Theorem 2.3.5 ([Vit06b, Vit06a]). *Consider the Cauchy sequences (ϕ_{H_i}, H_i) and (ϕ_{K_i}, K_i) in the L^∞-Hamiltonian metric, such that $(\phi^t_{H_i})^{-1} \circ (\phi^t_{K_i}) \to \mathrm{id}$, as $i \to \infty$, uniformly over $[0,1] \times M$. Then $\left\|\overline{H}_i \# K_i\right\|_\infty \to 0$, as $i \to \infty$. Therefore, the map $\overline{\iota^\infty_{\mathrm{ham}}}\colon \overline{\mathcal{P}^{\mathrm{ham}}_\infty\mathrm{Symp}}(M,\omega) \to \mathcal{P}\mathrm{Homeo}(M)$ is injective. That is, if (λ, H) and $(\lambda, K) \in \overline{\mathcal{P}^{\mathrm{ham}}_\infty\mathrm{Symp}}(M,\omega)$, then we have $H = K$.*

For open manifolds (see Section 3.1 for the corresponding definitions), a proof of this theorem appears, among many other results, in [Oh07b]. The normalization condition is of course crucial in the above statement. Theorem 2.3.5 together with Proposition 2.3.2 mean that the one-to-one correspondence between Hamiltonian paths and normalized Hamiltonians in the smooth category, *extends* to topological Hamiltonian paths and topological Hamiltonians, at least in the L^∞-case. Recall that many invariants of Hamiltonian paths are defined for their generating Hamiltonian functions, and then for Hamiltonian paths only via this one-to-one correspondence. The extension of the one-to-one correspondence therefore often allows to extend these invariants to topological Hamiltonians. See [Oh07a] for a discussion of several consequences of the uniqueness theorem. The following question is consequently of fundamental importance.

Question 2.3.6 (Uniqueness). Does the analog to Theorem 2.3.5 hold in the $L^{(1,\infty)}$-case?

Definition 2.3.7. We denote by

$$\overline{\mathrm{ev}_1}\colon \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \longrightarrow \mathrm{Homeo}(M), \qquad (\lambda, H) \longmapsto \lambda(1),$$

the (Lipschitz) continuous extension of the evaluation map ev_1. We denote by

$$\mathrm{Hameo}(M,\omega) \subset \mathrm{Homeo}(M)$$

the image of $\overline{\mathrm{ev}_1}$, equipped with the C^0-topology, and call an element thereof a *Hamiltonian homeomorphism*. I.e., a homeomorphism $h \in \mathrm{Homeo}(M)$ is a Hamiltonian homeomorphism, if and only if there exists a Cauchy sequence $(\phi_{H_i}, H_i) \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, with respect to the Hamiltonian metric, such that $h = \lim_{C^0} \phi_{H_i}^1$. We define the *Hamiltonian topology* on the set $\mathrm{Hameo}(M,\omega)$ to be the topology induced by the map $\overline{\mathrm{ev}_1}$, i.e. the largest (or strongest) topology such that $\overline{\mathrm{ev}_1}$ is continuous. We denote the resulting topological space by $\mathcal{H}\mathrm{ameo}(M,\omega)$.

In particular, we have $\mathrm{Ham}(M,\omega) \subset \mathrm{Hameo}(M,\omega) \subset \mathrm{Sympeo}(M,\omega)$ from the definitions. Recall that $\| \cdot \|_{(1,\infty)} \leq \| \cdot \|_\infty$, so any Cauchy sequence in the L^∞-Hamiltonian metric is also a Cauchy sequence in the $L^{(1,\infty)}$-Hamiltonian metric. In particular, $\mathrm{Hameo}_\infty(M,\omega) \subset \mathrm{Hameo}_{(1,\infty)}(M,\omega)$. We will see below that equality in fact holds, see Theorem 2.6.1. This is to a large extend a consequence of Polterovich's Lemma 1.1.2, or more precisely, its proof.

By definition, the map $\overline{\mathrm{ev}_1}\colon \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \to \mathcal{H}\mathrm{ameo}(M,\omega)$ is surjective, continuous, and the following diagram commutes

$$\begin{array}{ccc}
\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) & \longrightarrow & \mathcal{H}\mathrm{am}(M,\omega) \\
\big\downarrow & & \big\downarrow \\
\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) & \longrightarrow & \mathcal{H}\mathrm{ameo}(M,\omega),
\end{array}$$

where the vertical maps are the natural inclusions, and the horizontal maps are induced by the time-one evaluation map.

The way how we define $\mathrm{Hameo}(M,\omega)$, starting from the completion of the path space $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, seems natural, since $\mathrm{Ham}(M,\omega)$ itself is defined in a similar way (recall our earlier remark). In particular, every Hamiltonian homeomorphism is the time-one map of a (topological) Hamiltonian path. Note that $\mathrm{Hameo}(M,\omega)$ is not defined as the closure (or completion) of $\mathrm{Ham}(M,\omega) \subset \mathrm{Homeo}(M)$ with respect to some topology (or metric), compare to Section 3.2.

One crucial advantage of the Hamiltonian topology over the Hofer topology is that it enables one to extend the evaluation map

$$\mathrm{ev}_1 \colon \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \longrightarrow \mathrm{Ham}(M,\omega)$$

to the completion of $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ with respect to the Hamiltonian metric. Recall from Remark 2.2.3 that the evaluation map is not continuous if one equips $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ with the Hofer topology, and $\mathrm{Ham}(M,\omega)$ with the C^0-topology. It is also an interesting problem to understand the (abstract) completion of $\mathrm{Ham}(M,\omega)$ with respect to the Hofer metric, but this is much harder to study, partly because a general element in the completion would not be a continuous map. See [Bat94, Hum07] for some related remarks for compactly supported Hamiltonian diffeomorphisms of $(\mathbb{R}^{2n},\omega_0)$.

From our estimates in the previous chapter, we obtain the following important results, which will be used frequently throughout this work.

Proposition 2.3.8. *Every* $(\lambda, H) \in \overline{\mathcal{P}^{\rm ham}{\rm Symp}}(M,\omega)$ *can be approximated in the* $L^{(1,\infty)}$*-Hamiltonian metric by boundary flat smooth Hamiltonian paths* $(\phi_{H_i}, H_i) \in \mathcal{P}^{\rm ham}{\rm Symp}(M,\omega)$*. If* $H_0 \equiv 0 \equiv H_1$*, the same conclusion holds in the* L^∞*-Hamiltonian metric.*

Proof. The $L^{(1,\infty)}$-case is an immediate consequence of Lemma 1.2.8. The L^∞-case is proved similarly using Lemma 1.2.6. □

That is, any (smooth or topological) Hamiltonian path can be approximated by boundary flat smooth Hamiltonian paths in the $L^{(1,\infty)}$-Hamiltonian metric. This can be done in the L^∞-Hamiltonian metric if and only if $H_0 \equiv 0 \equiv H_1$, so this approximation procedure fails in general in the L^∞-case. As remarked above, this approximation procedure is an important property of the $L^{(1,\infty)}$-Hamiltonian topology.

Proposition 2.3.9. *Suppose* $H_i \in \mathcal{H}([0,1] \times M)$ *is a Cauchy sequence in the metric* $\|\cdot\|$*, and* $\lambda_i \in \mathcal{P}{\rm Homeo}(M)$ *is a Cauchy sequence of continuous paths in the* C^0*-metric. Then the sequence* $H_i \circ \lambda_i$ *is also Cauchy, i.e.*

$$\|H_i \circ \lambda_i - H_j \circ \lambda_j\| \to 0,$$

as $i,\ j \to \infty$*. If* H *denotes the limit of the sequence* H_i*, and* λ *the limit of the sequence* λ_i*, then* $H_i \circ \lambda_i$ *converges to* $H \circ \lambda$ *in the metric* $\|\cdot\|$*.*

Proof. If the H_i are in fact smooth, the statement follows immediately from Lemma 1.2.9. For general H_i, we compute

$$\begin{aligned}\|H_i \circ \lambda_i - H_j \circ \lambda_j\| &\leq \|H_i \circ \lambda_i - H \circ \lambda_i\| + \|H \circ \lambda_i - H \circ \lambda_j\| + \|H \circ \lambda_j - H_j \circ \lambda_j\| \\ &= \|H_i - H\| + \|H \circ \lambda_i - H \circ \lambda_j\| + \|H - H_j\|,\end{aligned}$$

where the first and third term converge to zero by assumption. So it suffices to show that the middle term converges to zero as well. Now since $H \in \mathcal{H}([0,1] \times M)$, there exists a sequence of smooth Hamiltonians K_i, converging to H in the metric $\|\cdot\|$, and thus

$$\begin{aligned}\|H\circ\lambda_i - H\circ\lambda_j\| &\leq \|H\circ\lambda_i - K_i\circ\lambda_i\| + \|K_i\circ\lambda_i - K_j\circ\lambda_j\| + \|K_j\circ\lambda_j - H\circ\lambda_j\| \\ &= \|H - K_i\| + \|K_i\circ\lambda_i - K_j\circ\lambda_j\| + \|K_j - H\| \to 0,\end{aligned}$$

since the first and third term again converge to zero by assumption, and the second term by the smooth version of this proposition proved above. The last statement is then obvious. $\square$

The proposition in fact still holds if we replace $\overline{d}$ by $\hat{d}$ in the hypothesis.

2.3.1 Topological groups

We now prove one of the key properties of the Hamiltonian topology.

Theorem 2.3.10. *The space* $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$ *forms a topological group.*

Proof. We first define composition in $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$ in a way that extends composition in $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, and show that inversion extends the one in $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ as well. The other group properties will follow immediately. We then show that composition and inversion are continuous with respect to the Hamiltonian topology.

Let (λ, H), $(\mu, K) \in \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$. We define composition by

$$(\lambda, H) \circ (\mu, K) = (\lambda \circ \mu, H \# K), \tag{2.6}$$

and note that

$$(\lambda, H)^{-1} = (\lambda^{-1}, \overline{H}) \tag{2.7}$$

is the inverse to (λ, H) with respect to this composition rule. The right hand sides are well-defined by Proposition 2.3.3, and we have to show that they are contained in $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$.

By definition, there are sequences (ϕ_{H_i}, H_i) and (ϕ_{K_i}, K_i), converging to (λ, H) and (μ, K), respectively, with respect to to the metric d_{ham}. That is, $\|H - H_i\|$, $\|K - K_i\| \to 0$, and $\overline{d}(\lambda, \phi_{H_i})$, $\overline{d}(\mu, \phi_{K_i}) \to 0$, as $i \to \infty$. Since $\mathcal{P}\mathrm{Homeo}(M)$ is a topological group with respect to the C^0-topology, $\overline{d}(\lambda \circ \mu, \phi_{H_i} \circ \phi_{K_i}) \to 0$, and $\overline{d}(\lambda^{-1}, \phi_{H_i}^{-1}) \to 0$, as $i \to \infty$. Recall from (1.6) that $H_i \# K_i = H_i + K_i \circ (\phi_{H_i})^{-1}$ generates the path $\phi_{H_i} \circ \phi_{K_i}$, and observe that

$$\|H_i + K_i \circ \phi_{H_i}^{-1} - H - K \circ \lambda^{-1}\| \leq \|H - H_i\| + \|K_i \circ \phi_{H_i}^{-1} - K \circ \lambda^{-1}\| \to 0,$$

as $i \to \infty$. Here the first term converges to zero by assumption, while the second term converges to zero by Proposition 2.3.9. We have thus verified that $(\phi_{H_i} \circ \phi_{K_i}, H_i \# K_i)$ converges to $(\lambda \circ \mu, H \# K)$ in the Hamiltonian metric, and in particular, that the latter is contained in $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$. Moreover, the proof shows that this limit does not depend on the choices of H_i and K_i, but only on (λ, H) and (μ, K), and that the definition of composition in $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$ restricts to composition in $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$.

For the inverse, we note that

$$\left\|\overline{H} - \overline{H}_i\right\| = \|H \circ \lambda - H_i \circ \lambda_i\| \to 0,$$

as $i \to \infty$, again by Proposition 2.3.9. Then the inverse $(\lambda, H)^{-1}$ is well-defined, independent of the choice of sequence (ϕ_{H_i}, H_i), and also restricts to the usual inversion in $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$. This proves that $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$ forms a group: it is straightforward to check that all group axioms are satisfied.

We have to show that the group operations in $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$ are continuous, i.e. that the maps

$$\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \times \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \to \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega),\ ((\lambda, H), (\mu, K)) \mapsto (\lambda \circ \mu, H \# K),$$

and

$$\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \longrightarrow \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega), \qquad (\lambda, H) \longmapsto (\lambda^{-1}, \overline{H}),$$

are continuous with respect to the metric d_{ham}.

For the composition, suppose we have two sequences (λ_i, H_i) and (μ_i, K_i) in the completion $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$, converging to (λ, H) and (μ, K), in the metric d_{ham} on $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$, respectively. We have to show that $\overline{d}(\lambda \circ \mu, \lambda_i \circ \mu_i) \to 0$, and $\|H_i \# K_i - H \# K\| \to 0$, as $i \to \infty$. The C^0-convergence is again immediate, while convergence in $\|\cdot\|$ follows from Proposition 2.3.9, since

$$\begin{aligned} \|H_i \# K_i - H \# K\| &= \|H_i + K_i \circ \lambda_i^{-1} - H - K \circ \lambda^{-1}\| \\ &\leq \|H_i - H\| + \|K_i \circ \lambda_i^{-1} - K \circ \lambda^{-1}\|. \end{aligned}$$

That proves continuity of composition.

For the inverse, we have $\overline{d}(\lambda^{-1}, \lambda_i^{-1}) \to 0$, as $i \to \infty$, and again by Proposition 2.3.9,

$$\left\|\overline{H} - \overline{H}_i\right\| = \|H_i \circ \lambda_i - H \circ \lambda\| \to 0,$$

as $i \to \infty$. That completes the proof. $\square$

From the above proof, we extract the following immediate corollary.

Corollary 2.3.11. *The space* $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M, \omega) \subset \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M, \omega)$ *forms a topological subgroup.*

In particular, in both groups, right and left translations are continuous, and therefore homeomorphisms.

Corollary 2.3.12. *The space* $\mathcal{P}^{\mathrm{ham}}\mathrm{Sympeo}(M, \omega) \subset \mathcal{P}\mathrm{Homeo}(M)$ *forms a topological subgroup.*

The following is an easy consequence that is valid in the context of general topological groups.

Corollary 2.3.13. *Consider the commutative diagram*

$$\begin{array}{ccc} \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M, \omega) & \longrightarrow & \mathcal{H}\mathrm{am}(M, \omega) \\ \downarrow & & \downarrow \\ \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M, \omega) & \longrightarrow & \mathcal{H}\mathrm{ameo}(M, \omega), \end{array}$$

where the horizontal maps are the time-one evaluation maps, and the vertical maps are the obvious continuous inclusion maps. The evaluation maps are surjective, continuous, and open, and thus induce the structure of topological groups on the spaces $\mathcal{H}\mathrm{am}(M, \omega)$ *and* $\mathcal{H}\mathrm{ameo}(M, \omega)$*, so that the evaluation maps become homomorphisms. Left and right translations of a neighborhood basis at the identity* $\mathrm{id} \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M, \omega)$ *or* $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M, \omega)$*, form a neighborhood basis at any path* λ*, as do projections via* ev_1 *or* $\overline{\mathrm{ev}_1}$*, of any neighborhood basis, and right and left translations in* $\mathcal{H}\mathrm{am}(M, \omega)$ *and* $\mathcal{H}\mathrm{ameo}(M, \omega)$*, of any neighborhood basis.*

Proof. The inclusion of $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ in $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$ is obviously continuous by Corollary 2.3.11. Now $\mathcal{H}\mathrm{am}(M,\omega)$ may not be a (topological) subspace of $\mathcal{H}\mathrm{ameo}(M,\omega)$, but the inclusion is nonetheless continuous. In other words, it is easy to see that the Hamiltonian topology on $\mathcal{H}\mathrm{am}(M,\omega)$ is larger than the subspace topology. If $\mathcal{U} \subset \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ is open, then

$$ev_1^{-1}(ev_1(\mathcal{U})) = \bigcup_\lambda L_\lambda(\mathcal{U}),$$

where the union is taken over all Hamiltonian loops, and L_λ denotes left translation by the path λ, is the union of open sets and thus itself open. By definition of the Hamiltonian topology on $\mathcal{H}\mathrm{am}(M,\omega)$, this shows ev_1 is open. Openness of $\overline{\mathrm{ev}_1}$ is proved verbatim. The remaining statements are easily verified. The composition in $\mathcal{H}\mathrm{am}(M,\omega)$ and $\mathcal{H}\mathrm{ameo}(M,\omega)$ is in fact just the usual composition of maps. □

Since as sets, $\mathrm{Hameo}(M,\omega)$ coincides with $\mathcal{H}\mathrm{ameo}(M,\omega)$, we also derive the following

Corollary 2.3.14. *The space* $\mathrm{Hameo}(M,\omega)$ *is a topological subgroup of* $\mathrm{Homeo}(M)$.

Definition 2.3.15. We extend the map $\mathrm{Tan}\colon \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \to C^\infty_m([0,1] \times M)$ to

$$\overline{\mathrm{Tan}}\colon \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \longrightarrow H([0,1] \times M)$$

in the obvious way by

$$(\lambda, H) \longmapsto H \circ \lambda,$$

and again call this map the *tangent map.*

The identity $\mathrm{Tan}(\phi_H) = -\mathrm{Dev}(\phi_H^{-1}) = -\overline{H}$ extends to the identity

$$\overline{\mathrm{Tan}}(\lambda, H) = -\overline{\mathrm{Dev}}((\lambda, H)^{-1}) = -\overline{H}. \tag{2.8}$$

We obtain commutative diagrams

$$\begin{array}{ccc} \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) & \longrightarrow & C^\infty_m([0,1] \times M) \\ \downarrow & & \downarrow \\ \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) & \longrightarrow & H([0,1] \times M), \end{array} \tag{2.9}$$

where the horizontal maps are either the developing or tangent maps, and the vertical maps are the obvious inclusions. In particular, the images of $\overline{\mathrm{Dev}}$ and $\overline{\mathrm{Tan}}$ contain $C^\infty_m([0,1] \times M)$.

Proposition 2.3.16. *The map* $\overline{\mathrm{Tan}}$ *is a continuous extension of the map* Tan. *In particular, all maps in the commutative diagram (2.9) are continuous.*

Proof. The first statement follows immediately from the identity (2.8), combined with continuity of inversion, which was proved in Theorem 2.3.10. The remaining parts of the second statement were already proved above or follow immediately from the definitions. $\square$

2.3.2 Hamilton's equation

The image of the map $\overline{\text{Dev}}$ contains the space $C^{1,1}_m([0,1] \times M)$ of normalized C^1-Hamiltonians with Lipschitz derivative.

Theorem 2.3.17. *The group* $\text{Hameo}(M,\omega)$ *contains all* $C^{1,1}$*-Hamiltonian diffeomorphisms. More precisely, if* ϕ *is the time-one map of Hamilton's equation* $\dot{x}(t) = X_H(t, x(t))$*, for a* C^1*-function* $H\colon [0,1] \times M \to \mathbb{R}$*, such that* X_H *is uniquely integrable (for example, if* $X_H(t,\cdot)$ *is Lipschitz, with Lipschitz constant independent of* $t \in [0,1]$*), then* $(\phi_H, H) \in \overline{\mathcal{P}^{\text{ham}}\text{Symp}}(M,\omega)$*, and in particular* $\phi \in \text{Hameo}(M,\omega)$*.*

Proof. If H is $C^{1,1}$, then by standard existence and uniqueness theorems for Lipschitz vector fields in the theory of ordinary differential equations, X_H integrates to a unique flow ϕ_H. On the other hand, H can be approximated by a sequence of smooth functions $H_i\colon [0,1] \times M \to \mathbb{R}$ in the C^1-topology, so that $\|H - H_i\| \to 0$, as $i \to \infty$, and the Lipschitz vector fields $X_{H_i}(t,x)$ converge to $X_H(t,x)$, uniformly over $t \in [0,1]$ and $x \in M$. Therefore, the flows $\phi_{H_i} \to \phi_H$, and in particular $\phi^1_{H_i} \to \phi^1_H$, in the C^0-metric, by standard continuity theorem in the theory of ordinary differential equations. Thus (ϕ_{H_i}, H_i) is a Cauchy sequence in $\mathcal{P}^{\text{ham}}\text{Symp}(M,\omega)$, converging to (ϕ_H, H), with $\lim_{C^0} \phi^1_{H_i} = \phi^1_H = \phi$, which implies $\phi \in \text{Hameo}(M,\omega)$. $\square$

Note that this theorem gives rise to non smooth topological Hamiltonians, and topological Hamiltonian paths $\lambda \in \mathcal{P}^{\text{ham}}\text{Sympeo}(M,\omega)$ that are not C^1 with respect to t. Therefore we have

$$C^\infty_m([0,1] \times M) \subsetneq \mathcal{H}([0,1] \times M)$$

and

$$\mathcal{P}^{\text{ham}}\text{Symp}(M,\omega) \subsetneq \mathcal{P}^{\text{ham}}\text{Sympeo}(M,\omega).$$

However, we cannot conclude that the time-one map of λ is non smooth (in the space variable).

We refer to Example 2.4.5 below for the construction of such non smooth Hamiltonian homeomorphisms. I do not know whether the images of the tangent and developing maps contain the whole $C^0_m([0,1] \times M)$.

Note that Hamilton's equation

$$\dot{x}(t) = X_H(t, x(t)), \tag{2.10}$$

does not make sense in general even for C^1-functions, because their Hamiltonian vector field would be only C^0, and so their flow ϕ_H may not exist. We now 'extend' Hamilton's equation to Hamiltonians with less regularity in the following sense: given an arbitrary function H, a continuous path λ is by definition a solution of (2.10), if there exists a sequence of smooth Hamiltonian functions H_i that converges to H in the norm $\|\cdot\|$, and such that the sequence ϕ_{H_i} of Hamiltonian paths converges in the C^0-metric to λ.

By definition, a solution for Hamilton's equation exists, if and only if H is a topological Hamiltonian, and the solution λ is nothing but a topological Hamiltonian path. By Theorem 2.3.17, every classical solution is a solution in the 'extended' sense. Moreover, by Corollary 2.3.2, solutions to this 'extended' differential equation (2.10) are unique. The question naturally arises which functions admit solutions. For example, does every continuous function admit such a solution, or at least every autonomous continuous function? Recall that this amounts to studying the image of the extension of the developing map.

2.4 Basic properties of the group of Hamiltonian homeomorphisms

In this section, we derive some basic properties of the group $\mathrm{Hameo}(M,\omega)$. We first recall that from their definitions

$$\mathrm{Ham}(M,\omega) \subset \mathrm{Hameo}(M,\omega) \subset \mathrm{Sympeo}(M,\omega).$$

In this section we begin the study of these inclusions.

The following theorem is the C^0-version of the well-known fact that $\mathrm{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Symp}(M,\omega)$.

Theorem 2.4.1. $\mathrm{Hameo}(M,\omega)$ *is a normal subgroup of* $\mathrm{Sympeo}(M,\omega)$*.*

Proof. We have to show $\psi^{-1} \circ h \circ \psi \in \mathrm{Hameo}(M,\omega)$, for any $h \in \mathrm{Hameo}(M,\omega)$, and $\psi \in \mathrm{Sympeo}(M,\omega)$. By definition, there are Cauchy sequences $(\phi_{H_i}, H_i) \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, with $\phi_i = \phi^1_{H_i}$, and $\psi_i \in \mathrm{Symp}(M,\omega)$, such that $h = \lim_{C^0} \phi_i$, and $\lim_{C^0} \psi_i = \psi$. Recall from (1.8) that $\psi_i^{-1} \circ \phi_{H_i} \circ \psi_i$ is generated by the Hamiltonian $H_i \circ \psi_i$, for all i. It therefore suffices to prove that $(\psi_i^{-1} \circ \phi_{H_i} \circ \psi_i, H_i \circ \psi_i)$ is a Cauchy sequence in $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, and $\lim_{C^0} \psi_i^{-1} \circ \phi_i \circ \psi_i = \psi^{-1} \circ h \circ \psi$. The C^0-convergence of the paths, and the time-one maps, is immediate. On the other hand, $\|H_i \circ \psi_i - H_j \circ \psi_j\| \to 0$, as i, $j \to \infty$, by Proposition 2.3.9. That completes the proof. □

The following is an important property of $\mathrm{Hameo}(M,\omega)$, which indicates that it is a 'good' C^0-counterpart to $\mathrm{Ham}(M,\omega)$.

Theorem 2.4.2. *The spaces* $\mathcal{H}\mathrm{am}(M,\omega)$ *and* $\mathcal{H}\mathrm{ameo}(M,\omega)$ *are path connected and locally path connected. Consequently,* $\mathrm{Hameo}(M,\omega)$ *is path connected, and we have*

$$\mathrm{Hameo}(M,\omega) \subset \mathrm{Sympeo}_0(M,\omega) \subset \mathrm{Sympeo}(M,\omega) \cap \mathrm{Homeo}_0^\Omega(M).$$

Proof. We only prove the statements about Hamiltonian homeomorphisms. The proof for $\mathcal{H}\mathrm{am}(M,\omega)$ is essentially the same.

Let $h \in \mathcal{H}\mathrm{ameo}(M,\omega)$. For path connectedness of $\mathcal{H}\mathrm{ameo}(M,\omega)$, it suffices to prove that h can be connected to the identity by a path $\ell\colon [0,1] \to \mathcal{H}\mathrm{ameo}(M,\omega)$, such that $\ell(0) = \mathrm{id}$, $\ell(1) = h$, and ℓ is continuous with respect to the Hamiltonian topology.

By definition, there exists a sequence $(\phi_{H_i}, H_i) \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, converging to an element $(\lambda, H) \in \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$, with $h = \overline{\mathrm{ev}_1}(\lambda, H) = \lambda(1) = \lim_{C^0} \phi^1_{H_i}$. Consider the Hamiltonians H_i^s, generating the Hamiltonian paths $t \mapsto \phi^t_{H_i^s} = \phi^{st}_{H_i}$, for all $s \in [0,1]$, and all i (see Section 1.2). We have

$$\overline{d}(\phi_{H_i^s}, \phi_{H_j^s}) = \max_{t\in[0,1]} \overline{d}(\phi^t_{H_i^s}, \phi^t_{H_j^s}) = \max_{t\in[0,s]} (\phi^t_{H_i}, \phi^t_{H_j}) \leq \overline{d}(\phi_{H_i}, \phi_{H_j}) \to 0,$$

and

$$\|H_i^s - H_j^s\| \leq \|H_i - H_j\| \to 0,$$

as i, $j \to \infty$. In the L^∞-case this follows from an argument similar to the case of $\overline{d}$ above, and in the $L^{(1,\infty)}$-case from a simple change of variables, namely $\tau = st$. So $(\phi_{H_i^s}, H_i^s)$ is Cauchy in the Hamiltonian metric. Denote by $(\lambda^s, H^s) \in \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$ its limit, and note that λ^s is nothing but the path $t \mapsto \lambda(st)$. By the above considerations, $\ell(s) = \overline{\mathrm{ev}_1}(\lambda^s, H^s) = \lambda(s) \in \mathcal{H}\mathrm{ameo}(M,\omega)$, for all $s \in [0,1]$, and $\ell(0) = \mathrm{id}$, $\ell(1) = h$.

It only remains to show that ℓ is continuous with respect to the Hamiltonian topology. Note that ℓ factors through

$$[0,1] \longrightarrow \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \longrightarrow \mathcal{H}\mathrm{ameo}(M,\omega), \qquad s \longmapsto (\lambda^s, H^s) \longmapsto \ell(s),$$

where the second map is the time-one evaluation map $\overline{\mathrm{ev}_1}$. By definition of the Hamiltonian topology on $\mathcal{H}\mathrm{ameo}(M,\omega)$, it suffices to show that the first map is continuous, that is, that $s \mapsto (\lambda^s, H^s)$ is continuous, with respect to the standard metric on $[0,1]$, and the Hamiltonian metric d_{ham} on $\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$. Let $\epsilon > 0$. Consider, for r, $s \in [0,1]$, the functions $\zeta_1(t) = tr$

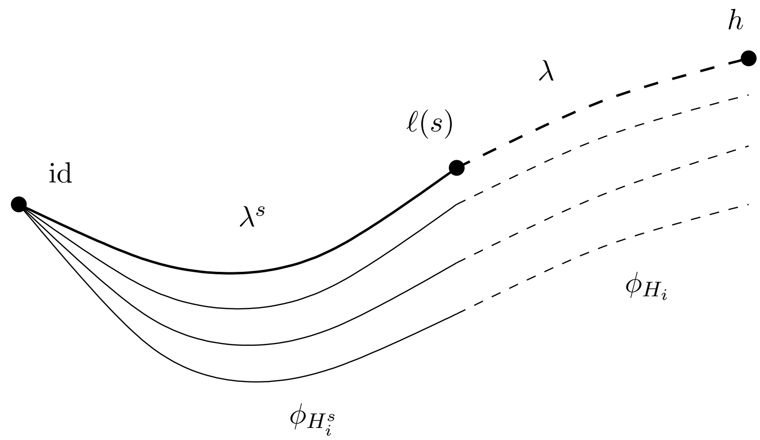

Figure 2: A continuous path in $\text{Hameo}(M, \omega)$ connecting h to the identity

and $\zeta_2(t) = ts$, which satisfy $\|\zeta_1 - \zeta_2\|_{\text{ham}} = 2|r - s|$. By Lemma 1.2.8, there exist $\delta > 0$ and $i_0 > 0$, such that, if $|r - s| < \delta$, then

$$\|H_i^r - H_i^s\|_{(1,\infty)} < \frac{\epsilon}{2},$$

for all $i \geq i_0$. In the special case of functions ζ_1, ζ_2 considered here, the same conclusion in fact also holds in the L^∞-case, see Lemma 1.2.1 for the required estimate. Thus

$$\|H^r - H^s\| = \lim_{i \to \infty} \|H_i^r - H_i^s\| < \frac{\epsilon}{2},$$

provided $|r - s| < \delta$. Since $(t, x) \mapsto \lambda_t(x)$ and $(t, x) \mapsto \lambda_t^{-1}(x)$ are uniformly continuous functions on $[0, 1] \times M$, we find, by making $\delta > 0$ smaller if necessary,

$$\overline{d}(\lambda^r, \lambda^s) < \frac{\epsilon}{2},$$

for all r, $s \in [0, 1]$ satisfying $|r - s| < \delta$. Then

$$d_{\text{ham}}\Big((\lambda^r, H^r), (\lambda^s, H^s)\Big) < \epsilon,$$

for all r, $s \in [0, 1]$ with $|r - s| < \delta$. That proves (uniform) continuity of ℓ, and hence completes the proof of path connectedness of $\mathcal{H}\text{ameo}(M, \omega)$.

In fact, ℓ is a topological Hamiltonian path. By replacing H by H^ζ, and H_i by H_i^ζ, for some fixed reparameterization function $\zeta\colon [0,1] \to [0,1]$, with $\zeta \equiv 0$ near $t = 0$ and $\zeta \equiv 1$ near $t = 1$, we can connect any $h \in \mathcal{H}\mathrm{ameo}(M,\omega)$ to the identity by a boundary flat topological Hamiltonian path. The concatenation of two such topological Hamiltonian paths is again a topological Hamiltonian path. That shows that any two Hamiltonian homeomorphisms can be connected inside $\mathcal{H}\mathrm{ameo}(M,\omega)$ by a topological Hamiltonian path.

Since as a set, $\mathrm{Hameo}(M,\omega)$ coincides with $\mathcal{H}\mathrm{ameo}(M,\omega)$, and any topological Hamiltonian path is continuous with respect to the C^0-topology, we have also proved path connectedness of $\mathrm{Hameo}(M,\omega)$. The remaining statements about $\mathrm{Hameo}(M,\omega)$ are obvious.

For local path connectedness, recall that $\mathcal{H}\mathrm{ameo}(M,\omega)$ is a topological group, so it suffices to show that it is locally path connected at the identity. Applying the above argument to the neighborhood basis $\overline{\mathrm{ev}_1}(\mathcal{U}(\mathrm{id},\epsilon))$ of the identity, where we recall that $\mathcal{U}(\mathrm{id},\epsilon) \subset \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$ denotes the metric ball, in the Hamiltonian metric d_{ham}, of radius $\epsilon > 0$, centered at the identity, completes the proof. $\square$

Question 2.4.3. Is the space $\mathrm{Hameo}(M,\omega)$ locally path connected?

Note that the argument in the proof of Theorem 2.4.2 fails in this case, since the time-one evaluation map is not necessarily open with respect to the C^0-topology.

By the above theorem, we have $\mathrm{Hameo}(M,\omega) \subset \mathrm{Sympeo}_0(M,\omega)$, but a priori it is not clear whether this inclusion is proper. We point out the following situation in which $\mathrm{Hameo}(M,\omega)$ is a proper subgroup of $\mathrm{Sympeo}_0(M,\omega)$. This question will be taken up again in Section 2.5.

Theorem 2.4.4. *Any C^0-limit (or more generally, any limit in the metric $\hat{d}$) of Hamiltonian diffeomorphism has a fixed point. In particular, any Hamiltonian homeomorphism has a fixed point. Therefore, if (M,ω) carries a symplectic diffeomorphism $\psi \in \mathrm{Symp}_0(M,\omega)$ (or*

$\mathrm{Sympeo}_0(M,\omega)$*) that has no fixed point, then* $\psi \notin \mathrm{Hameo}(M,\omega)$*, and in particular we have*

$$\mathrm{Hameo}(M,\omega) \subsetneq \mathrm{Sympeo}_0(M,\omega).$$

Proof. Let $h = \lim_{C^0} \phi_i$, for a sequence $\phi_i \in \mathrm{Ham}(M,\omega)$. We prove the theorem by contradiction. Suppose h has no fixed point. Define $m = \inf_{x\in M} d(x, h(x)) > 0$, by compactness of M. But each ϕ_i must have a fixed point x_i by the Arnold Conjecture, which was proved in [FO99, LT98, Rua99]. Hence

$$\overline{d}(h,\phi_i) \geq d(h(x_i),\phi_i(x_i)) = d(h(x_i),x_i) \geq m > 0,$$

for all i, which gives rise to a contradiction, since we assumed $\overline{d}(h,\phi_i) \to 0$ as $i \to \infty$. ☐

An example of a closed symplectic manifold (M,ω), satisfying the hypotheses of the theorem, is the torus T^{2n}, with the standard symplectic form ω_0. Recall that we may identify T^{2n} with a subgroup of $\mathrm{Symp}_0(T^{2n},\omega_0) \subset \mathrm{Diff}(T^{2n})$. Then by Theorem 2.4.4,

$$T^{2n} \cap \mathrm{Hameo}(T^{2n},\omega_0) = \{\mathrm{id}\},$$

and thus $\mathrm{Hameo}(T^{2n},\omega_0) \subsetneq \mathrm{Sympeo}_0(T^{2n},\omega_0)$. More generally, the same argument applies to any closed Lie group G that admits a G-invariant symplectic structure.

Similarly, if $\mathrm{Ham}(M,\omega) \subsetneq \mathrm{Symp}_0(M,\omega)$, and if $\mathrm{Ham}(M,\omega)$ is C^0-closed in $\mathrm{Symp}_0(M,\omega)$, or in other words, if the C^0-Flux conjecture holds for (M,ω), then of course $\mathrm{Hameo}(M,\omega) \subsetneq \mathrm{Sympeo}_0(M,\omega)$. A list of manifolds for which the C^0-Flux conjecture holds can be found in [LMP98].

2.4.1 Examples of continuous Hamiltonian homeomorphisms

From the definition of Hamiltonian homeomorphisms, we have $\mathrm{Ham}(M,\omega) \subset \mathrm{Hameo}(M,\omega)$, and moreover, by Theorem 2.3.17, every Hamiltonian diffeomorphism generated by a $C^{1,1}$-Hamiltonian function is contained in $\mathrm{Hameo}(M,\omega)$. We now provide examples of Hamiltonian homeomorphisms on any symplectic manifold (M,ω) that are not differentiable or even Lipschitz. Therefore, we get the proper inclusion relation

$$\mathrm{Ham}(M,\omega) \subsetneq \mathrm{Hameo}(M,\omega) \subset \mathrm{Sympeo}_0(M,\omega),$$

for any symplectic manifold (M,ω).

Example 2.4.5. Let (x,y) be rectangular, and (r,θ) be polar coordinates, on the unit disc $D^2 \subset \mathbb{R}^2$, such that $x = r\cos\theta$ and $y = r\sin\theta$. The standard area or symplectic form $\Omega = \omega_0$ is given by $\Omega = dx \wedge dy = r dr \wedge d\theta$. Refer to Section 3.1 for the definition of Hamiltonian homeomorphisms in the case of symplectic manifolds with boundary.

We consider maps $\phi_\rho \colon D^2 \to D^2$ of the form

$$\phi_\rho(r,\theta) = \begin{cases} (r, \theta + \rho(r)) & \text{if } r \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where 0 denotes the origin in $\mathbb{R}^2$, and $\rho \colon (0,1] \to \mathbb{R}$ is a continuous function. In rectangular coordinates, this map is given by

$$\phi_\rho(x,y) = \begin{cases} \Big(x\cos\rho(r) - y\sin\rho(r), y\cos\rho(r) + x\sin\rho(r)\Big) & \text{if } (x,y) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $r = \sqrt{x^2+y^2}$.

Clearly, if ρ is C^k, $0 \leq k \leq \infty$, then ϕ_ρ is C^k everywhere on D^2 except possibly at the origin. However, ϕ_ρ is always continuous at the origin, and hence everywhere on D^2. One readily computes that ϕ_ρ is differentiable at the origin, if and only if ρ continuously extends over $[0,1]$ to a continuous function $\overline{\rho}\colon [0,1] \to \mathbb{R}$, and in that case, $d\phi_\rho(0)$ is the rotation about the origin through the angle $\overline{\rho}(0)$. Moreover, if ρ is C^1 and extends to a C^1 function $\overline{\rho}\colon [0,1] \to \mathbb{R}$, then ϕ_ρ is C^1 everywhere on D^2. On the other hand, if ρ is differentiable but $\rho'(r)$ is unbounded, then ϕ_ρ is not C^1 near the origin.

Obviously $\phi_{-\rho}$ is the inverse of ϕ_ρ, so that ϕ_ρ is a homeomorphism. And ϕ_ρ is compactly supported in the interior of the disc, if and only if $\rho \equiv 0 \pmod{2\pi}$ near $r = 1$. From now on, assume $\rho \equiv 0$ near $r = 1$. If ρ is C^1, a direct computation shows that $\phi_\rho^*(rdr \wedge d\theta) = rdr \wedge d\theta$ on $D^2 - \{0\}$. In particular, ϕ_ρ preserves the measure induced by Ω, so that $\phi_\rho \in \mathrm{Homeo}^\Omega(D^2, \partial D^2)$.

For now, consider $\rho \in C^\infty((0,1])$, and assume it extends smoothly to $\overline{\rho}\colon [0,1] \to \mathbb{R}$. For simplicity, assume ρ is constant near $r = 0$. Then ϕ_ρ is smooth on D^2, and in fact, $\phi_\rho \in \mathrm{Symp}(D^2, \partial D^2, \omega_0) = \mathrm{Ham}(D^2, \partial D^2, \omega_0)$ is a Hamiltonian diffeomorphism of the unit disc (since a smooth measure-preserving map preserves the corresponding area form) that is compactly supported in the interior of the disc. The time-independent Hamiltonian $H_\rho\colon D^2 \to \mathbb{R}$ given by

$$H_\rho(r,\theta) = -\int_1^r s\overline{\rho}(s)ds,$$

generates the Hamiltonian isotopy $t \mapsto \phi_{t\rho}$, with time-one map ϕ_ρ. If we in addition assume $\rho \geq 0$, i.e. the rotation is always counterclockwise, then the norm of H_ρ becomes

$$\|H_\rho\|_{(1,\infty)} = \|H_\rho\|_\infty = \int_0^1 s\overline{\rho}(s)ds.$$

We now show that for a suitable choice of ρ, the map ϕ_ρ is a Hamiltonian homeomorphism that is neither differentiable nor Lipschitz on D^2. Namely, choose ρ such that it is smooth on $(0,1]$, with $\rho \equiv 0$ near $r = 1$, $\rho \geq 0$ and $\rho' \leq 0$ near $r = 0$, and

$$\int_{0^+}^1 s\rho(s)ds < \infty, \tag{2.11}$$

but

$$\lim_{r\to 0^+} \rho(r) = +\infty, \qquad \lim_{r \to 0^+} r\rho'(r) = -\infty.$$

For example, take $\rho(r) = 1/\sqrt{r}$ near $r = 0$, or more generally, $\rho(r) = r^{-2+\epsilon}$ for $0 < \epsilon < 2$. Then choose a sequence $\rho_n \in C^\infty([0,1])$ approximating ρ uniformly on compact subsets of $(0,1]$. In fact, we can find ρ_n so that $\rho_n(r) = \rho(r)$ for $r > 1/n$, and $0 \leq \rho_n \leq \rho_{n+1} \leq \rho$ for all n.

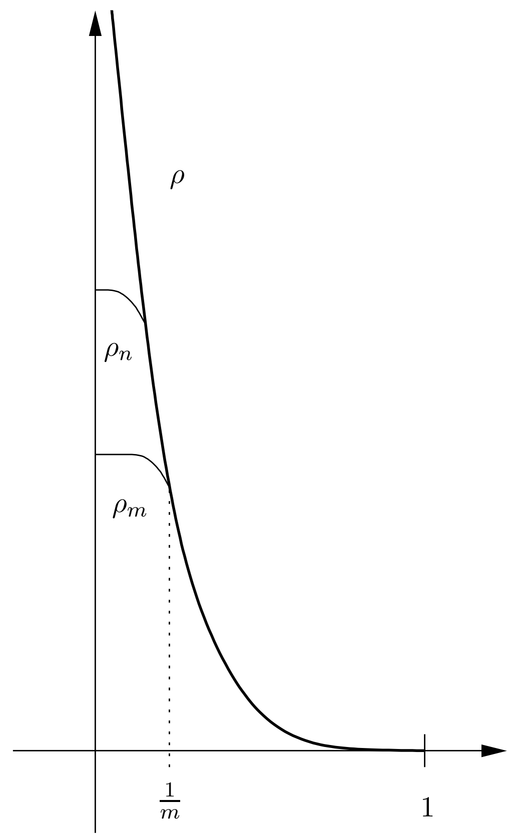

Figure 3: An approximating sequence ρ_n

Then clearly

$$\overline{d}(\phi_{\rho_n}, \phi_{\rho_m}) \leq \frac{2}{m} \to 0,$$

as $n \geq m \to \infty$, and moreover,

$$\|H_{\rho_n} - H_{\rho_m}\| = \int_0^{\frac{1}{m}} s(\rho_n - \rho_m)ds \leq \int_{0^+}^{\frac{1}{m}} s\rho(s)ds \to 0,$$

as $n \geq m \to \infty$, by the finiteness assumption (2.11). That is, the sequence $(\phi_{H_{\rho_n}}, H_{\rho_n})$

is Cauchy in the Hamiltonian metric. Since $\phi^1_{H_{\rho n}} \to \phi_\rho$ as $n \to \infty$, we have proved $\phi_\rho \in \mathrm{Hameo}(D^2, \partial D^2, \omega_0)$ is a Hamiltonian homeomorphism.

However, it follows from the assumptions on ρ and the above discussion that ϕ_ρ is not differentiable. Moreover, for any $L > 0$, by the mean value theorem and our assumptions on ρ, there exist $r > s > 0$, such that $|r - s| < 2s/L$, and

$$|\phi_\rho(r, \theta) - \phi_\rho(s, \theta)| = r + s > 2s > L|r - s|,$$

by choosing r, s so that $\rho(s) - \rho(r) = \pi \pmod{2\pi}$. So ϕ_ρ is *not* Lipschitz.

Finally note that we can also choose a function ρ that is only piecewise smooth on $(0, 1]$, and so for every $c \in (0, 1)$ such that ρ is not differentiable at $r = c$, the corresponding Hamiltonian homeomorphism ϕ_ρ is not differentiable on the circle $\{r = c\}$. In fact, to ensure that the sequence $(\phi_{H_{\rho n}}, H_{\rho_n})$ is Cauchy in the Hamiltonian metric, it suffices to assume the finiteness condition (2.11), and in addition that $\rho_n \nearrow \rho$ uniformly on compact subsets of $(0, 1]$.

Let Σ be any orientable surface. Choose $D^2 \subset \Sigma$ inside the domain of some Darboux chart, and define a homeomorphism h of Σ by $h = \phi_\rho$ on D^2, and $h = \mathrm{id}$ outside D^2, where ϕ_ρ is constructed as above. If Σ is closed, we may have to adjust H_{ρ_n} by adding a constant, so that it is normalized. This does not affect the convergence since the 'normalization constants' converge as well. Then h is a Hamiltonian homeomorphism that is neither differentiable nor Lipschitz.

Let (M^{2n}, ω) be any symplectic manifold of dimension $2n$. By choosing $D^2(\epsilon) \times D^{2n-2}(\epsilon)$, or $D^2(\epsilon) \times \ldots \times D^2(\epsilon)$, inside the domain of some Darboux chart in M, we can construct similar examples on (M, ω). Of course, we can also consider h such that $h \neq \mathrm{id}$ on more than one (in fact, infinitely many) such domains.

The preceding example proves

$$\mathrm{Ham}(M, \omega) \subsetneq \mathrm{Hameo}(M, \omega) \subset \mathrm{Sympeo}_0(M, \omega)$$

on any general symplectic manifold (M, ω), where we refer to Section 3.1 for the precise definition of Hamiltonian homeomorphisms in the case of noncompact manifolds or manifolds with nonempty boundary. Moreover, the Hamiltonian path λ and its generating Hamiltonian are non smooth as well, so that we have the proper inclusions

$$\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M, \omega) \subsetneq \mathcal{P}^{\mathrm{ham}}\mathrm{Sympeo}(M, \omega)$$

and

$$C_m^\infty([0,1] \times M) \subsetneq \mathcal{H}([0,1] \times M).$$

The above example can be modified, so that the (then time-dependent) Hamiltonian is discontinuous in t, and therefore

$$\mathcal{H}_\infty([0,1] \times M) \subsetneq \mathcal{H}_{(1,\infty)}([0,1] \times M).$$

For example, take $K(t, r, \theta) = +H(r, \theta)$ for $0 \leq t \leq 1/2$, and $K(t, r, \theta) = -H(r, \theta)$ for $1/2 < t \leq 1$, corresponding to the loop $t \mapsto \phi_{t\rho}$ for $0 \leq t \leq 1/2$, and $t \mapsto \phi_{(1-t)\rho}$ for $1/2 < t \leq 1$. It is easy to see that K can be approximated by smooth Hamiltonians in the $\| \cdot \|_{(1,\infty)}$-norm, and the corresponding Hamiltonian paths converge to the above loop in the C^0-metric. An affirmative answer to Question 2.3.6 would yield the analogous statement for topological Hamiltonian paths.

The following question seems to be one of fundamental importance.

Question 2.4.6. In Example 2.4.5, consider ρ such that

$$\int_{0^+}^1 s\rho(s)ds = +\infty.$$

Is the homeomorphism ϕ_ρ still contained in $\mathrm{Hameo}(M, \omega)$?

Note that such a homeomorphism always lies in the C^0-closure of $\mathrm{Ham}(M, \omega)$ in $\mathrm{Homeo}(M)$. An example of a ϕ_ρ that is not contained in $\mathrm{Hameo}(M, \omega)$ would imply $\mathrm{Hameo}(M, \omega) \subsetneq \mathrm{Sympeo}_0(M, \omega)$, and that $\mathrm{Hameo}(M, \omega)$ is not C^0-closed in $\mathrm{Sympeo}_0(M, \omega)$.

2.5 Mass flow and flux homomorphisms

We will use the mass flow homomorphism and the flux homomorphisms to study the relation between $\mathrm{Sympeo}_0(M, \omega)$ and its (normal) subgroup $\mathrm{Hameo}(M, \omega)$. This will also lead to a conjecture about the kernel of the mass flow in the case of orientable surfaces.

2.5.1 The flux homomorphisms

We briefly review the definitions of the flux homomorphisms for symplectic and volume-preserving diffeomorphisms given in [Cal70, Ban78, Thu73], see also [Ban97, MS98].

Recall that $\mathrm{Symp}(M, \omega)$ denotes the group of symplectic diffeomorphisms, and $\mathrm{Symp}_0(M, \omega)$ the path component of the identity in $\mathrm{Symp}(M, \omega)$, equipped with the C^∞-topology. Denote by $\mathcal{P}\mathrm{Symp}(M, \omega)$ the set of symplectic isotopies, i.e. smooth paths $\lambda\colon [0, 1] \to \mathrm{Symp}(M, \omega)$, with $\lambda(0) = \mathrm{id}$. Each $\lambda \in \mathcal{P}\mathrm{Symp}(M, \omega)$ defines a smooth map $\Lambda\colon [0, 1] \times M \to M$, by $\Lambda(t, \cdot) = \lambda(t)$, and we give $\mathcal{P}\mathrm{Symp}(M, \omega)$ the C^∞-topology as a subspace of $C^\infty([0, 1] \times M, M)$. This naturally forms a group. Recall that if

$$X_t = \frac{d}{dt}\lambda_t \circ \lambda_t^{-1}$$

denotes the infinitesimal generator of a path $\lambda\colon [0, 1] \to \mathcal{P}\mathrm{Diff}(M)$, then $\lambda \in \mathcal{P}\mathrm{Symp}(M, \omega)$, if and only if $\mathcal{L}_{X_t}\omega = 0$, where $\mathcal{L}$ denotes the Lie derivative, or equivalently (by Cartan's formula), $\iota(X_t)\omega$ is a closed 1-form, for all $0 \le t \le 1$. Recall that $\mathrm{Symp}(M, \omega)$ is locally contractible [Wei71], and so the universal covering space $\widetilde{\mathrm{Symp}}_0(M, \omega)$ of $\mathrm{Symp}_0(M, \omega)$ is represented by isotopy classes, relative to fixed end points, of smooth paths $\lambda \in \mathcal{P}\mathrm{Symp}(M, \omega)$. We denote the equivalence class represented by a path λ by $[\lambda]$. We give $\widetilde{\mathrm{Symp}}_0(M, \omega)$ the quotient topology and the group structure induced by the obvious map

$$\mathcal{P}\mathrm{Symp}(M, \omega) \longrightarrow \widetilde{\mathrm{Symp}}_0(M, \omega),$$

and denote by

$$p\colon \widetilde{\mathrm{Symp}_0}(M,\omega) \longrightarrow \mathrm{Symp}_0(M,\omega)$$

the covering projection induced by the time-one evaluation map. This is a surjective continuous group homomorphism.

To define the flux homomorphism

$$\mathcal{P}\mathrm{Symp}(M,\omega) \longrightarrow H^1(M,\mathbb{R}),$$

recall that for a symplectic isotopy $\lambda \in \mathcal{P}\mathrm{Symp}(M,\omega)$, the 1-form $\iota(X_t)\omega$ is closed for all $0 \leq t \leq 1$, where X_t is the infinitesimal generator of the path λ. Then we define

$$\mathrm{Flux}_\omega(\lambda) = \left[\int_0^1 \iota(X_t)\omega dt\right].$$

This depends only on the isotopy class, relative to the end points, of the path λ, and therefore projects down to the universal covering space $\widetilde{\mathrm{Symp}_0}(M,\omega)$ of $\mathrm{Symp}_0(M,\omega)$. We obtain a surjective continuous homomorphism

$$\mathrm{Flux}_\omega\colon \widetilde{\mathrm{Symp}_0}(M,\omega) \longrightarrow H^1(M,\mathbb{R}),$$

called the *flux homomorphism*.

Note that $\ker p = \pi_1\left(\mathrm{Symp}_0(M,\omega)\right)$, and denote by

$$\Gamma_\omega = \mathrm{Flux}_\omega\left(\pi_1\left(\mathrm{Symp}_0(M,\omega)\right)\right)$$

the image of $\ker p$ in $H^1(M,\mathbb{R})$. By the C^∞-Flux Conjecture [Ono06], the subgroup $\Gamma_\omega \subset H^1(M,\mathbb{R})$ is discrete (in the natural topology of $H^1(M,\mathbb{R})$ as a finite dimensional real vector space). Various special cases of the flux conjecture had been established before Ono verified it in complete generality, see for example [Ban78, LMP98], or the references in [Ono06]. We obtain by passing to the quotient a continuous surjective group homomorphism

$$\mathrm{flux}_\omega\colon \mathrm{Symp}_0(M,\omega) \longrightarrow H^1(M,\mathbb{R})/\Gamma_\omega,$$

which is also called the *flux homomorphism*. It is shown in [Ban78] that $\ker \mathrm{flux}_\omega = \mathrm{Ham}(M, \omega)$. In particular, we see that $\mathrm{Ham}(M, \omega) = \mathrm{Symp}_0(M, \omega)$ if and only if $H^1(M, \mathbb{R}) = 0$.

The flux homomorphism for volume-preserving diffeomorphisms is defined similarly. We restrict to the case of interest to us, where Ω is a volume form on (M, ω), e.g. the Liouville volume form. Denote by $\mathrm{Diff}^\Omega(M) \subset \mathrm{Diff}(M)$ the subgroup of volume-preserving diffeomorphisms of M, and by $\mathrm{Diff}_0^\Omega(M)$ the path component of the identity in $\mathrm{Diff}^\Omega(M)$, equipped with the C^∞-topology. Denote by $\mathcal{P}\mathrm{Diff}^\Omega(M)$ the set of volume-preserving isotopies, i.e. smooth paths $\lambda\colon [0, 1] \to \mathrm{Diff}^\Omega(M)$, with $\lambda(0) = \mathrm{id}$, endowed with the C^∞-topology defined as above, and the same natural group structure as $\mathcal{P}\mathrm{Symp}(M, \omega)$. Since $\mathrm{Diff}_0^\Omega(M)$ is a smooth deformation retract of $\mathrm{Diff}_0(M)$ (by Moser's isotopy method [Mos65]), it is locally contractible. Thus the universal covering space $\widetilde{\mathrm{Diff}}_0^\Omega(M)$ of $\mathrm{Diff}_0^\Omega(M)$ is represented by isotopy classes, relative to fixed end points, of smooth paths $\lambda \in \mathcal{P}\mathrm{Diff}^\Omega(M)$. We again denote the equivalence class represented by a path λ by $[\lambda]$. We give $\widetilde{\mathrm{Diff}}_0^\Omega(M)$ the quotient topology and group structure induced by the obvious map

$$\mathcal{P}\mathrm{Diff}^\Omega(M) \longrightarrow \widetilde{\mathrm{Diff}}_0^\Omega(M),$$

and again denote by

$$p\colon \widetilde{\mathrm{Diff}}_0^\Omega(M) \longrightarrow \mathrm{Diff}_0^\Omega(M)$$

the covering projection induced by the time-one evaluation map, which is a surjective continuous group homomorphism as well.

To define the flux homomorphism

$$\mathcal{P}\mathrm{Diff}^\Omega(M) \longrightarrow H^{2n-1}(M, \mathbb{R}),$$

where $\dim M = 2n$, note that since $\lambda \in \mathcal{P}\mathrm{Diff}^\Omega(M)$ is a volume-preserving isotopy, the $(2n-1)$-form $\iota(X_t)\Omega$ is closed for all $0 \le t \le 1$, where X_t again denotes the infinitesimal generator of the path λ. Then we define

$$\mathrm{Flux}_\Omega(\lambda) = \left[\int_0^1 \iota(X_t)\Omega dt\right].$$

This depends only on the isotopy class, relative to the end points, of the path λ, and therefore projects down to the universal covering space $\widetilde{\mathrm{Diff}_0^\Omega}(M)$ of $\mathrm{Diff}_0^\Omega(M)$. We obtain a surjective continuous group homomorphism

$$\mathrm{Flux}_\Omega \colon \widetilde{\mathrm{Diff}_0^\Omega}(M) \longrightarrow H^{2n-1}(M, \mathbb{R}),$$

which is again called the *flux homomorphism*.

Again $\ker p = \pi_1\left(\mathrm{Diff}_0^\Omega(M)\right)$, and we denote by

$$\Gamma_\Omega = \mathrm{Flux}_\Omega\left(\pi_1\left(\mathrm{Diff}_0^\Omega(M)\right)\right)$$

the image of $\ker p$ in $H^{2n-1}(M, \mathbb{R})$. Thurston [Thu73] has shown that the subgroup $\Gamma_\Omega \subset H^{2n-1}(M, \mathbb{R})$ is discrete. We obtain by passing to the quotient a continuous surjective group homomorphism

$$\mathrm{flux}_\Omega \colon \mathrm{Diff}_0^\Omega(M) \longrightarrow H^{2n-1}(M, \mathbb{R})/\Gamma_\Omega,$$

which is still called the *flux homomorphism*.

To distinguish the flux homomorphisms for symplectic and volume-preserving diffeomorphisms, we sometimes add the prefix symplectic or volume-preserving where necessary.

2.5.2 The mass flow homomorphism

We briefly review the construction from [Fat80] of the mass flow homomorphism for homeomorphisms preserving a good measure. We restrict to the case important to us, where the measure is induced by integrating a volume form Ω on a connected smooth orientable manifold M, e.g. the Liouville volume form on a symplectic manifold (M, ω). For more details see [Fat80], or [Her79, Sch57] for different versions of this homomorphism.

Recall that $\mathrm{Homeo}^\Omega(M) \subset \mathrm{Homeo}(M)$ denotes the group of measure-preserving homeomorphisms of M, and $\mathrm{Homeo}_0^\Omega(M)$ the path component of the identity in $\mathrm{Homeo}^\Omega(M)$, equipped with the C^0-topology. Denote by $\mathcal{P}\mathrm{Homeo}^\Omega(M) \subset \mathcal{P}\mathrm{Homeo}(M)$ the set of measure-preserving isotopies, i.e. continuous paths $\lambda\colon [0,1] \to \mathrm{Homeo}^\Omega(M)$, with $\lambda(0) = \mathrm{id}$, endowed with the C^0-topology. Of course this also carries a natural group structure. Since $\mathrm{Homeo}^\Omega(M)$ is locally contractible [Fat80], the universal covering space $\widetilde{\mathrm{Homeo}}_0^\Omega(M)$ of $\mathrm{Homeo}_0^\Omega(M)$ is represented by isotopy classes, relative to fixed end points, of paths $\lambda \in \mathcal{P}\mathrm{Homeo}^\Omega(M)$. We denote the equivalence class represented by a path λ by $[\lambda]$. We give $\widetilde{\mathrm{Homeo}}_0^\Omega(M)$ the quotient topology and group structure induced by the obvious map

$$\mathcal{P}\mathrm{Homeo}^\Omega(M) \longrightarrow \widetilde{\mathrm{Homeo}}_0^\Omega(M),$$

and denote by

$$p\colon \widetilde{\mathrm{Homeo}}_0^\Omega(M) \longrightarrow \mathrm{Homeo}_0^\Omega(M)$$

the covering projection induced by the time-one evaluation map. This is a surjective continuous group homomorphism.

To define the *mass flow homomorphism*

$$\widetilde{\theta}\colon \widetilde{\mathrm{Homeo}}_0^\Omega(M) \longrightarrow H_1(M,\mathbb{R}), \tag{2.12}$$

we use the fact that $H_1(M,\mathbb{R})$ is isomorphic to $\mathrm{Hom}([M,S^1],\mathbb{R})$, where $[M,S^1]$ is the set of homotopy classes of maps from M to S^1. Denote by $C^0(M,S^1)$ the set of continuous maps $M \to S^1$, equipped with the compact-open topology, and note that this, and therefore also $[M,S^1]$, naturally forms a group. Identifying S^1 with $\mathbb{R}/\mathbb{Z}$, we can write the group law on S^1 and thus on $C^0(M,S^1)$ additively. Given $\lambda \in \mathcal{P}\mathrm{Homeo}^\Omega(M)$, $\lambda(t) = h_t \in \mathrm{Homeo}_0^\Omega(M)$, we define a continuous group homomorphism

$$\widetilde{\theta}(\lambda)\colon C^0(M,S^1) \longrightarrow \mathbb{R},$$

in the following way: let $f\colon M \to S^1 = \mathbb{R}/\mathbb{Z}$ be continuous. The homotopy $t \mapsto fh_t - f\colon M \to S^1$ satisfies $fh_0 - f = 0$, hence we can lift it to a unique homotopy $t \mapsto \overline{fh_t - f}\colon M \to \mathbb{R}$, such that $\overline{fh_0 - f} = 0$. Then we define

$$\widetilde{\theta}(\lambda)(f) = \int_M \overline{fh_1 - f}\Omega.$$

The number $\widetilde{\theta}(\lambda)(f)$ depends only on the homotopy class of f, and $\widetilde{\theta}(\lambda)$ is a homomorphism, which depends only on the equivalence class $[\lambda]$ of λ. Moreover, $\widetilde{\theta}$ is a homomorphism. Therefore it induces a surjective group homomorphism (2.12), which is continuous if $\mathrm{Hom}([M,S^1],\mathbb{R})$ is given the weak topology [Fat80]. That is, for each given $f\colon M \to S^1$, the assignment $\lambda \mapsto \widetilde{\theta}(\lambda)(f)$ is continuous. This topology coincides with the natural topology of $\mathrm{Hom}([M,S^1],\mathbb{R}) \cong H_1(M,\mathbb{R})$ as a finite dimensional real vector space: after choosing any basis, both topologies are given by usual convergence in $\mathbb{R}^k$, that is, convergence of the coefficients with respect to the (dual) bases, with respect to the standard metric on $\mathbb{R}^k$, where k is the rank of $H_1(M,\mathbb{R})$ over $\mathbb{R}$. By elementary linear algebra, the induced topology is independent of the choice of basis (the resulting metrics are related by the operator norms of the change of basis matrices).

Note that $\ker p = \pi_1\left(\mathrm{Homeo}_0^\Omega(M)\right)$, and define

$$\Gamma_{\widetilde{\theta}} = \widetilde{\theta}\left(\pi_1\left(\mathrm{Homeo}_0^\Omega(M)\right)\right) \subset H_1(M,\mathbb{R}).$$

We obtain by passing to the quotient a surjective continuous group homomorphism

$$\theta\colon \mathrm{Homeo}_0^\Omega(M) \longrightarrow H_1(M,\mathbb{R})/\Gamma_{\widetilde{\theta}},$$

which is also called the *mass flow homomorphism*. The group $\Gamma_{\widetilde{\theta}}$ is discrete (in the natural topology of $H_1(M,\mathbb{R})$) because, after normalizing Ω so that $\int_M \Omega = 1$, it is contained in $H_1(M,\mathbb{Z})$ [Fat80, Proposition 5.1].

If $n \geq 3$, Fathi [Fat80] showed that the group $\ker\theta = [\ker\theta,\ker\theta]$ is perfect and simple, and in particular equals the commutator subgroup of $\mathrm{Homeo}_0^\Omega(M)$. The following still remains an open problem concerning the algebraic structure of area-preserving homeomorphism groups in two dimensions.

Question 2.5.1. Is $\ker\theta$ simple when $n = 2$? In particular, is $\mathrm{Homeo}_0^\Omega(S^2)$ a simple group?

2.5.3 Duality and other identities

We now recall that the flux homomorphism for volume-preserving diffeomorphisms is Poincaré dual to the mass flow homomorphism, and is also related to the flux homomorphism for symplectic diffeomorphisms. This allows us to compare the mass flow to the symplectic flux homomorphism. In particular in two dimensions, the two flux homomorphisms coincide, and the mass flow homomorphism is Poincaré dual to the symplectic flux. One crucial point of considering the mass flow homomorphism instead of the flux homomorphism is that it is defined for any isotopy of measure-preserving homeomorphisms, not just for diffeomorphisms. This has immediate applications to C^0-symplectic topology, which will be outlined below.

Let Ω as above denote a volume form on M, normalized so that $\int_M \Omega = 1$.

Proposition 2.5.2 ([Fat80],Appendix A.5)**.** *For any $\lambda \in \mathcal{P}\mathrm{Diff}^\Omega(M)$, the cohomology class $[\mathrm{Flux}_\Omega(\lambda)] \in H^{2n-1}(M,\mathbb{R})$ is the Poincaré dual of the homology class $[\widetilde{\theta}(\lambda)] \in H_1(M,\mathbb{R})$.*

Note that with the usual identification of $H_1(M,\mathbb{R})$ with $\mathrm{Hom}([M,S^1],\mathbb{R})$, i.e. via the identification of $H^1(M,\mathbb{Z})$ with $[M,S^1]$, applying the $\mathrm{Hom}(-,\mathbb{Z})$ functor, and then tensoring with $\mathbb{R}$, the above proposition takes the following simple form [Fat80, Appendix A.5]: let σ denote the canonical volume form on S^1 given by the natural orientation of the circle. Then for any $f\colon M \to S^1$, we have

$$\int_M \mathrm{Flux}_\Omega(\lambda) \wedge f^*\sigma = \widetilde{\theta}(\lambda)(f). \tag{2.13}$$

Now let $\Omega = \omega^n/n!$ be the Liouville volume form. Then there are obvious inclusions $\mathcal{P}\mathrm{Symp}(M,\omega) \subset \mathcal{P}\mathrm{Diff}^\Omega(M)$ and $\mathrm{Symp}_0(M,\omega) \subset \mathrm{Diff}_0^\Omega(M)$. A straightforward calculation, see [Ban78], shows that

$$\mathrm{Flux}_\Omega(\lambda) = \mathrm{Flux}_\omega(\lambda) \wedge \frac{\omega^{n-1}}{(n-1)!}. \tag{2.14}$$

If we assume that ω is normalized, in the sense that we again have $\int_M \Omega = 1$, then by combining (2.13) and (2.14), we can compare the mass flow to the symplectic flux.

Consider the map

$$\wedge\omega^{n-1}\colon H^1(M,\mathbb{R}) \longrightarrow H^{2n-1}(M,\mathbb{R}), \qquad \beta \longmapsto \beta \wedge \omega^{n-1}. \tag{2.15}$$

It coincides, up to (the nonsingular) cup product pairing, with the pairing (in the form it takes in de Rham cohomology)

$$H^1(M,\mathbb{R}) \times H^1(M,\mathbb{R}) \longrightarrow \mathbb{R}, \qquad (\alpha,\beta) \longmapsto \int_M \alpha \wedge \beta \wedge \omega^{n-1}. \tag{2.16}$$

It is interesting to know when (2.15) is nontrivial. If it is an isomorphism, M is said to be of Lefschetz type. The list of Lefschetz manifolds includes all Kähler manifolds, such as even-dimensional tori, and orientable surfaces.

2.5.4 Applications to C^0-symplectic topology

Let $\phi \in \mathrm{Ham}(M,\omega)$, and choose a path $\lambda \in \mathcal{P}\mathrm{Symp}(M,\omega)$ with $\lambda(1) = \phi$. We can w.l.o.g. assume that $\mathrm{Flux}_\omega(\lambda) = 0$. Indeed, by definition $\mathrm{flux}_\omega(\phi) = \mathrm{Flux}_\omega(\lambda) \,(\mathrm{mod}\, \Gamma_\omega)$, and since $\mathrm{Ham}(M,\omega) = \ker \mathrm{flux}_\omega$, we have $\mathrm{Flux}_\omega(\lambda) \in \Gamma_\omega$. Now by definition of Γ_ω, we can find a loop $\mu \in \mathcal{P}\mathrm{Symp}(M,\omega)$ with $\mathrm{Flux}_\omega(\mu) = \mathrm{Flux}_\omega(\lambda)$. Then $(\mu^{-1} \circ \lambda)(1) = \phi$, and since Flux_ω is a homomorphism, we have $\mathrm{Flux}_\omega(\mu^{-1} \circ \lambda) = 0$. By (2.13) and (2.14), we see that $\lambda \in \ker \widetilde{\theta}$, and therefore $\phi \in \ker \theta$. We derive

$$\mathrm{Ham}(M,\omega) \subset \ker \theta \cap \mathrm{Symp}_0(M,\omega),$$

for any closed symplectic manifold (M,ω). By continuity of $\widetilde{\theta}$ (or θ itself), we obtain

$$\mathrm{Hameo}(M,\omega) \subset \ker \theta \cap \mathrm{Sympeo}_0(M,\omega). \tag{2.17}$$

Suppose now that $H^1(M,\mathbb{R}) \neq 0$, and that the map (2.15) is nontrivial, i.e. not identically zero. Then, since $\Gamma_{\widetilde{\theta}}$ is discrete, and by surjectivity of Flux_ω, also using (2.13) and (2.14), we can find $\lambda \in \mathcal{P}\mathrm{Symp}(M,\omega)$ with $\widetilde{\theta}(\lambda) \notin \Gamma_{\widetilde{\theta}}$. But then $\theta(\lambda(1)) \neq 0$, i.e. $h = \lambda(1) \notin \ker\theta$. By (2.17), this implies $h \notin \mathrm{Hameo}(M,\omega)$. That shows that, under the above assumptions, $\ker\theta \cap \mathrm{Sympeo}_0(M,\omega) \subsetneq \mathrm{Sympeo}_0(M,\omega)$, and in particular, $\mathrm{Hameo}(M,\omega) \subsetneq \mathrm{Sympeo}_0(M,\omega)$. We have proved the following

Theorem 2.5.3. *Let (M,ω) be a closed symplectic manifold of dimension $2n$. Then*

$$\mathrm{Hameo}(M,\omega) \subset \ker\theta \cap \mathrm{Sympeo}_0(M,\omega).$$

Suppose in addition that $H^1(M,\mathbb{R}) \neq 0$, and that the map $\wedge\omega^{n-1}$ (2.15) is not identically zero. Then

$$\mathrm{Hameo}(M,\omega) \subsetneq \mathrm{Sympeo}_0(M,\omega).$$

This in particular holds for even-dimensional tori T^{2n}, and for surfaces of genus $g > 0$. The theorem does not apply if $H^1(M,\mathbb{R})$ has rank 1, since the above pairing (2.16), and hence the map (2.15), is identically zero (for degree reasons) in this case. In fact, the above discussion shows that then $\mathrm{Sympeo}_0(M,\omega) \subset \ker\theta$.

Recall from Theorem 2.4.1 that $\mathrm{Hameo}(M,\omega)$ is a normal subgroup of $\mathrm{Sympeo}_0(M,\omega)$, and by Corollary 2.1.6, $\mathrm{Sympeo}_0(M,\omega) = \mathrm{Homeo}_0^\Omega(M)$ if $\dim M = 2$.

Corollary 2.5.4. *Let M be a closed orientable surface, and let ω be any area form on it. Then $\mathrm{Hameo}(M,\omega)$ is a normal subgroup of $\ker\theta$, where θ is the mass flow homomorphism associated to the measure induced by integrating the area form ω. Moreover, if $M \neq S^2$, then*

$$\mathrm{Hameo}(M,\omega) \subsetneq \mathrm{Sympeo}_0(M,\omega)$$

is a proper normal subgroup.

We propose the following

Conjecture 2.5.5. $\mathrm{Hameo}(M,\omega)$ is a proper subgroup of $\ker\theta$ in general. Therefore, if $\dim M = 2$, $\ker\theta$ is not a simple group. In particular, for $M = S^2$, with $\Omega = \omega$ an area form on S^2, $\mathrm{Hameo}(S^2,\omega)$ is a proper normal subgroup of $\ker\theta = \mathrm{Homeo}_0^\Omega(S^2) = \mathrm{Sympeo}_0(S^2,\omega)$.

The affirmative answer to this conjecture will answer Question 2.5.1 negatively, and settle the simpleness question of $\mathrm{Homeo}_0^\Omega(S^2)$, which has been open since the paper [Fat80] appeared. In fact, a positive answer to this conjecture would be an immediate corollary of the following more concrete conjecture.

Conjecture 2.5.6. The answer to Question 2.4.6 on S^2 is negative, at least for a suitable choice of ρ.

If $H^1(M,\mathbb{R}) = 0$, then $\mathrm{Ham}(M,\omega) = \mathrm{Symp}_0(M,\omega)$, and neither the methods of this section, nor Theorem 2.4.4, apply to show that $\mathrm{Hameo}(M,\omega) \subsetneq \mathrm{Sympeo}_0(M,\omega)$, since they are of topological nature. More refined methods have to be developed to verify that $\mathrm{Hameo}(M,\omega)$ is a proper subgroup of $\mathrm{Sympeo}_0(M,\omega)$ in general, and also to give an affirmative answer to Conjecture 2.5.5.

2.6 $\mathrm{Hameo}_{(1,\infty)}(M,\omega) = \mathrm{Hameo}_{\infty}(M,\omega)$

In this section, we prove that the group of Hamiltonian homeomorphisms does not depend on the choice of norm in its construction. More precisely, we have the following theorem.

Theorem 2.6.1. $\mathrm{Hameo}_{(1,\infty)}(M,\omega) = \mathrm{Hameo}_{\infty}(M,\omega)$.

This result already appeared in [Mül07]. In view of this theorem, we may, and will henceforth, omit the subscripts.

As remarked earlier, the inclusion $\mathrm{Hameo}_{\infty}(M,\omega) \subset \mathrm{Hameo}_{(1,\infty)}(M,\omega)$ is obvious. The converse is more delicate. Our proof is mainly based on Polterovich's Lemma 1.1.2. In fact, Polterovich proved the following slightly stronger Lemma 2.6.3. Lemma 1.1.2 is an immediate consequence of this more technical result.

Remark 2.6.2. In this section, we want to allow more general Hamiltonian paths that are not necessarily based at the identity. That is, unless explicit mention is made to the contrary, we consider paths $\lambda = \phi \circ \phi_H$, where ϕ_H is a Hamiltonian path in the previous sense, with $\phi_H^0 = \mathrm{id}$, and $\phi \in \mathrm{Ham}(M,\omega)$. It is easy to see that λ solves Hamilton's equation (2.10), with Hamiltonian $H \circ \phi$, and initial condition $\lambda(0) = \phi$. We therefore call the Hamiltonian $K = H \circ \phi$ the generating Hamiltonian of λ. Note that we could as well work with paths of the form $\lambda = \phi_H \circ \phi$, which solve Hamilton's equation with Hamiltonian H, and initial condition $\lambda(0) = \phi$. It turns out that the former will be more convenient for the computations below. We will often simply write $\lambda = \phi_K$. If one path starts where another one ends, we may consider their concatenation, and if both paths are boundary flat, then that concatenation is a smooth Hamiltonian path, whose generating Hamiltonian agrees with (a reparameterization of) the Hamiltonian of the first path for some time, and with (a reparameterization of) the Hamiltonian of the second path for the remaining time. Of course, when computing the Hofer norm of an element $\phi \in \mathrm{Ham}(M,\omega)$, we only allow paths ϕ_H with $\phi_H^0 = \mathrm{id}$ and $\phi_H^1 = \phi$.

Lemma 2.6.3. *Let $H\colon [0,1]\times M \to \mathbb{R}$ be a normalized Hamiltonian function, generating the Hamiltonian path $\phi_H\colon t\mapsto \phi_H^t$. Let $\epsilon > 0$ be given. Then there exists a normalized Hamiltonian function $F\colon [0,1]\times M\to\mathbb{R}$, such that the following holds*

(i) $\phi_F^0 = \phi_H^0$ *and* $\phi_F^1 = \phi_H^1$*,*

(ii) $\|F\|_\infty < \|H\|_{(1,\infty)} + \epsilon$*, and*

(iii) $\overline{d}(\phi_F, \phi_H^0) < \overline{d}(\phi_H, \phi_H^0) + \epsilon$*.*

In (iii), ϕ_H^0 denotes the constant path $t\mapsto\phi_H^0$.

Proof. We first consider the path $t\mapsto \phi_K^t = \phi^t\circ\phi_H^t$, where $t\mapsto\phi^t$ is a loop in $\mathrm{Ham}(M,\omega)$, $\phi^0 = \phi^1 = \mathrm{id}$. Clearly $\phi_K^0 = \phi_H^0$ and $\phi_K^1 = \phi_H^1$. We may choose the loop ϕ^t such that it is arbitrarily close to the constant loop id in the C^0-metric, its generating Hamiltonian is arbitrarily small in the $\|\cdot\|_{(1,\infty)}$ norm, and such that $\mathrm{osc}(K_t)\neq 0$, for all $t\in[0,1]$, see [Pol01, 5.2]. Therefore we may choose the Hamiltonian K such that

$$\|K\|_{(1,\infty)} < \|H\|_{(1,\infty)} + \frac{\epsilon}{2}, \qquad \overline{d}(\phi_K,\phi_H^0) < \overline{d}(\phi_H,\phi_H^0)+\epsilon.$$

To see the second inequality, write

$$\overline{d}(\phi_K,\phi_H^0) \le \overline{d}(\phi_K,\phi_H) + \overline{d}(\phi_H,\phi_H^0),$$

and note that the first term on the right hand side of the inequality can be made as small as we want, since the space of continuous paths $\mathcal{P}\mathrm{Homeo}(M)$ forms a topological group.

We may normalize K if necessary without losing any of the above properties. Now consider the Hamiltonian K^ζ, where ζ is the inverse of (here we use $\mathrm{osc}(K_t)\neq 0$ for all $t\in[0,1]$)

$$\xi\colon [0,1]\longrightarrow[0,1], \qquad t\longmapsto \frac{\int_0^t \mathrm{osc}(K_s)ds}{\int_0^1 \mathrm{osc}(K_s)ds}.$$

Note that ζ fixes 0 and 1, so that ϕ_{K^ζ} has the same end points as ϕ_K. By the chain rule,

$$\zeta'(t) = \frac{\int_0^1 \mathrm{osc}(K_s)ds}{\mathrm{osc}\left(K_{\zeta(t)}\right)}.$$

Hence for every t,

$$\operatorname{osc}\left(K_t^{\zeta}\right) = \zeta'(t)\operatorname{osc}(K_{\zeta(t)}) = \int_0^1 \operatorname{osc}(K_s)ds = \|K\|_{(1,\infty)},$$

and therefore

$$\|K^{\zeta}\|_\infty = \|K\|_{(1,\infty)} < \|H\|_{(1,\infty)} + \frac{\epsilon}{2}.$$

Now ζ (and therefore K^ζ and ϕ_{K^ζ}) may not be smooth, but only C^1. We approximate ζ in the C^1-topology by a smooth diffeomorphism ρ of $[0,1]$ that also fixes 0 and 1, to obtain a smooth normalized Hamiltonian $F = K^\rho$, with $\|F\|_\infty < \|K^\zeta\|_\infty + \epsilon/2$. Then F clearly satisfies (i) and (ii). Since ϕ_F is just a reparameterization of ϕ_K, we also have

$$\overline{d}(\phi_F, \phi_H^0) = \overline{d}(\phi_K, \phi_H^0) < \overline{d}(\phi_H, \phi_H^0) + \epsilon.$$

That proves (iii), and hence finishes the proof. $\square$

The above lemma (or more precisely, its proof) can be rephrased as follows: we have $\|\cdot\|_{(1,\infty)} \leq \|\cdot\|_\infty$, and the former is invariant under reparameterization, while the latter is from being invariant. Although our earlier example shows that (or why) the norms $\|\cdot\|_{(1,\infty)}$ and $\|\cdot\|_\infty$ are not equivalent, a generic Hamiltonian path ϕ_H (in the C^∞-topology on $C^\infty_m([0,1] \times M)$ or $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ [Pol01, 5.2], and thus in particular in either of the Hamiltonian topologies) can be reparameterized so that the derivative of the reparameterization function is 'small' whenever the oscillation is 'large', and vice versa (so that the time-one maps coincide), and thus the maximum oscillation of the Hamiltonian F generating the reparameterized path is 'close to' its mean oscillation, i.e. $\|F\|_\infty \approx \|F\|_{(1,\infty)}$.

Proof of Lemma 1.1.2. For every Hamiltonian H we have $\|H\|_{(1,\infty)} \leq \|H\|_\infty$. So the inequality $\|\phi\|_{(1,\infty)} \leq \|\phi\|_\infty$ is obvious. For the converse, let $\epsilon > 0$ be arbitrary. Choose a Hamiltonian H generating ϕ, such that $\|H\|_{(1,\infty)} < \|\phi\|_{(1,\infty)} + \epsilon$. By Lemma 2.6.3, we can find a Hamiltonian F generating ϕ, such that $\|F\|_\infty < \|H\|_{(1,\infty)} + \epsilon < \|\phi\|_{(1,\infty)} + 2\epsilon$. But then $\|\phi\|_\infty \leq \|F\|_\infty <$

$\|\phi\|_{(1,\infty)} + 2\epsilon$. Since $\epsilon > 0$ was arbitrary, this implies $\|\phi\|_\infty \leq \|\phi\|_{(1,\infty)}$. That completes the proof. $\square$

To prove Theorem 2.6.1, it only remains to show $\mathrm{Hameo}_{(1,\infty)}(M,\omega) \subset \mathrm{Hameo}_\infty(M,\omega)$. Let $h \in \mathrm{Hameo}_{(1,\infty)}(M,\omega)$. By definition, there exists a sequence (ϕ_{H_i}, H_i) of normalized Hamiltonian functions H_i, generating the Hamiltonian paths ϕ_{H_i}, such that (ϕ_{H_i}, H_i) converges in the $L^{(1,\infty)}$-Hamiltonian metric, and $\phi_{H_i}^1 \to h$ in the C^0-metric. As remarked earlier, we cannot expect the sequence H_i to be Cauchy in the L^∞-metric in general. Our goal is to modify the sequence (ϕ_{H_i}, H_i) to a sequence that is Cauchy in the L^∞-Hamiltonian metric. Our strategy will be as follows. The given sequence gives a 'short' path from the end point ϕ_i of the path ϕ_{H_i}, to the end point ϕ_{i+1} of the path $\phi_{H_{i+1}}$, for all i. We will construct a sequence ϕ_{F_i} of Hamiltonian paths, so that $\phi_{F_{i+1}}$ coincides with its predecessor ϕ_{F_i} for some time, followed by the path from ϕ_i to ϕ_{i+1} (see Figure 4 below). We will have to apply Lemma 2.6.3 to pass from the $L^{(1,\infty)}$-norm to the L^∞-norm, and we have to make the pieces we paste together boundary flat, so that the elements of the constructed sequence are smooth. Along the way, we will have to keep track of the closeness of the paths and their Hamiltonians in the Hamiltonian metric. Note that the 'image' in $\mathrm{Homeo}(M)$ of the limit path of the modified sequence will be very different from the 'image' of the limit path of the original sequence. We cannot apply Lemma 2.6.3 directly to the sequence H_i, and expect these Hamiltonians, and the paths they generate, to be Cauchy in the L^∞-Hamiltonian metric in general.

Before giving the proof of Theorem 2.6.1, we make the following useful remark.

Remark 2.6.4. Given a Cauchy sequence (ϕ_{H_i}, H_i) in $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, any subsequence has the same limit $(\lambda, H) \in \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$. In particular, given any (decreasing) sequence $\epsilon_i > 0$ of positive numbers, with $\epsilon_i \to 0$ as $i \to \infty$, by passing to a subsequence, we may assume that the given sequence satisfies $\|H_j - H_k\| < \epsilon_i$ and $\overline{d}(\phi_{H_j}, \phi_{H_k}) < \epsilon_i$, for all $j, k \geq i$, for all i. Similarly, we may assume that the given sequence satisfies $\|H_j - H\| < \epsilon_i$ and

$\overline{d}(\phi_{H_j}, \lambda) < \epsilon_i$ for all $j \geq i$, and all i, or any combination of the two. For instance, we may assume that the given sequence satisfies $\|H_j - H_k\| < \epsilon_i$ for all j, $k \geq i$, and $\overline{d}(\phi_{H_j}, \lambda) < \epsilon_i$, for all $j \geq i$, and for all i. It is often convenient to consider a sequence ϵ_i such that $\sum \epsilon_i$ converges, or $\sum_{i=j}^{k} \epsilon_i \to 0$, as $k \geq j \to \infty$, for example $\epsilon_i = 1/2^i$.

Proof of Theorem 2.6.1. It only remains to prove $\mathrm{Hameo}_{(1,\infty)}(M, \omega) \subset \mathrm{Hameo}_{\infty}(M, \omega)$. Let $h \in \mathrm{Hameo}_{(1,\infty)}(M, \omega)$. By definition, there exists a sequence (ϕ_{H_i}, H_i) of normalized Hamiltonian functions H_i, generating the Hamiltonian paths ϕ_{H_i}, with $\phi_{H_i}^0 = \mathrm{id}$, such that

- $\overline{d}(\phi_{H_i}, \phi_{H_j}) \to 0$, as i, $j \to \infty$,
- $\left\|\overline{H_i} \# H_j\right\|_{(1,\infty)} = \|H_i - H_j\|_{(1,\infty)} \to 0$, as i, $j \to \infty$, and
- $\overline{d}(\phi_{H_i}^1, h) \to 0$ as $i \to \infty$.

Denote by λ the C^0-limit of the sequence of paths ϕ_{H_i}, and $\phi_i = \phi_{H_i}^1$.

Let $\epsilon_i > 0$ be a decreasing sequence of real numbers with $\epsilon_i \to 0$ as $i \to \infty$. Since h is uniformly continuous, there exists a sequence $\delta_i > 0$, such that for all i: $d(h(x), h(y)) < \epsilon_i$, for all x, $y \in M$ with $d(x, y) < \delta_i$. W.l.o.g. we may assume that $\delta_i \leq \epsilon_i$ for all i. By remark 2.6.4, by passing to a subsequence if necessary, we may assume that

$$\left\|\overline{H_i} \# H_{i+1}\right\|_{(1,\infty)} < \delta_i \leq \epsilon_i, \qquad \overline{d}(\phi_{H_i}, \lambda) < \delta_i,$$

for all i. We will specify the sequence ϵ_i later in the proof.

For convenience, denote by H_0 the Hamiltonian $H_0 \equiv 0$, which generates the constant loop id. Define the sequence K_i of smooth Hamiltonians by

$$K_i = \left(\overline{H_{i-1}} \# H_i\right) \circ \phi_{i-1},$$

for all $i \geq 1$. Since the Hamiltonians H_i are normalized for all i, the functions K_i are normalized as well, and the Hamiltonian paths they generate can be chosen to be the paths (see Remark

2.6.2)

$$\phi_{K_i} = \phi_{i-1} \circ \left(\phi_{H_{i-1}}\right)^{-1} \circ \phi_{H_i}$$

from ϕ_{i-1} to ϕ_i, for all $i \geq 1$. Here and in the following, we denote by ϕ_i either the diffeomorphism itself, or the constant path $t \mapsto \phi_i$.

By assumption,

$$\|K_i\|_{(1,\infty)} < \epsilon_{i-1},$$

for all $i > 1$. Moreover, we claim that the assumption on the sequence ϕ_{H_i} implies

$$\overline{d}(\phi_{K_i}, \phi_{i-1}) \leq 4\epsilon_{i-1},$$

for all $i > 1$. Indeed, note that by definition

$$\begin{aligned}
\overline{d}(\phi_{K_i}, \phi_{i-1}) &= \max\left(\hat{d}\left(\phi_{K_i}, \phi_{i-1}\right), \hat{d}\left((\phi_{K_i})^{-1}, \phi_{i-1}^{-1}\right)\right) \\
&= \max\left(\hat{d}\left(\phi_{i-1} \circ \left(\phi_{H_{i-1}}\right)^{-1} \circ \phi_{H_i}, \phi_{i-1}\right), \hat{d}\left((\phi_{H_i})^{-1} \circ \phi_{H_{i-1}} \circ \phi_{i-1}^{-1}, \phi_{i-1}^{-1}\right)\right).
\end{aligned}$$

For the second term, we use that the metric $\hat{d}$ is right invariant, to see that

$$\begin{aligned}
\hat{d}\left((\phi_{H_i})^{-1} \circ \phi_{H_{i-1}} \circ \phi_{i-1}^{-1}, \phi_{i-1}^{-1}\right) &= \hat{d}\left((\phi_{H_i})^{-1} \circ \phi_{H_{i-1}}, id\right) \\
&= \hat{d}\left((\phi_{H_i})^{-1}, \left(\phi_{H_{i-1}}\right)^{-1}\right) \\
&\leq \overline{d}\left(\phi_{H_i}, \phi_{H_{i-1}}\right) \\
&\leq \overline{d}\left(\phi_{H_i}, \lambda\right) + \overline{d}\left(\lambda, \phi_{H_{i-1}}\right) \\
&\leq \epsilon_i + \epsilon_{i-1} \leq 2\epsilon_{i-1}.
\end{aligned}$$

For the first term, we compute

$$\begin{aligned}
\hat{d}\left(\phi_{i-1} \circ \left(\phi_{H_{i-1}}\right)^{-1} \circ \phi_{H_i}, \phi_{i-1}\right) &\leq \hat{d}\left(\phi_{i-1} \circ \left(\phi_{H_{i-1}}\right)^{-1} \circ \phi_{H_i}, h\right) + \hat{d}(h, \phi_{i-1}) \\
&\leq \hat{d}\left(\phi_{i-1} \circ \left(\phi_{H_{i-1}}\right)^{-1}, h \circ (\phi_{H_i})^{-1}\right) + \hat{d}(\lambda, \phi_{H_{i-1}}) \\
&\leq \hat{d}\left(\phi_{i-1} \circ \left(\phi_{H_{i-1}}\right)^{-1}, h \circ \left(\phi_{H_{i-1}}\right)^{-1}\right) + \hat{d}\left(h \circ \left(\phi_{H_{i-1}}\right)^{-1}, h \circ \lambda^{-1}\right) \\
&\qquad + \hat{d}\left(h \circ \lambda^{-1}, h \circ (\phi_{H_i})^{-1}\right) + \epsilon_{i-1} \\
&\leq \hat{d}(\phi_{i-1}, h) + \epsilon_{i-1} + \epsilon_i + \epsilon_{i-1} \\
&\leq \hat{d}(\phi_{H_{i-1}}, \lambda) + 3\epsilon_{i-1} \leq 4\epsilon_{i-1}.
\end{aligned}$$

Therefore we have

$$\overline{d}(\phi_{K_i}, \phi_{i-1}) \leq 4\epsilon_{i-1},$$

for all $i > 1$, as claimed.

Now apply Lemma 2.6.3 to each K_i, to obtain a sequence of normalized Hamiltonians L_i, such that $\phi^0_{L_i} = \phi^0_{K_i} = \phi_{i-1}$, $\phi^1_{L_i} = \phi^1_{K_i} = \phi_i$, for all i, and

$$\|L_i\|_\infty < \|K_i\|_{(1,\infty)} + \epsilon_{i-1} \leq 2\epsilon_{i-1}, \qquad \overline{d}(\phi_{L_i}, \phi_{i-1}) < \overline{d}(\phi_{K_i}, \phi_{i-1}) + \epsilon_{i-1} \leq 5\epsilon_{i-1},$$

for all $i > 1$.

Then using Lemma 1.2.6 to reparameterize each L_i, we obtain a normalized boundary flat Hamiltonian M_i, such that $\phi^0_{M_i} = \phi^0_{L_i} = \phi_{i-1}$, $\phi^1_{M_i} = \phi^1_{L_i} = \phi_i$, for all i, and

$$\|M_i\|_\infty \leq \epsilon_{i-1} + 2\|L_i\|_\infty \leq 5\epsilon_{i-1}, \qquad \overline{d}(\phi_{M_i}, \phi_{L_i}) < \epsilon_{i-1},$$

for all $i > 1$. In particular,

$$\overline{d}(\phi_{M_i}, \phi_{i-1}) \leq \overline{d}(\phi_{M_i}, \phi_{L_i}) + \overline{d}(\phi_{L_i}, \phi_{i-1}) < 6\epsilon_{i-1},$$

for all $i > 1$.

Finally, let $t_i = 1 - \frac{1}{2^i}$, for all $i \geq 0$. In particular, $0 = t_0 < t_1 < t_2 < \ldots < 1$. Then for $i \geq 1$, define the sequence N_i of smooth normalized boundary flat Hamiltonians, defined on

$[t_{i-1}, t_i]$, by $N_i = M_i^{\zeta_{t_{i-1},t_i}}$. As remarked above, we have

$$\|N_i\|_\infty = \frac{1}{t_i - t_{i-1}} \|M_i\|_\infty = 2^i \|M_i\|_\infty < 5 \cdot 2^i \epsilon_{i-1},$$

for all $i > 1$. By choosing ϵ_i sufficiently small, for example $\epsilon_{i-1} = \frac{1}{5}\frac{1}{4^i}$ for $i > 1$, we get $\|N_i\|_\infty < \frac{1}{2^i}$, and since M_i is just a reparameterization of N_i,

$$\overline{d}(\phi_{N_i}, \phi_{i-1}) = \overline{d}(\phi_{M_i}, \phi_{i-1}) < 6\epsilon_{i-1} < \frac{1}{2^i}.$$

The sequence F_i of smooth normalized Hamiltonians is then defined as follows. Let $F_1 = N_1$ on $[0, t_1]$, and $F_1 = 0$ on $[t_1, 1]$, and for $i > 1$, define

$$\begin{aligned} F_i &= F_{i-1} && \text{on } [0, t_{i-1}], \\ F_i &= N_i && \text{on } [t_{i-1}, t_i], \text{ and} \\ F_i &= 0 && \text{on } [t_i, 1]. \end{aligned}$$

The Hamiltonians F_i are indeed smooth due to boundary flatness of the functions N_i. We see that $\|F_i - F_{i-1}\|_\infty = \|N_i\|_\infty < \frac{1}{2^i}$. In particular, by the triangle inequality, $\|F_i - F_j\|_\infty \to 0$, as i, $j \to \infty$.

It follows from the definition that F_1 generates a reparameterization of the path ϕ_{H_1}, and for $i > 1$, the path generated by F_i is equal to the one generated by F_{i-1} on the interval $[0, t_{i-1}]$, equal to the path ϕ_{N_i} on the interval $[t_{i-1}, t_i]$, and is constant on the remaining interval $[t_i, 1]$. In particular, the paths ϕ_{F_i} are continuous, and due to the boundary flatness of the N_i, in fact smooth. Moreover, the paths $\phi_{F_{i-1}}$ and ϕ_{F_i} agree everywhere except on the interval $[t_{i-1}, 1]$. Since both paths are constant on the interval $[t_i, 1]$, their maximum distance (with respect to the C^0-metric) is achieved on the interval $[t_{i-1}, t_i]$. On that interval, $\phi_{F_{i-1}}$ is just the constant path ϕ_{i-1}, while ϕ_{F_i} is the path ϕ_{N_i} from ϕ_{i-1} to ϕ_i. By the above this implies that

$$\overline{d}(\phi_{F_{i-1}}, \phi_{F_i}) = \overline{d}(\phi_{N_i}, \phi_{i-1}) < \frac{1}{2^i}.$$

In particular, $\overline{d}(\phi_{F_i}, \phi_{F_j}) \to 0$, as i, $j \to \infty$.

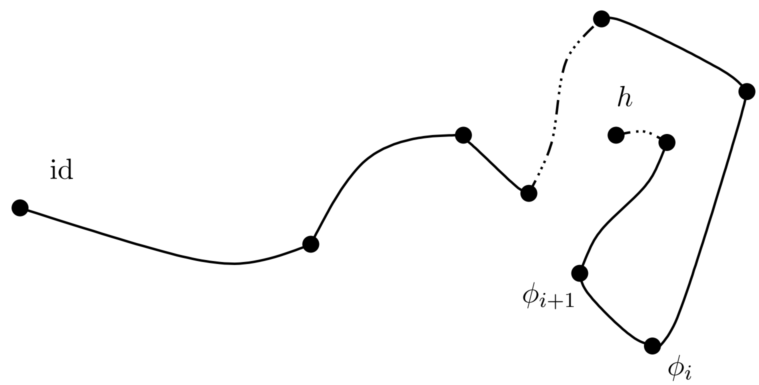

Figure 4: The sequence of paths ϕ_{F_i}

That is, the sequence (ϕ_{F_i}, F_i) is Cauchy in $\mathcal{P}^{\mathrm{ham}}_{\infty}\mathrm{Symp}(M,\omega)$, i.e. in the L^{∞}-Hamiltonian metric. Since $\phi^1_{F_i} = \phi_i \to h$, as $i \to \infty$, we conclude that $h \in \mathrm{Hameo}_{\infty}(M,\omega)$. Hence $\mathrm{Hameo}_{(1,\infty)}(M,\omega) \subset \mathrm{Hameo}_{\infty}(M,\omega)$, and since the other inclusion was already proved above, we have completed the proof of the theorem. □

2.6.1 Hofer norm of Hamiltonian homeomorphisms

We expand the discussion from [Oh07a] on the length or Hofer norm of a topological Hamiltonian path, and the resulting Hofer norm of a Hamiltonian homeomorphism. Let $(\lambda, H) \in \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$. By definition, there is a sequence $(\phi_{H_i}, H_i) \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, so that $\phi_{H_i} \to \lambda$ in the C^0-metric, and H_i converges in the norm $\|\cdot\|$ to the topological Hamiltonian H. We then define the *Hofer norm* of (λ, H) by

$$\|(\lambda, H)\| = \lim_{i\to\infty} \|\phi_{H_i}\| = \lim_{i\to\infty} \|H_i\| = \|H\|.$$

By the Uniqueness Theorem 2.3.5, this is in fact an invariant of the topological Hamiltonian path λ, at least in the L^{∞}-case. This definition agrees with the usual definition of the Hofer norm of a smooth Hamiltonian path. For $h \in \mathrm{Hameo}(M,\omega)$ a Hamiltonian homeomorphism, we define the *Hofer norm* of h to be

$$\|h\| = \inf\left\{ \|(\lambda, H)\| \mid (\lambda, H) \in \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega),\ \overline{\mathrm{ev}_1}(\lambda, H) = h \right\}. \tag{2.18}$$

This is obviously well-defined.

Let us denote by $\|\cdot\|_{\mathrm{Ham}}$ the usual Hofer norm (1.16) on $\mathrm{Ham}(M,\omega)$, and by $\|\cdot\|_{\mathrm{Hameo}}$ the Hofer norm (2.18) on $\mathrm{Hameo}(M,\omega)$ defined above. Then both norms are defined on $\mathrm{Ham}(M,\omega) \subset \mathrm{Hameo}(M,\omega)$. To avoid confusion, we will sometimes use these subscripts to denote the Hofer norms of a Hamiltonian diffeomorphism, but omit the subscript Hameo when we mean the Hofer norm of a general Hamiltonian homeomorphism.

Theorem 2.6.5 ([Oh07a])**.** *The Hofer norm (2.18) satisfies the following properties: For any h, $g \in \mathrm{Hameo}(M,\omega)$, we have*

(nondegeneracy) $\|h\| \geq 0$*, and* $\|h\| = 0$ *if and only if* $h = \mathrm{id}$*,*

(symmetry) $\|h\| = \|h^{-1}\|$*,*

(triangle inequality) $\|h \circ g\| \leq \|h\| + \|g\|$*, and*

(symplectic invariance) $\|\psi^{-1} \circ h \circ \psi\| = \|h\|$ *for any* $\psi \in \mathrm{Sympeo}(M,\omega)$*,*

and therefore the Hofer norm is indeed a norm. The Hofer norm is continuous with respect to the Hamiltonian topology, i.e. as a map $\mathcal{H}\mathrm{ameo}(M,\omega) \to \mathbb{R}$*. The Hofer norm induces a bi-invariant metric on* $\mathrm{Hameo}(M,\omega)$ *by defining* $\rho(h,g) = \|h^{-1} \circ g\|$*.*

Proof. Nonnegativity is obvious. Symmetry, the triangle inequality, and symplectic invariance, are proved precisely as in the smooth case. By definition of the Hamiltonian topology on $\mathrm{Hameo}(M,\omega)$, the function $\mathcal{H}\mathrm{ameo}(M,\omega) \to \mathbb{R}$ is continuous if and only if

$$\overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \xrightarrow{\overline{\mathrm{ev}_1}} \mathcal{H}\mathrm{ameo}(M,\omega) \xrightarrow{\|\cdot\|} \mathbb{R}$$

is continuous with respect to the Hamiltonian metric. To prove the latter, suppose (λ_i, H_i), $(\lambda, H) \in \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$, and $(\lambda_i, H_i) \to (\lambda, H)$ in the Hamiltonian metric. Then by the triangle inequality

$$|\|\lambda_i(1)\| - \|\lambda(1)\|| \leq \left\|\left(\lambda_i^{-1} \circ \lambda\right)(1)\right\| \leq \left\|(\lambda_i^{-1} \circ \lambda, \overline{H}_i \# H)\right\| = \left\|\overline{H}_i \# H\right\| \to 0,$$

as $i \to \infty$, proving continuity. To prove nondegeneracy, suppose $h \neq \mathrm{id}$. Then as in the prove of Theorem 2.2.1, h displaces a small compact ball B of positive displacement energy $\epsilon = e(B) > 0$. Let $(\lambda, H) \in \overline{\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}}(M, \omega)$, with $\lambda(1) = h$, such that $\|h\| > \|(\lambda, H)\| - \epsilon/3$. Then choose $(\phi_F, F) \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M, \omega)$ sufficiently close to (λ, H) in the Hamiltonian metric, i.e. so that $\|(\lambda, H)\| > \|F\| - \epsilon/3$, and so that $\phi = \phi_F^1$ still displaces B. Then by definition of the displacement energy,

$$\|h\| > \|(\lambda, H)\| - \frac{\epsilon}{3} > \|F\| - \frac{2\epsilon}{3} \geq \|\phi\|_{\mathrm{Ham}} - \frac{2\epsilon}{3} \geq \frac{\epsilon}{3} > 0.$$

It is now a standard exercise to check that ρ defines a left invariant metric on $\mathrm{Hameo}(M, \omega)$. The right invariance of ρ follows immediately from the symplectic invariance of $\|\cdot\|$. The proof is complete. $\square$

We clearly have $\|\cdot\|_{\mathrm{Hameo}} \leq \|\cdot\|_{\mathrm{Ham}}$. It seems likely that they are in fact equal.

Question 2.6.6. Does $\|\cdot\|_{\mathrm{Hameo}} = \|\cdot\|_{\mathrm{Ham}}$ hold on $\mathrm{Ham}(M, \omega)$?

The answer to this question is not known. The difficulty is that if, for some $\phi \in \mathrm{Ham}(M, \omega)$, the Hofer norm of (λ, H) 'approximates' $\|\phi\|_{\mathrm{Hameo}}$, then (λ, H) can in turn be approximated by smooth Hamiltonian paths in the Hamiltonian metric. However, the (right) end points of these paths are in general different from ϕ, and therefore these paths are not admissible to compute the Hofer norm $\|\phi\|_{\mathrm{Ham}}$ of ϕ. Note that there is a 'short' topological Hamiltonian path from each such end point to ϕ, but there need not be such a smooth path. A related question is thus whether we can choose above sequence of smooth Hamiltonian paths, so that the (right) end point equals ϕ for each path, or equivalently, can each topological Hamiltonian loop based at the identity be approximated by smooth Hamiltonian loops with the same base point? This problem seems to lie at the heart of topological Hamiltonian geometry.

In view of Theorem 2.6.1 and Lemma 1.1.2, one can then ask whether we have $\|\cdot\|_\infty = \|\cdot\|_{(1,\infty)}$ for the Hofer norms (2.18) on $\mathrm{Hameo}(M, \omega)$. The next Proposition is the precise analog to Lemma 1.1.2, and provides an affirmative answer to this question.

Proposition 2.6.7. $\|h\|_{(1,\infty)} = \|h\|_\infty$ *for each* $h \in \mathrm{Hameo}(M,\omega)$.

Note that if the answer to Question 2.6.6 were affirmative, then this together with Lemma 1.1.2, and the continuity of the Hofer norm with respect to the Hamiltonian topology, would immediately imply this equality.

Proof. The inequality $\|h\|_{(1,\infty)} \leq \|h\|_\infty$ is again obvious. Let $\epsilon > 0$, and choose (λ, H) in $\overline{\mathcal{P}^{\mathrm{ham}}_{(1,\infty)}\mathrm{Symp}}(M,\omega)$, with $\lambda(1) = h$ and $\|(\lambda, H)\|_{(1,\infty)} < \|h\|_{(1,\infty)} + \epsilon$. Choose $(\phi_F, F) \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, and construct $(\mu, K) \in \overline{\mathcal{P}^{\mathrm{ham}}_\infty\mathrm{Symp}}(M,\omega)$ as in the proof of Theorem 2.6.1, such that μ is a topological Hamiltonian path from id to $(\phi^1_F)^{-1} \circ h$, $\|F\|_{(1,\infty)} < \|(\lambda, H)\|_{(1,\infty)} + \epsilon < \|h\|_{(1,\infty)} + 2\epsilon$, and $\|(\mu, K)\|_\infty < \epsilon$. Then by Lemma 2.6.3, we can find $(\phi_G, G) \in \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, such that ϕ_G has the same end points as ϕ_F, and $\|G\|_\infty < \|F\|_{(1,\infty)} + \epsilon$.

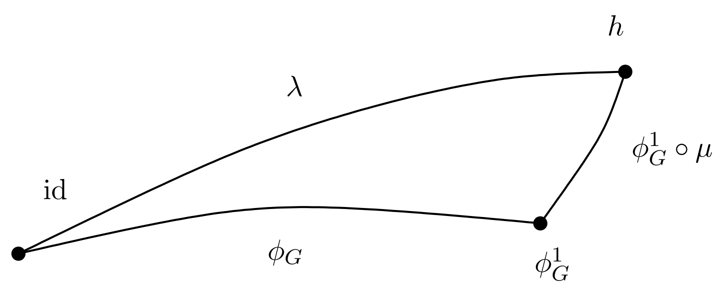

Figure 5: $L^{(1,\infty)}$ and L^∞ 'short' paths connecting h to the identity

Combining all of the above, we obtain

$$\|h\|_\infty \leq \|(\phi_G \circ \mu, G\#K)\|_\infty \leq \|G\|_\infty + \|(\mu, K)\|_\infty < \|F\|_{(1,\infty)} + 2\epsilon < \|h\|_{(1,\infty)} + 4\epsilon.$$

Since ϵ was arbitrary, that implies $\|h\|_\infty \leq \|h\|_{(1,\infty)}$, and hence the proof. $\square$

The above can be expressed as follows: given $h \in \mathrm{Hameo}(M,\omega)$, and $\epsilon > 0$, there exists (λ, H) so that the path λ is a topological Hamiltonian path in both the $L^{(1,\infty)}$-sense and the

L^∞-sense, and such that

$$\|h\|_\infty = \|h\|_{(1,\infty)} \leq \|(\lambda, H)\|_{(1,\infty)} \leq \|(\lambda, H)\|_\infty < \|h\|_\infty + \epsilon.$$

In other words, Theorem 2.6.1 states that each end point of an $L^{(1,\infty)}$-topological Hamiltonian path is also the end point of some (possibly different) L^∞-topological Hamiltonian path. And since $\|\cdot\|_\infty = \|\cdot\|_{(1,\infty)}$ on $\mathrm{Hameo}(M, \omega)$, the same statement holds for 'short' topological Hamiltonian paths as well.

Chapter 3

Open manifolds and other Hamiltonian topologies

In the next section we discuss the Hamiltonian topology on noncompact manifolds and manifolds with nonempty boundary. Then we consider possible variations of our definition of the Hamiltonian topology used in the rest of this work, and finish with a discussion of the essential features of the Hamiltonian topology in the context of C^0-Hamiltonian geometry and C^0-symplectic topology.

3.1 The case of open manifolds

So far we have assumed that M is closed. In this section we discuss the case of open manifolds, i.e. noncompact manifolds and manifolds with nonempty boundary $\partial M \neq \emptyset$. In particular, this includes the case $M = D^2 \subset \mathbb{R}^2$, and therefore completes the discussion of Example 2.4.5.

Suppose that M is noncompact and / or has nonempty boundary $\partial M \neq \emptyset$. We face two main difficulties not present for closed manifolds: firstly, the flow of a Hamiltonian vector field may no longer be defined for all times. More precisely, the time for which the integral curves of X_H are defined will in general depend on each initial condition, so that the flow ϕ_H may not be defined globally for all times, and possibly for no $t \neq 0$. Secondly, the various metrics $\overline{d}$, $\| \cdot \|$, and d_{ham} are not well-behaved. For instance, they are no longer complete metrics on the appropriate ambient spaces. We will now explain the necessary changes to be made in the open case to overcome these difficulties.

First of all, we require that all Hamiltonians H are compactly supported in the interior $\mathrm{Int}(M)$ of M. Then the Hamiltonian vector field X_H is compactly supported in $\mathrm{Int}(M)$, so that the flow exists and is unique for all times, and is supported in $\mathrm{Int}(M)$. The path ϕ_H is thus globally defined for all times, and supported in $\mathrm{Int}(M)$. We replace the normalization condition for closed manifolds by the requirement that the generating Hamiltonian is compactly supported in the interior of M. Then to each Hamiltonian path there corresponds a unique normalized Hamiltonian, and vice versa.

In the remainder of this section, K will always denote a compact subset $K \subset \mathrm{Int}(M)$. Denote by $C_K^\infty([0,1] \times M) \subset C^\infty([0,1] \times M)$ the space of Hamiltonians supported in K, and by $\mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ the space of Hamiltonians generated by elements of $C_K^\infty([0,1] \times M)$. We define the Hamiltonian metric d_{ham} on $\mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ just as before, and equip these sets with the induced topology, which we again call the Hamiltonian topology. Denote by

$$\mathcal{P}_c^{\mathrm{ham}}\mathrm{Symp}(M,\omega) = \bigcup_K \mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega),$$

where K ranges over all compact $K \subset \mathrm{Int}(M)$, the set of *compactly supported Hamiltonian paths*, and by

$$\iota_K \colon \mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \hookrightarrow \mathcal{P}_c^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$$

the obvious inclusions. The compact subsets $K \subset \mathrm{Int}(M)$ form a directed set under set inclusion, and the spaces $\mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, where K ranges over all compact $K \subset \mathrm{Int}(M)$, form a direct system, with morphisms the above inclusions ι_K. We give $\mathcal{P}_c^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ the direct limit topology, i.e. as a space

$$\mathcal{P}_c^{\mathrm{ham}}\mathrm{Symp}(M,\omega) = \varinjlim \mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega).$$

That is, $\mathcal{P}_c^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ is the set-theoretic limit of the sets $\mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$, equipped with the final topology, i.e. the largest (or strongest) topology such that all the inclusion maps ι_K are continuous. That means that $\mathcal{U} \subset \mathcal{P}_c^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ is open, if and only if $\mathcal{U} \cap \mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ is open in $\mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ for all K. The final topology can also be characterized as follows: a function $f\colon \mathcal{P}_c^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \to Z$, where Z is a topological space, is continuous, if and only if $f \circ \iota_K\colon \mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \to Z$ is continuous for all K.

Similarly, we put the direct limit or final topology on the set of *compactly supported Hamiltonians*, that is,

$$C_c^\infty([0,1] \ \times \ M) = \varinjlim C_K^\infty([0,1] \ \times \ M).$$

In general, if M is open, we only consider maps $M \to M$ and functions on M that are compactly supported in the interior of M. Note that we could also allow maps that are the identity on the boundary, instead of imposing they equal the identity near the boundary. However, our choice is consistent with the requirement that all Hamiltonian paths are the identity near the boundary, so we adopt this convention for all maps. Alternatively, one could also consider Hamiltonians that are the identity on the boundary, but some care has to be taken in the definitions: if M has nonempty boundary, then (by definition) (M,ω) extends to a symplectic manifold $(\widetilde{M},\widetilde{\omega})$ without boundary. Then consider only Hamiltonians on M that extend smoothly to $\widetilde{M}$ and are compactly supported in M. The corresponding Hamiltonian vector field on $\widetilde{M}$ is uniquely integrable, and its flow is supported in M, hence can be considered as a flow on M. We will only focus on the first case in our treatment here.

We denote the group of diffeomorphisms that are compactly supported in $\mathrm{Int}(M)$ by $\mathrm{Diff}^c(M,\partial M)$, and similarly for the group of homeomorphisms of M, and all their subgroups. We similarly define the sets $\mathcal{P}_c\mathrm{Diff}(M,\partial M)$, $\mathcal{P}_c\mathrm{Homeo}(M,\partial M)$, and their subgroups. A *compactly supported Hamiltonian diffeomorphism* is by definition the time-one map of a compactly supported Hamiltonian path. We denote the set of compactly supported Hamiltonian diffeomorphisms by $\mathrm{Ham}^c(M,\partial M,\omega) \subset \mathrm{Symp}_0^c(M,\partial M,\omega)$.

Denote by $\mathrm{Diff}^K(M) \subset \mathrm{Diff}^c(M,\partial M)$ the subgroup of diffeomorphism that are supported in the compact subset K, equipped with the C^∞-topology, and similarly for $\mathcal{P}_K\mathrm{Diff}(M)$, $\mathrm{Homeo}^K(M)$, and $\mathcal{P}_K\mathrm{Homeo}(M)$, and their various subgroups, with one exception, namely that $\mathrm{Ham}^K(M,\omega)$ is defined as the set of Hamiltonian diffeomorphisms that admit a generating Hamiltonian that is compactly supported in K. In other words, $\mathrm{Ham}^K(M,\omega)$ is the image under the time-one evaluation map of the space $\mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$. We give all these spaces the same topologies as in the case of closed manifolds. For example, the compact-open topology on $\mathrm{Homeo}^K(M)$ and $\mathcal{P}_K\mathrm{Homeo}(M)$ is induced by the complete metric $\overline{d}$. We then equip all of the above sets of compactly supported diffeomorphisms, homeomorphisms, and paths, with the direct limit topology, which is defined just as in the case of Hamiltonian paths and functions above. In particular, the group of symplectic homeomorphisms is defined as

$$\mathrm{Sympeo}^c(M,\partial M,\omega) = \varinjlim \mathrm{Sympeo}^K(M,\omega),$$

where $\mathrm{Sympeo}^K(M,\omega)$ is defined as the closure of $\mathrm{Symp}^K(M,\omega) \subset \mathrm{Homeo}^K(M)$ with respect to the C^0-topology, compare to Definition 2.1.1. We remark that for any continuous path defined on the interval $[0,1]$, or more generally, any compact interval, there is a compact $K \subset \mathrm{Int}(M)$, containing the support of the entire path. This means, for example, that any continuous path in the space $\mathrm{Symp}^c(M,\partial M,\omega)$, lies in $\mathcal{P}_c\mathrm{Symp}(M,\partial M,\omega)$, or $\mathcal{P}\mathrm{Symp}^c(M,\partial M,\omega) = \mathcal{P}_c\mathrm{Symp}(M,\partial M,\omega)$.

Note that if M is closed, these direct limit topologies all agree with the usual topologies we used in the rest of this work, so that we could adopt the definitions in this section as the general definition for all manifolds M.

We define the *Hamiltonian topology* on the set $\mathrm{Ham}^c(M,\partial M,\omega)$ to be the direct limit topology induced by the direct system

$$\mathcal{H}\mathrm{am}^c(M,\partial M,\omega) = \bigcup_K \mathcal{H}\mathrm{am}^K(M,\omega),$$

where $\mathcal{H}\mathrm{am}^K(M,\omega)$ denotes the set $\mathrm{Ham}^K(M,\omega)$ with the Hamiltonian topology, which is defined as the largest (or strongest) topology such that the map $\mathrm{ev}_1 \colon \mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \to \mathrm{Ham}^K(M,\omega)$ is continuous.

The space $\mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ can again be identified with a subset of $\mathcal{P}_K\mathrm{Homeo}(M) \times C_K^\infty([0,1] \times M)$, via the inclusion ι_{ham} and the developing map, and we can consider its completion $\overline{\mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$ with respect to the Hamiltonian metric. We then define

$$\overline{\mathcal{P}_c^{\mathrm{ham}}\mathrm{Symp}}(M,\partial M,\omega) = \bigcup_K \overline{\mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}}(M,\omega),$$

topologized as a direct limit. Note that, despite the suggestive notation used here, the set $\overline{\mathcal{P}_c^{\mathrm{ham}}\mathrm{Symp}}(M,\partial M,\omega)$ is not defined as the completion of the space $\mathcal{P}_c^{\mathrm{ham}}\mathrm{Symp}(M,\partial M,\omega)$ of compactly supported Hamiltonian paths with respect to the Hamiltonian metric. The latter may not be contained in a complete metric space (other than its abstract completion), as it was the case for closed manifolds.

We have the following obvious (Lipschitz) continuous extended maps

$$\overline{\iota_{\mathrm{ham}}} \colon \overline{\mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \longrightarrow \mathcal{P}_K\mathrm{Homeo}(M),$$

and

$$\overline{\mathrm{Dev}} \colon \overline{\mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \longrightarrow H_K([0,1] \times M),$$

and the time-one evaluation map

$$\overline{\mathrm{ev}_1} \colon \overline{\mathcal{P}_K^{\mathrm{ham}}\mathrm{Symp}}(M,\omega) \longrightarrow \mathrm{Homeo}^K(M).$$

We define the spaces $\mathcal{P}_K^{\mathrm{ham}}\mathrm{Sympeo}(M,\omega)$, $\mathcal{H}_K([0,1] \times M)$, and $\mathrm{Hameo}^K(M,\omega)$ as the images of the map $\overline{\iota_{\mathrm{ham}}}$, the developing map $\overline{\mathrm{Dev}}$, and the time-one evaluation map $\overline{\mathrm{ev}_1}$ defined above, respectively, equipped with the subspace topologies. Then define the spaces $\mathcal{P}_c^{\mathrm{ham}}\mathrm{Sympeo}(M,\partial M,\omega)$, $\mathcal{H}_c([0,1] \times M)$, and $\mathrm{Hameo}^c(M,\partial M,\omega)$ as direct limits, and call their elements *compactly supported topological Hamiltonian path*, *compactly supported topological Hamiltonian functions*, or for short *compactly supported topological Hamiltonians*, and *compactly supported Hamiltonian homeomorphisms*, respectively. We define the Hamiltonian topology on the set $\mathrm{Hameo}^K(M,\omega)$ as before, that is, as the largest (or strongest) topology such that the time-one evaluation map $\overline{\mathrm{ev}_1}\colon \overline{\mathcal{P}^{\mathrm{ham}}_K\mathrm{Symp}}(M,\omega) \to \mathrm{Hameo}^K(M,\omega)$ is continuous, and denote the resulting topological space by $\mathcal{H}\mathrm{ameo}^K(M,\omega)$. We then define the *Hamiltonian topology* on the set $\mathrm{Hameo}^c(M,\partial M,\omega)$ as the direct limit

$$\mathrm{Hameo}^c(M,\partial M,\omega) = \varinjlim \mathcal{H}\mathrm{ameo}^K(M,\omega).$$

With these definitions, analogs to all the results previously stated for the case of closed manifolds still hold. For example, we have the following theorem. We give a brief sketch of the proof to indicate how the proofs in the previous chapters can be generalized to open manifolds.

Theorem 3.1.1. *The group* $\mathrm{Hameo}^c(M,\partial M,\omega)$ *is path connected, and a normal subgroup of* $\mathrm{Sympeo}^c(M,\partial M,\omega)$.

Proof. Let $h \in \mathrm{Hameo}^c(M,\partial M,\omega)$. By definition, there exists a compact $K \subset \mathrm{Int}(M)$, and $(\lambda,H) \in \overline{\mathcal{P}^{\mathrm{ham}}_K\mathrm{Symp}}(M,\omega)$, such that $h = \overline{\mathrm{ev}_1}(\lambda,H)$. In particular, (λ,H) is the limit of a sequence (ϕ_{H_i},H_i) with respect to the Hamiltonian metric d_{ham}, where $(\phi_{H_i},H_i) \in \mathcal{P}^{\mathrm{ham}}_K\mathrm{Symp}(M,\omega)$ for all i. The proof that λ is a continuous path in $\mathrm{Hameo}^c(M,\partial M,\omega)$, connecting h to the identity, is now the same as in the closed case. For any two paths (λ,H) and $(\mu,L) \in \overline{\mathcal{P}^{\mathrm{ham}}_c\mathrm{Symp}}(M,\omega)$, there is by definition a compact $K \subset \mathrm{Int}(M)$ such that (λ,H), $(\mu,L) \in \overline{\mathcal{P}^{\mathrm{ham}}_K\mathrm{Symp}}(M,\omega)$. Then the group operations on $\overline{\mathcal{P}^{\mathrm{ham}}_c\mathrm{Symp}}(M,\omega)$ can be defined just as before. $\mathrm{Hameo}^c(M,\partial M,\omega)$ evidently becomes a subgroup of $\mathrm{Sympeo}^c(M,\partial M,\omega)$. Normality is proved similarly. $\square$

In relation to Conjecture 2.5.5, we also state the following analogs to Theorem 2.1.5 and Corollary 2.1.6.

Theorem 3.1.2. *Let M be a smooth n-manifold, equipped with a measure induced by some volume form Ω on M. If a compactly supported measure-preserving homeomorphism h can be*

C^0-approximated by diffeomorphisms (e.g. if $n \leq 3$), then it can be C^0-approximated by volume-preserving diffeomorphisms that are supported in some fixed compact subset $K \subset \mathrm{Int}(M)$.

Proof. Let $h \in \mathrm{Homeo}_c^\Omega(M, \partial M)$. Then there exists a compact subset $K' \subset \mathrm{Int}(M)$ with $\mathrm{supp}(h) \subset K'$. We can find approximating diffeomorphisms as in the proof of Theorem 2.1.5 that are compactly supported in some fixed compact K, with $K' \subset K \subset \mathrm{Int}(M)$. Then the same arguments as in the proof in the closed case apply, in particular, the vector field X_t is uniquely integrable, and generates a flow that is compactly supported in K. Thus h can be approximated by volume-preserving diffeomorphisms that are compactly supported in K. $\square$

Corollary 3.1.3. *Let M be an orientable surface, and $\omega = \Omega$ be any area form on M, then*

$$\mathrm{Sympeo}^c(M, \partial M, \omega) = \mathrm{Homeo}_c^\Omega(M, \partial M), \qquad \mathrm{Sympeo}_0^c(M, \partial M, \omega) = \mathrm{Homeo}_{c,0}^\Omega(M).$$

3.2 Other Hamiltonian topologies

In this section we discuss various modifications of our definition of the Hamiltonian topology. There are in fact many possible variations (which can often be combined in more than one way), so we only discuss the most interesting ones from our point of view. We will complete our considerations with a discussion of the essential features a 'Hamiltonian topology' should possess for the study of C^0-Hamiltonian geometry and C^0-symplectic topology in our sense.

3.2.1 Weak Hamiltonian topology

We can define the notion of a weak Hamiltonian topology similarly to the Hamiltonian topology, which we for emphasis temporarily call the strong Hamiltonian topology. In the sets (2.1), we just replace the C^0-distance of the whole paths by the C^0-distance of the time-one maps only, and similarly in the definition of the Hamiltonian metric (Definition 2.2.6). In the weak Hamiltonian topology, we do not have any control over the C^0-convergence of the whole paths other than the time-one maps. Although this seems natural in light of Theorem 2.2.1, it turns out that the weak Hamiltonian topology does not behave as nicely as the strong Hamiltonian topology. For example, it is unlikely that the map Tan is continuous with respect to the weak Hamiltonian topology, and that the analogously defined spaces $\overline{\mathcal{P}_w^{\mathrm{ham}}\mathrm{Symp}}(M,\omega)$ and $\mathrm{Hameo}_w(M,\omega)$ form groups. Many of the proofs given above fail for the weak Hamiltonian topology, essentially because we do not have a version of Proposition 2.3.9 at our disposal. The strong Hamiltonian topology is obviously larger (or stronger) than the weak one, but it is an open question whether they are indeed different in general, i.e. whether the strong Hamiltonian topology is strictly larger.

3.2.2 Induced metrics

We can define a function $\mathrm{Ham}(M,\omega) \times \mathrm{Ham}(M,\omega) \to \mathbb{R}$ by setting

$$d_{\mathrm{ham}}(\phi,\psi) = \inf\left\{d_{\mathrm{ham}}(\phi_H,\phi_K) \mid H \mapsto \phi, K \mapsto \psi\right\},$$

and similarly a weak version d_{ham}^w defined using the weak Hamiltonian metric on the path space $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$.

Let us discuss this in the general context of metric spaces and topological groups. Let (X,d) denote a metric space, equipped with the topology induced by the metric. Let Y be a set, and $p\colon X \to Y$ be a surjective function. We can define a function $\rho\colon Y \times Y \to \mathbb{R}$ by

$$\rho(y,y') = \inf\left\{d(x,x') \mid p(x) = y, p(x') = y'\right\}.$$

This function is obviously nonnegative and symmetric, but it may fail to be nondegenerate and / or to satisfy the triangle inequality. In the situation considered above, the function d_{ham} is nondegenerate, so we will focus on the triangle inequality here. To get a handle on this, it makes sense to assume in addition that X is a group (in fact, as in our motivating situation, a topological group with respect to the metric topology). If the metric d is left or right invariant, then ρ does satisfy the triangle inequality, as is readily verified. However, the Hofer norm on $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ is only left (but not right) invariant, and the metric $\hat{d}$ is only right (but not left) invariant, so the metric $\overline{d}$ is neither left nor right invariant (the second term in the definition of $\overline{d}$ is only right (but not left) invariant). Note that the Hofer norm (1.16) on $\mathrm{Ham}(M,\omega)$ is in fact bi-invariant. It is very natural to consider bi-invariant metrics. In our set-up, we trade in this naturality for the completeness of the metric $\overline{d}$: recall that the metric $\hat{d}$ is not complete.

Here is a simple example that illustrates our dilemma.

Example 3.2.1. Let $p\colon \mathbb{R} \to \mathbb{R}/2\mathbb{Z} = S^1$ be the universal covering space of the circle, equipped with the standard topologies and group structures. Then p becomes a continuous and open group homomorphism, and in fact, the topology on S^1 coincides with the quotient topology induced by the covering projection. Even in this seemingly 'nice' situation, we can put a nonstandard metric on $\mathbb{R}$ so that ρ fails to be a pseudo-metric. Define d so that it is equal to the standard metric on each interval $[n, n+1/2]$ up to a rescaling factor. Choose this factor to be $1/2$ on $[0, 1/2]$ and $[-1, -1/2]$, and $3/2$ on $[1/2, 1]$ and $[-1/2, 0]$, and so that the distance between any two consecutive integers equals 1. Then in S^1, the distance between 0 and 1 equals 1, while the distance from each of these two points to $1/2$ is only $1/4$. Therefore, ρ violates the triangle inequality. Note that d induces the standard topology on $\mathbb{R}$, in fact, it is equivalent to the standard metric: $1/2 \cdot d \leq |\cdot| \leq 3/2 \cdot d$. By changing the rescaling factor as we 'move toward' ∞, we can easily achieve that the distance between 0 and $1/2$ is zero in S^1, so that ρ becomes degenerate. This rescaled metric still induces the standard topology on the real line, however, it is of course not equivalent to the standard metric.

If ρ is indeed a metric, it is natural to ask whether the induced metric topology coincides with the quotient topology on Y induced by the function p. Note that if ρ fails to satisfy the triangle inequality, or is degenerate, then the ϵ-balls with respect to ρ only form a subbasis of a topology, and this topology is non-Hausdorff, respectively, so we are not really interested in these situations. Since p is continuous with respect to the metrics d and ρ, the quotient topology is always larger (or stronger) than the metric topology. In our original example, the Hamiltonian topology on the set $\mathrm{Ham}(M, \omega)$ is defined as the quotient topology induced by the projection $\mathrm{ev}_1 \colon \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M, \omega) \to \mathrm{Ham}(M, \omega)$, see Definition 2.2.4. One advantage of this quotient topology is that a function $f \colon \mathcal{H}\mathrm{am}(M, \omega) \to Z$, where Z is a topological space, is continuous, if and only if the composition $f \circ \mathrm{ev}_1$ is continuous. If Z is in addition a metric space, one only has to check whether the map $f \circ \mathrm{ev}_1$ is a continuous map between metric spaces, which is in practice often easier to verify than continuity of maps between general topological spaces. Similar remarks apply to the space $\mathcal{H}\mathrm{ameo}(M, \omega)$. (Compare to the proof of Theorem

2.6.5.) Remark further that in the situation considered in the beginning, the space $\mathrm{Ham}(M,\omega)$ is infinite dimensional and thus has no local compactness properties, and the function ev_1 is far from being proper: it is a homomorphism, and its kernel is the space of Hamiltonian loops, which is 'very noncompact'. These two facts pose a major difficulty in answering the above questions. Note that if d is left or right invariant, and p is a homomorphism, then it is not too hard to see that the metric topology indeed coincides with the quotient topology. This is for example the case if $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ is given the Hofer topology, and we consider the map $\mathrm{ev}_1\colon \mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega) \to \mathrm{Ham}(M,\omega)$.

In general I do not know if the (nondegenerate) function d_{ham} defines a metric on the set $\mathrm{Ham}(M,\omega)$, and if it does, whether the metric topology is equivalent to the quotient topology. Moreover, even if d_{ham} does define a metric, which in addition induces the quotient topology on $\mathrm{Ham}(M,\omega)$, it is not clear whether each Cauchy sequence in $\mathrm{Ham}(M,\omega)$ lifts to a Cauchy sequence in the path space $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ (unless again, the metric were left or right invariant). Therefore, the metric completion of $\mathrm{Ham}(M,\omega)$ is a priori larger than the group of Hamiltonian homeomorphisms. While each Hamiltonian homeomorphism is the end point of a topological Hamiltonian path, this need not be true for a general element in the metric completion, unless the two sets coincide. For these reasons, we define the group of Hamiltonian homeomorphisms as time-one maps of topological Hamiltonian paths, and use the quotient topology as the definition of the Hamiltonian topology on the spaces $\mathcal{H}\mathrm{am}(M,\omega)$ and $\mathcal{H}\mathrm{ameo}(M,\omega)$, see Definitions 2.2.4 and 2.3.7 respectively.

3.2.3 $\mathbf{L}^p$-norms

One might consider replacing the oscillation of the Hamiltonians H_t in the above definitions by L^p norms

$$\|H\|_p = \left(\int_M |H|^p \Omega\right)^{1/p},$$

for $H \in C^\infty_m(M)$, and $1 \leq p < \infty$, where Ω denotes the Liouville volume form. However, a typical element of the abstract completion of $C^\infty_m([0,1]\times M)$ with respect to the norms $\int_0^1 \|\cdot\|_p dt$ or $\max_{t\in[0,1]} \|\cdot\|_p$ would only be well-defined a.e. on $[0,1]\times M$, and could be discontinuous on a dense subset of $[0,1]\times M$. And these norms do not appear to have any properties that might make them preferable to the $\|\cdot\|_{(1,\infty)}$ norm. In relation to this, note that if one equips $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$ with the metric induced by $\|\cdot\|_p$ only, not with the Hamiltonian metric, then the induced Hofer metric on $\mathrm{Ham}(M,\omega)$ is degenerate [EP93]. We would like to refer to [EP93] or Chapter 2 in [Pol01] for a more detailed discussion of L^p-norms and invariant norms on $C^\infty_m(M)$. As pointed out there, there may be many such invariant norms, and it is not understood which ones give rise to genuine metrics on $\mathrm{Ham}(M,\omega)$. This makes the question of which possible norms give rise to 'good' Hamiltonian topologies very difficult as well.

3.2.4 Metrics on $\mathrm{Ham}(M,\omega)$

It is possible to define a 'Hamiltonian metric' directly on the group $\mathrm{Ham}(M,\omega)$, not via the infimum over a distance defined on the path space $\mathcal{P}^{\mathrm{ham}}\mathrm{Symp}(M,\omega)$. This metric should be of the form $\|\phi^{-1}\circ\psi\|+\overline{d}(\phi,\psi)$, for ϕ, $\psi \in \mathrm{Ham}(M,\omega)$, where $\|\cdot\|$ is a Hofer or 'Hofer-like' metric. As remarked in the previous section, any L^p-norm, $1 \leq p < \infty$, gives rise to a degenerate pseudo-metric. If M is closed, as a consequence of Banyaga's theorem (i.e. $\mathrm{Ham}(M,\omega)$ is simple) [Ban78], the induced pseudo-metrics on $\mathrm{Ham}(M,\omega)$ vanish identically (see [Pol01]). Thus the above metrics would simply give rise to the C^0-metric. By Polterovich's Lemma 1.1.2 [Pol01], one obtains the same metric regardless of whether one starts with the L^∞ or $L^{(1,\infty)}$-norm on the space $C^\infty_m([0,1]\times M)$.

Using generating functions, one could replace Hofer convergence on $\mathrm{Ham}(M,\omega)$ by c-convergence, i.e. convergence in Viterbo's distance γ, see [Vit92, Hum07], or the distance $\widetilde{\gamma}$, see [CV07, Hum07], for the definitions and some related results. It is known that $\widetilde{\gamma}(\phi^1_H) \leq \|H\|_\infty$, so that $\widetilde{\gamma} \leq \|\cdot\|$ on $\mathrm{Ham}(M, \omega)$ [CV07, Proposition 2.6].

In relation to this, we would like to mention the following result by Hofer [Hof93, HZ94] on $\mathbb{R}^{2n}$:

$$\|\phi \circ \psi^{-1}\| \leq C \cdot \mathrm{diam}\,\left(\mathrm{supp}(\phi \circ \psi^{-1})\right) \|\phi \circ \psi^{-1}\|_{C^0},$$

where $C \leq 128$ is a constant. This in particular implies that the C^0-topology is larger (or stronger) than the Hofer topology on $\mathrm{Ham}^c(\mathbb{R}^{2n}, \omega_0)$ if $\mathrm{supp}(\phi \circ \psi^{-1})$ is controlled. Observe that

$$\|\phi \circ \psi^{-1}\|_{C^0} = \hat{d}(\phi \circ \psi^{-1}, \mathrm{id}) = \hat{d}(\phi, \psi).$$

See Section 4 of [Vit92] for some similar inequalities involving Viterbo's distance.

The following is a slight variation of an interesting observation by Bates [Bat94].

Theorem 3.2.2. *Any* $\phi \in \mathrm{Symp}^c(\mathbb{R}^{2n}, \omega_0)$ *can be approximated in the* C^0*-topology by diffeomorphisms in* $\mathrm{Ham}^c(\mathbb{R}^{2n}, \omega_0) = \mathrm{Symp}^c_0(\mathbb{R}^{2n}, \omega_0)$*. In fact, the direct limit of the* C^0*-closures of the spaces* $\mathrm{Ham}^K(\mathbb{R}^{2n}, \omega_0)$ *in* $\mathrm{Symp}^K(\mathbb{R}^{2n}, \omega_0)$*, coincides with* $\mathrm{Symp}^c(\mathbb{R}^{2n}, \omega_0)$*.*

Proof. Let $\psi \in \mathrm{Symp}^c(\mathbb{R}^{2n}, \omega_0)$. By conjugating ψ by a compactly supported Hamiltonian diffeomorphism that moves the support of ψ away from the origin, we may without loss of generality assume $0 \notin \mathrm{supp}(\psi)$. Then there exists an annulus

$$A = A(R, r) = \left\{x \in \mathbb{R}^{2n} \mid r < \|x\| < R\right\} \subset \mathbb{R}^{2n},$$

where $0 < r < R < \infty$, and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^{2n}$, such that $\mathrm{supp}(\psi) \subset A$. For $t \in \mathbb{R}$, denote by R_t multiplication by $t+1$ in $\mathbb{R}^{2n}$. Consider the sequence $\phi_n \in \mathrm{Ham}^c(\mathbb{R}^{2n}, \omega_0)$, where ϕ_t is the symplectic isotopy

$$\phi_t = \left[R_t^{-1}, \psi^{-1}\right] = R_t^{-1} \circ \psi^{-1} \circ R_t \circ \psi,$$

with $\phi_0 = \mathrm{id}$. Then

$$\frac{d}{dt}\phi_t \circ \phi_t^{-1} = R_t^* \left(\psi^* X - X\right),$$

where X denotes the radial vector field in $\mathbb{R}^{2n}$, which integrates to the flow $t \mapsto R_t$. In particular, this vector field is supported in $\operatorname{supp}(\psi)$. Therefore, $\operatorname{supp}(\phi_t) \subset K$ independent of t (recall the definition of the space of compactly supported diffeomorphisms as direct limits). For t sufficiently large, we have

$$\phi_t(x) = \begin{cases} \psi(x) & \text{if } x \in \operatorname{supp}(\psi), \\ (R_t^{-1} \circ \psi \circ R_t)(x) & \text{if } x \notin \operatorname{supp}(\psi). \end{cases}$$

Since

$$\operatorname{supp}\left(R_t^{-1} \circ \psi \circ R_t\right) \subset R_t^{-1}\left(\operatorname{supp}(\psi)\right) \subset A\left(\frac{R}{t+1}, \frac{r}{t+1}\right),$$

we see that

$$|\psi(x) - \phi_t(x)| < \frac{2R}{t+1}.$$

That proves $\hat{d}(\psi, \phi_n) \to 0$ as $n \to \infty$. The proof of $\hat{d}(\psi^{-1}, \phi_n^{-1}) \to 0$ as $n \to \infty$ is similar. $\square$

Since $\operatorname{supp}(\psi \circ \phi_n^{-1}) \subset A$ for all n, Hofer's inequality cited above implies the same result if we replace the C^0-distance by $\|\cdot\| + \overline{d}$ in the above statement. By considering diagonal subsequences, we immediately obtain the following corollary.

Corollary 3.2.3. *Define a subset of* $\mathrm{Homeo}^c(\mathbb{R}^{2n})$ *as the direct limit of the closures of the spaces* $\mathrm{Ham}^K(\mathbb{R}^{2n}, \omega_0)$ *in* $\mathrm{Homeo}^K(\mathbb{R}^{2n})$ *with respect to the compact-open topology. Then this set is the whole* $\mathrm{Sympeo}^c(\mathbb{R}^{2n}, \omega_0)$.

As we just recalled, the closure in $\mathrm{Symp}^c(\mathbb{R}^{2n}, \omega_0)$ of the identity component $\mathrm{Ham}^c(\mathbb{R}^{2n}, \omega_0)$ of $\mathrm{Symp}^c(\mathbb{R}^{2n}, \omega_0)$, with respect to the C^0-topology, is the whole $\mathrm{Symp}^c(\mathbb{R}^{2n}, \omega_0)$. However, as was pointed out in [MS98], this does not necessarily prove that $\mathrm{Ham}^c(\mathbb{R}^{2n}, \omega_0)$ is not closed in $\mathrm{Symp}^c(\mathbb{R}^{2n}, \omega_0)$, since it is not known whether $\mathrm{Symp}^c(\mathbb{R}^{2n}, \omega_0)$ is disconnected. Nonetheless, Corollary 3.2.3 gives some indication that the C^0-closure of $\mathrm{Ham}(M, \omega)$ in $\mathrm{Homeo}(M)$ could be 'too big'. In contrast, the Hamiltonian homeomorphism group is always path connected (Theorem 2.4.2).

It is also possible to study abstract completions of $\mathrm{Ham}(M, \omega)$ with respect to Hofer's distance without the C^0-distance. See [Bat94] for some remarks in this direction for $\mathbb{R}^{2n}$, and [Hum07] for completions with respect to Viterbo's distance.

Note however that for all these metrics defined on $\mathrm{Ham}(M, \omega)$, we lose the property that a Hamiltonian homeomorphism is the end point of a Hamiltonian path (in a suitable sense).

3.2.5 Features of the Hamiltonian topology

As the considerations in this section show, there are different workable definitions of Hamiltonian topology one could adopt. From our point of view, the Hamiltonian homeomorphism group should be a 'good' C^0-counterpart to the Hamiltonian diffeomorphism group. In particular, it is reasonable to expect the Hamiltonian homeomorphism group to be a topological group, and a normal subgroup of the symplectic homeomorphism group. It should also be path connected, and moreover, a Hamiltonian homeomorphisms should be the end point of a Hamiltonian path in some suitable sense. Note that this excludes some of the ideas introduced above.

Finally note that Banyaga [Ban08a, Ban08b] has recently proposed an alternate definition of the symplectic homeomorphism group. In the present work, this is the closure of the group of symplectic diffeomorphisms with respect to the C^0-topology (Definition 2.1.1), simply for the lack of an analog to Hofer's distance for a general isotopy of symplectic diffeomorphisms. Banyaga recently defined such an analog of Hofer's distance, and used it to define what he calls strong symplectic homeomorphisms. We refer to [Ban08a, Ban08b] for more details.

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