Finite-Size Scaling in Complex Networks

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A finite-size-scaling (FSS) theory is proposed for various models in complex networks. In particular, we focus on the FSS exponent, which plays a crucial role in analyzing numerical data for finite-size systems. Based on the droplet-excitation (hyperscaling) argument, we conjecture the values of the FSS exponents for the Ising model, the susceptible-infected-susceptible model, and the contact process, all of which are confirmed reasonably well in numerical simulations.

Critical phenomena in complex networks have attracted much attention recently and traditional models in statistical physics have been examined on diverse networks [1–7]. Interesting questions in such probes would be the existence of phase transitions and the network-dependent critical behavior near the transition. Until now, various equilibrium systems such as Ising, Potts, and *XY* models [1–4], as well as nonequilibrium systems such as percolation [5], directed percolation [6], and synchronization models [7], have been studied by means of the mean-field (MF) approach [2], the replica method [3], and the thermodynamic potential hypothesis [1]. It is predicted that phase transitions in complex networks exhibit mostly the *standard* MF critical behavior except in highly heterogeneous scale-free (SF) networks where the interesting heterogeneity-dependent (but still MF-type) critical behavior appears [1–3].

The MF prediction has been brought to the test by extensive numerical simulations and passed it reasonably well in cases of less heterogeneous networks like random, small-world, and even some SF networks [4–7]. However, for the highly heterogeneous networks (like most of the SF networks found in nature), the asymptotic scaling regime could not be reached easily in numerical tests due to huge finite-size effects and consequently there is no reasonably solid numerical analysis reported as of yet. Therefore, it is essential to understand the finite-size-scaling (FSS) behavior analytically in networks, not only to analyze numerical data for finite-size networks, but also to explore the physics of correlated size scales in networks.

In the present Letter, we propose a FSS theory for various models in complex networks, based on a droplet-excitation (hyperscaling) argument. Our conjecture for the FSS exponent values is confirmed via numerical simulations for the Ising model and the contact process [8].

We first start with the standard FSS theory in low dimensional systems. As a typical example, consider the ferromagnetic Ising model. Its critical behavior near the transition is characterized by the singular behavior in the magnetization $m \sim \epsilon^\beta$, the susceptibility $\chi \sim |\epsilon|^{-\gamma}$, and the correlation length $\xi \sim |\epsilon|^{-\nu}$, where ϵ is the reduced temperature defined by $\epsilon \equiv (T_c - T)/T_c$.

The standard FSS theory for the singular part of the free energy f reads [9]

$$f(\epsilon, h, L^{-1}) = b^{-d} f(b^{y_T}\epsilon, b^{y_H}h, bL^{-1}), \quad (1)$$

where b is the scale factor, d the spatial dimension, L the system linear size, and h the external field. The two scaling dimensions, y_T and y_H, determine all thermodynamic exponents such as $\beta = (d - y_H)/y_T$ and $\gamma = (2y_H - d)/y_T$. The FSS behavior of the spontaneous magnetization is derived from Eq. (1) as

$$m = L^{-\beta y_T} g(L^{y_T}\epsilon), \quad (2)$$

where g is the scaling function and $d/y_T = 2\beta + \gamma$. If the correlation length scales as $\xi(\epsilon, h) = b\xi(b^{y_T}\epsilon, b^{y_H}h)$, the hyperscaling relation appears as $y_T = 1/\nu$ [9]. In this case, the FSS variable $L^{y_T}\epsilon$ can be expressed as $(L/\xi)^{y_T}$, implying that the correlation length competes with the system size. In low dimensions where the hyperscaling holds, the above FSS theory works perfectly well for most equilibrium models and even for nonequilibrium ones (like directed percolation models) with appropriate modifications [10].

In high dimensions where the MF theory becomes valid, the FSS becomes a little bit tricky. It has been argued that a ''dangerous irrelevant variable'' comes into play in high dimensions which breaks the hyperscaling ($y_T \neq 1/\nu$) and modifies the FSS [11]. Nevertheless, for block-shape samples with periodic boundary conditions, Eqs. (1) and (2) are still valid with the trivial MF values of $y_T = d/2$ and $y_H = 3d/4$, which correspond to the MF bulk exponents such as $\beta = 1/2$ and $\gamma = 1$. The broken hyperscaling implies that there must be another length scale competing with the system size, rather than the usual Gaussian correlation length.

Consider the Landau free energy of the order parameter m near the transition in the MF regime as

$$f(m) = -\epsilon m^2 + um^4 + \mathcal{O}(m^6), \quad (3)$$

where the Gaussian spatial fluctuation term $(\nabla m)^2$ is ignored. In the ordered phase ($\epsilon > 0$), the free energy has a minimum at $m = m^* \sim \epsilon^{1/2}$ with $f \sim -\epsilon^2$. One may ask

a typical size ξ_T of a disordered droplet excitation out of the uniformly ordered environment. As the free energy cost by the droplet excitation is compensated by the thermal energy, $(\Delta f)\xi_T^d \sim k_B T$, we find $\xi_T \sim \epsilon^{-\nu_T}$ with $\nu_T = 2/d = 1/y_T$. The Gaussian length scale diverges as $\xi_G \sim \epsilon^{-\nu_G}$ with $\nu_G = 1/2$.

For $d > d_u = 4$ (the upper critical dimension), ξ_G dominates over ξ_T, which leads to the correlation length exponent $\nu = \nu_G$ and the MF theory is valid. However, the FSS variable $L^{y_T}\epsilon$ becomes $(L/\xi_T)^{y_T}$, implying that the competing length scale is not the dominant correlation length but the droplet size. Substituting the linear size L by the volume $N \sim L^d$, Eq. (2) reads

$$m = N^{-\beta/\bar{\nu}}\psi(N^{1/\bar{\nu}}\epsilon), \qquad (4)$$

where the FSS (droplet volume) exponent $\bar{\nu} = d\nu_T = 2$ in the MF regime. For the general ϕ^q MF theory ($f = -\epsilon m^2 + u m^q$), we find that $\bar{\nu} = d_u \nu_G$ with $d_u = 2q/(q-2)$, which is consistent with the earlier result by Botet *et al.* for models with infinite-range interactions [12].

We are now ready to explore the FSS in networks. Networks have no space dimensionality and may be considered as a limiting case of $d \to \infty$. So we expect that any model in networks displays a MF-type critical behavior. In particular, the MF FSS exponent $\bar{\nu}$ is independent of d, which leads to the natural conjecture that Eq. (4) also applies in networks. These predictions have been confirmed by numerical simulations for various models in random networks, small-world networks, and complete graphs. Moreover, the relation of $\bar{\nu} = d_u \nu_G$ has been exploited to calculate the value of d_u via simulations in networks for complex nonequilibrium models [13].

In SF networks with the degree distribution $P(k) \sim k^{-\lambda}$, there appears a nontrivial λ-dependent MF critical scaling for $\lambda < \lambda_u$ (highly heterogeneous networks) while the standard MF theory applies for $\lambda > \lambda_u$ [1]. Naturally, we expect a nontrivial FSS theory associated with the nontrivial MF scaling for $\lambda < \lambda_u$. Previous studies pay attention to the MF analysis in the thermodynamic limit and hardly discuss the FSS in the general context. Recently, a few numerical efforts have been attempted to confirm the MF predictions, but huge finite-size effects and the lack of the FSS theory disallowed any decisive conclusion for highly heterogeneous networks [4,7]. Most recently, even a non-MF scaling has been claimed for the contact process [14,15] and the question arises as to whether the cutoff in degree k influences the FSS.

We start with the phenomenological MF free energy for the SF networks proposed in [1,2]

$$f(m) = -\epsilon m^2 + u m^4 + v|m|^{\lambda-1} + \mathcal{O}(m^6), \qquad (5)$$

where the λ-dependent term originates from the singular behavior of the higher moments of degree in SF networks. For $\lambda > \lambda_u = 5$, the λ-dependent term is irrelevant and we recover the usual ϕ^4 MF theory, yielding $\beta = 1/2$ and

$\bar{\nu} = 2$. For $3 < \lambda < 5$, the λ-dependent term becomes relevant and we find the ϕ^q MF theory with $q = \lambda - 1$. A simple algebra leads to $\beta = 1/(\lambda - 3)$ and the free energy density in the ordered phase is $f \sim -\epsilon^{1+2\beta}$. One can estimate the typical droplet volume $N_T \sim (\Delta f)^{-1}$, yielding $N_T \sim \epsilon^{-\bar{\nu}}$ with $\bar{\nu} = 1 + 2\beta = (\lambda - 1)/(\lambda - 3)$. By including the external field term hm in Eq. (5), one can show $\gamma = 1$ for all $\lambda > 3$.

The results for β, γ, and the FSS exponent $\bar{\nu}$ for the Ising model in SF networks are then summarized as

$$(\beta, \gamma, \bar{\nu}) = \begin{cases} \frac{1}{\lambda-3}, 1, \frac{\lambda-1}{\lambda-3} & \text{for } 3 < \lambda < 5 \\ \frac{1}{2}, 1, 2 & \text{for } \lambda > 5. \end{cases} \qquad (6)$$

For $\lambda < 3$, no phase transitions occur at finite temperatures and, at $\lambda = 5$, a multiplicative logarithmic correction is expected [1]. It is interesting to notice that a naive power counting for the ϕ^q local theory with the Gaussian spatial fluctuation term $(\nabla m)^2$ yields the same result for $\bar{\nu}$ by using the relation of $\bar{\nu} = d_u \nu_G$ [16]. Our conjecture for $\bar{\nu}$ bears no reference to the degree cutoff k_c caused by the finite system size N. We will argue later that the cutoff is irrelevant if it is not too strong: $k_c > N^{1/\lambda}$ [15].

We check our conjecture via numerical simulations. Two typical SF networks are considered, namely, the static model [17] and the uncorrelated configuration model (UCM) [18]. As these networks have different degree cutoffs (natural cutoff $k_c \sim N^{1/(\lambda-1)}$ versus forced sharp cutoff $k_c \sim N^{1/2}$) in finite systems, one may look for a possibility of the cutoff-dependent FSS behavior if any. It turns out that both cutoffs are not strong enough to influence the FSS for $\lambda > 2$.

We performed Monte Carlo simulations at various values of λ up to $N = 10^7$. We measure the magnetization m, the fluctuation $\chi' = N(\Delta m)^2$, and the Binder cumulant B and average over $\sim 10^3$ network realizations. The transition temperature T_c is estimated by the asymptotic limit of the crossing points of B for successive system sizes as well as of the peak points of χ'. At criticality, Eq. (4) leads to $m \sim N^{-\beta/\bar{\nu}}$ and similarly $\chi' \sim N^{\gamma'/\bar{\nu}}$ with $\chi' \sim |\epsilon|^{-\gamma'}$ in the thermodynamic limit. This power-law behavior in N provides an alternative check for the criticality as well as the estimates for the exponent ratios. In equilibrium systems, the fluctuation-dissipation theorem guarantees $\gamma' = \gamma$. By collapsing the data over the range of temperatures, we estimate the value of the FSS exponent $\bar{\nu}$. Our numerical data for m and χ' collapse very well for all values of λ in both static and UCM networks. In Fig. 1, the data collapse is shown for $\lambda = 3.87$ in static networks. We summarize in Table I the numerical estimates for $\beta/\bar{\nu}$, $\bar{\nu}$, and $\gamma'/\bar{\nu}$ at various values of λ in static and UCM networks. All data agree reasonably well with our predictions.

We also measure the degree-dependent quantities like the magnetization on vertices of degree k, m_k, and its fluctuation $(\Delta m_k)^2$. These quantities are found to satisfy
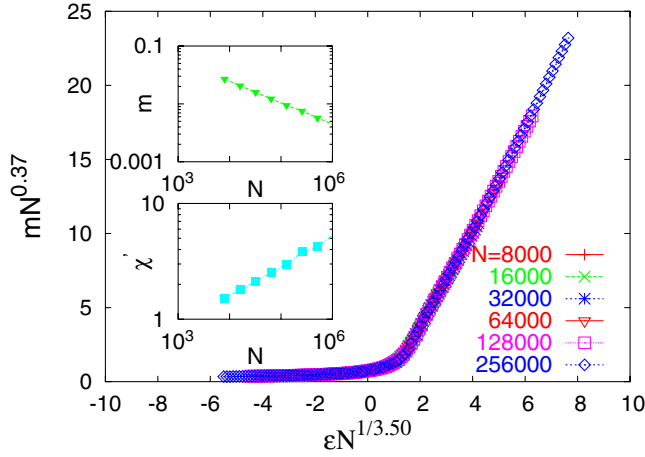
FIG. 1 (color online).   Data collapse of m for the static network with $\lambda = 3.87$, using $\beta/\bar{\nu} = 0.37$ and $\bar{\nu} = 3.5$. Insets: Double logarithmic plots of the critical decay of m and χ' against N with slopes $\beta/\bar{\nu} = 0.37(5)$ and $\gamma'/\bar{\nu} = 0.26(4)$.

a scaling relation with the scale variable $kN^{-1/\lambda}$ (not shown here) [16]. For $k > N^{1/\lambda}$, the distribution $P(k)$ becomes almost flat for each realization of networks and the degree exponent λ loses its identity. Therefore, vertices of such a high degree contribute in a trivial way and the cutoff beyond this range ($k_c > N^{1/\lambda}$) should not be distinguishable [15]. This argument is supported by our numerical results which cannot differentiate the FSS scaling in the static and UCM networks.

Now we move to a typical model exhibiting a nonequilibrium phase transition, namely, the directed percolation (DP) system [19]. It is well known that most of the nonequilibrium models showing an absorbing-type phase tran-

TABLE I.   Critical exponents of the Ising model on the static and the UCM networks, and the CP on the UCM networks, compared with our MF predictions.

Network	λ	$\beta/\bar{\nu}$	$\bar{\nu}$	$\gamma'/\bar{\nu}$
			Ising	
MF	$\lambda > 5$	$1/4$	2	$1/2$
	$3 < \lambda < 5$	$\frac{1}{\lambda-1}$	$\frac{\lambda-1}{\lambda-3}$	$\frac{\lambda-3}{\lambda-1}$
Static	7.08	$0.26(4)$	$2.0(2)$	$0.45(5)$
	4.45	$0.28(2)$	$2.4(2)$	$0.45(3)$
	3.87	$0.37(5)$	$3.5(3)$	$0.26(4)$
UCM	6.50	$0.24(4)$	$2.0(2)$	$0.51(5)$
	4.25	$0.31(1)$	$2.5(1)$	$0.39(1)$
	3.75	$0.38(6)$	$3.9(2)$	$0.24(3)$
			CP	
Network	λ	$\beta/\bar{\nu}$	$\bar{\nu}$	$\gamma'/\bar{\nu}$
MF	$\lambda > 3$	$1/2$	2	0
	$2 < \lambda < 3$	$\frac{1}{\lambda-1}$	$\frac{\lambda-1}{\lambda-2}$	$\frac{\lambda-3}{\lambda-1}$
UCM	4.0	$0.49(1)$	$2.1(1)$	$0.00(5)$
	2.75	$0.58(1)$	$2.4(1)$	$-0.16(2)$
	2.25	$0.78(1)$	$4.0(5)$	$-0.55(5)$

sition belong to the DP universality class. Among such models, we here consider the contact process (CP) and the susceptible-infected-susceptible (SIS) model [6].

The CP is an interacting particle model on a lattice. A particle creates another particle in one of its neighboring sites with rate p and a particle annihilates with rate 1. In the SIS model, the particle creation is attempted in all neighboring sites. A particle-particle interaction comes in through disallowance of multiple occupancy at a site. As p increases, the system undergoes a phase transition at p_c from a quiescent vacuum (absorbing) phase to a noisy many-particle (active) phase in the steady state. Near the absorbing phase transition, the order parameter (particle density) $\rho \sim \epsilon^{\beta}$, the fluctuations $\chi' = N(\Delta\rho)^2 \sim \epsilon^{-\gamma'}$, the susceptibility $\chi \sim |\epsilon|^{-\gamma}$, the correlation length $\xi \sim |\epsilon|^{-\nu}$, the relaxation time $\tau \sim |\epsilon|^{-\nu_t}$, and the survival probability $P_s \sim \epsilon^{\beta'}$ with the reduced coupling constant $\epsilon = (p - p_c)/p_c$. It is known that $\beta = \beta'$ due to the time-reversal symmetry in the DP systems [19] and $\gamma' \neq \gamma$ in general nonequilibrium systems.

Consider the droplet (cluster) excitation starting from a localized seed in the absorbing phase. The average space-time size S of a cluster is estimated as

$$S \sim \tau_\ell \xi_c^d \sim |\epsilon|^{-\sigma}, \qquad (7)$$

where τ_ℓ and ξ_c are the average lifetime and typical size of a droplet, respectively. Usually τ_ℓ diverges near the transition as $\tau_\ell \sim |\epsilon|^{-\nu_t + \beta'}$ for $\nu_t > \beta'$ [19], but τ_ℓ is a $\mathcal{O}(1)$ constant otherwise. In the MF regime, it is shown later that the latter always applies. The droplet size diverges as $\xi_c \sim |\epsilon|^{-\nu_T}$, which leads to $\sigma = d\nu_T + \max\{\nu_t - \beta', 0\}$. It is well known that the susceptibility is proportional to the cluster mass, which yields $\gamma = \sigma - \beta$ [19]. Finally, we arrive at the generalized exponent relation as

$$\gamma = d\nu_T - \beta + \max\{\nu_t - \beta', 0\}. \qquad (8)$$

The fluctuation exponent γ' satisfies the standard hyperscaling relation as $\gamma' = d\nu_T - 2\beta$.

In SF networks, we propose a phenomenological modification of the MF Langevin equation describing the DP models, similar to the free energy modification of the Ising model in Eq. (5):

$$\frac{d}{dt}\rho(t) = \epsilon\rho - b\rho^2 - d\rho^{\theta-1} + \sqrt{\rho}\,\eta(t), \qquad (9)$$

where $\rho(t)$ is the particle density at time t and $\eta(t)$ is a Gaussian noise. Our modification to the standard MF theory comes in by the third $\rho^{\theta-1}$ term and it is straightforward to show that the exponent $\theta = \lambda$ for the CP and $\theta = \lambda - 1$ for the SIS.

By dropping the noise term, one may easily get the MF steady-state solution for ρ. We find that $\beta = 1$ for $\theta > 3$ and $\beta = 1/(\theta - 2)$ for $2 < \theta < 3$. For $\theta < 2$, there is no phase transition at finite p. The same result may be obtained from the well-established k-dependent noiseless MF
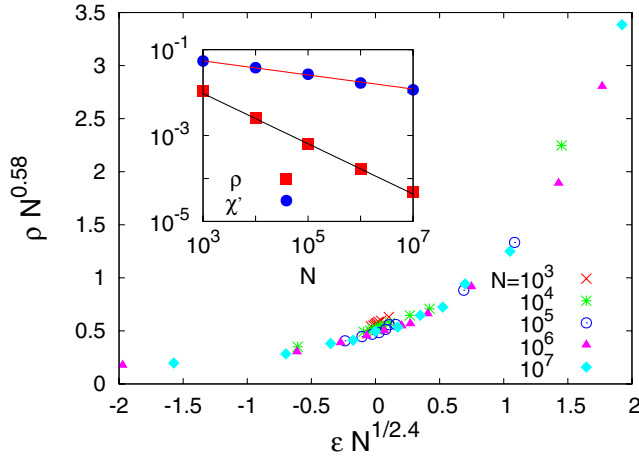
FIG. 2 (color online). Data collapse of ρ for the UCM network with $\lambda = 2.75$, using $\beta/\bar{\nu} = 0.58$ and $\bar{\nu} = 2.4$. Inset: Double logarithmic plots of the critical decay of ρ and χ' against N with slopes $\beta/\bar{\nu} = 0.58(1)$ and $\gamma'/\bar{\nu} = -0.16(2)$.

theory [6]. Moreover, the time-dependent solution leads to $\nu_t = 1$, and the inclusion of a weak external field leads to $\gamma = 1$ for all $\theta > 2$. Notice that $\nu_t \leq \beta = \beta'$.

Utilizing the exponent relation of Eq. (8), we summarize the results for β, γ, and the FSS exponent $\bar{\nu} = d\nu_T = 1 + \beta$ as

$$(\beta, \gamma, \bar{\nu}) = \begin{cases} \frac{1}{\theta-2}, 1, \frac{\theta-1}{\theta-2} & \text{for } 2 < \theta < 3 \\ 1, 1, 2 & \text{for } \theta > 3, \end{cases} \quad (10)$$

where $\theta = \lambda$ for the CP and $\theta = \lambda - 1$ for the SIS. For both cases, $\nu_t = 1$, $\sigma = \bar{\nu}$, and $\gamma' = 1 - \beta$. As in the Ising model, a naive power counting for the local Langevin equation with the spatial fluctuation term $\nabla^2 \rho$ yields the same result for $\bar{\nu}$ by using $\bar{\nu} = d_u \nu_G$ [16].

We performed numerical simulations for the CP on the UCM networks at various values of λ up to $N = 10^7$. We measure ρ and $\chi' = N(\Delta\rho)^2$. The transition point p_c is estimated by the power-law temporal dependence of ρ and also by the power-law size dependency of ρ in the steady state. At criticality, the temporal dependence is given as $\rho \sim t^{-\beta/\nu_t}$ and, in the steady state, $\rho \sim N^{-\beta/\bar{\nu}}$ and $\chi' \sim N^{\gamma'/\bar{\nu}}$. By collapsing the off critical data in the steady state, we can estimate the value of $\bar{\nu}$.

In Fig. 2, the data collapse is shown for $\lambda = 2.75$ in the UCM networks. Our estimates for the exponents summarized in Table I agree well with our predictions, in general. Our data become a little bit weaker close to $\lambda = 2$ where high heterogeneity in networks yields big corrections to scaling. We also confirmed the relation of $\sigma = \bar{\nu}$ by investigating the external field effect at criticality. Simulations were also performed on the static networks, where no cutoff dependence is found. The SIS model shows bigger finite-size corrections, which hinder us from estimating the transition point accurately. Never-

theless, our estimates for the steady-state exponent ratios are consistent with our predictions within errors [16].

In summary, we have explored the finite-size scaling behavior in complex networks. Based on the droplet-excitation argument, we conjectured the FSS exponent values, which have been confirmed by extensive numerical simulations on the static and UCM networks. Real networks found in many biological, economical, and social systems are highly heterogeneous and also quite small in size. Our results provide essential information on analyzing the data on these networks.

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