Relevance of Abelian Symmetry and Stochasticity in Directed Sandpiles

Hang-Hyun Jo[1] and Meesoon Ha[2,*]

[1]*School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea*

[2]*Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea*

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We provide a comprehensive view of the role of Abelian symmetry and stochasticity in the universality class of directed sandpile models, in the context of the underlying spatial correlations of metastable patterns and scars. It is argued that the relevance of Abelian symmetry may depend on whether the dynamic rule is stochastic or deterministic, by means of the interaction of metastable patterns and avalanche flow. Based on the new scaling relations, we conjecture critical exponents for an avalanche, which is confirmed reasonably well in large-scale numerical simulations.

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Since a prototype of sandpile models was first introduced by Bak, Tang, and Wiesenfeld, over the last two decades a lot of variants have been tested and become successful in figuring out the common underlying mechanism of ubiquitous scale invariance in nature [1]. In such models, grains are slowly added, redistributed (toppled) instantly whenever the instability threshold is overcome, and finally dissipated at boundaries. The most interesting issue under debate is whether the universality class of critical avalanche dynamics can be changed by the modification of local toppling rules, such as the breaking of Abelian symmetry [2] and the consideration of stochasticity [3], with still conflicting numerical results [4]. Abelian symmetry here means that the order of toppling the unstable sites does not affect the final state. Contrary to undirected models, directed sandpile models (DSMs) with a preferred direction of toppling turn out to be more tractable analytically as long as they have Abelian symmetry [5–7]. It is because metastable patterns in the Abelian DSMs are fully uncorrelated.

Once the Abelian symmetry is broken in DSMs by some specific way, long-range spatial correlations emerge in their metastable patterns. Such correlations are often observed in nature, like a fractal structure in the crust of the earth formed by seismic events. Two non-Abelian DSMs with spatially correlated metastable patterns were introduced by Hughes and Paczuski [8] for a stochastic version and by Pan *et al.* [9] for a deterministic version. In the stochastic version it is claimed that Abelian symmetry is not relevant to avalanche dynamics, while in the deterministic version it is. Although this difference might be attributed to the existence of stochasticity, it is not clear enough to say which factor governs the scaling property of metastable patterns. Therefore, it is quite crucial to clarify the role of spatially correlated metastable patterns in the universality class of DSMs, which has been hardly discussed up to now.

In this Letter, we discuss how the critical avalanche dynamics of non-Abelian models are entangled with spatially correlated metastable patterns. Based on the formation of metastable patterns and scars (trace of avalanche boundary sites) with the mapping onto particle dynamics, we give intuitive arguments about the scaling relations in terms of scar exponent, and conjecture a possible scenario for the universality class in DSMs. Finally, we reinterpret the earlier known results for the Abelian case by our conjecture, and confirm those for non-Abelian case by large-scale numerical simulations with various data analysis techniques developed so far.

Consider DSMs defined on a $(1+1)$-dimensional tilted square lattice of size (L, T). The preferred direction of avalanche propagation is denoted by the "layer" $t = 0, \ldots, T-1$ with open boundary conditions, and the transverse direction by $i = 0, \ldots, L-1$ with periodic boundary conditions. Initially, to each site of the lattice an integer value (the number of grains), $z_i(t) \in [0, z_c)$, is assigned, where we set the instability threshold $z_c = 2$. Given a stable configuration where all sites are stable, new grains are added one by one at a randomly chosen site on the top layer, $z_i(0) \to z_i(0) + 1$, until one of them becomes unstable. For any unstable site with $z_i(t) \geq 2$, grains at that site topple down to its left and right nearest-neighboring sites on the next layer, $t+1$:

$$z_i(t) \to z_i(t) - \Delta_{ii}, \quad z_{i\pm1}(t+1) \to z_{i\pm1}(t+1) + \Delta_{i,i\pm1}, \tag{1}$$

where $\Delta_{ii} = \Delta_{i,i-1} + \Delta_{i,i+1}$ (the local conservation of grains). Toppled grains at the unstable sites on the bottom layer $t = T-1$ are dissipated out of the system. Only after another stable configuration is recovered by a series of toppling events, denoting an avalanche, a new grain is added to keep generating another avalanche. By setting $\{\Delta_{ij}\}$ one may consider several variants of DSMs. In contrast to Abelian DSMs where Δ_{ii} is constant (used to be set as z_c), we set $\Delta_{ii} = z_i(t)$ as non-Abelian DSMs. All grains at the unstable site topple to the next layer and the toppled site becomes completely empty. For any given Δ_{ii}, the values of $\Delta_{i,i\pm1}$ can be determined in either stochastic or deterministic way. Besides the well-known Abelian deterministic or stochastic DSMs [5,6] (AD/AS in short) and the non-Abelian stochastic DSM [8] (NS in short), we explore the following three versions as the non-Abelian deterministic DSM (ND in short):

$$\text{(i) } \Delta_{i,i\pm1} = \begin{cases} k & \text{if } z_i(t) = 2k \\ k + \delta_{i\pm1,a_i(t)} & \text{if } z_i(t) = 2k+1, \end{cases}$$

$$\text{(ii) } \Delta_{i,i\pm1} = \begin{cases} k & \text{if } z_i(t) = 2k \\ k + \delta_{i\pm1,i+1} & \text{if } z_i(t) = 2k+1, \end{cases}$$

$$\text{(iii) } \Delta_{i,i\pm1} = z_i(t)/2,$$

where k is a positive integer and δ_{ij} denotes the Kronecker delta-function. We call (i) the alternatively biased version (aND) [9], (ii) the fully biased version (bND), and (iii) the continuous version without bias (cND), respectively. For the aND, an "arrow" $a_i(t)$ of each site initially points to one of its neighbors, say $i-1$ in Fig. 1(a). Whenever each grain is toppled at that site, the direction of the arrow flips to the other neighbor, see Fig. 1, which shows the case of $z_i(t) = 3$.

Each avalanche can be characterized by the following quantities: mass s (the number of toppled grains), duration t (the number of affected layers), area a (the number of distinct toppled sites), width w (the mean distance between left and right boundaries of avalanche), and height h (the mean number of toppled grains per toppled site). The avalanche distribution functions in DSMs show no characteristic scale except for T as long as L is sufficiently larger than the maximum width. They follow the simple scaling form as $P(x) \sim x^{-\tau_x} f(x/T^{D_x})$ for $x \in \{s, t, a, w, h\}$. Moreover, two quantities x and y scale as $\langle y\rangle \sim x^{\gamma_{yx}}$ with $\gamma_{yx} = \frac{\tau_x - 1}{\tau_y - 1} = \frac{D_y}{D_x}$ from $P(x)dx = P(y)dy$. Taking full advantage of the relations, $D_t = 1$ and $\langle s\rangle \sim T$ in DSMs, with the reasonable assumption of compactness of avalanche, i.e., $a \sim wt$ and $s \sim ah$, we obtain the following scaling relations: $\gamma_{xt} = D_x$ for any x, $D_s(2 - \tau_s) = 1$, $D_a = D_w + 1$, and $D_s = D_a + D_h$. As a result, there are only two independent exponents left. Concerning the metastable state, we define two scaling exponents more. Along the propagation direction, one can measure grain density as $\rho(t) \equiv \langle \frac{1}{L}\sum_i z_i(t)\rangle \sim t^{-\alpha}$ with the grain density exponent α for large t. The scar density $\rho_{sc}(t)$ and the scar exponent α_{sc} are defined by the same definition of grain density only with $z_i(t)$ replaced by $b_i(t)$, which takes a value of 1 for trace or 0 otherwise. The scar exponent is immediately related to the avalanche width exponent as $\alpha_{sc} = D_w$ because the density of avalanche boundary sites is inversely proportional to the typical avalanche width, i.e., $\rho_{sc}(t) \simeq w(t)^{-1}$. Since grains can remain only at the avalanche boundary sites for the non-Abelian DSMs, it is found that $\alpha = \alpha_{sc}$.

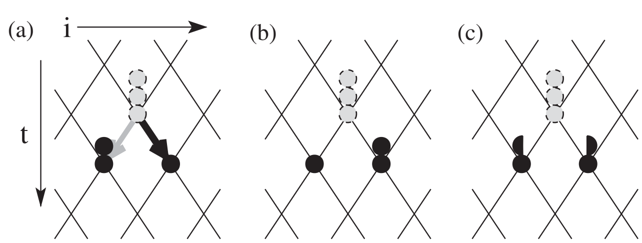

FIG. 1. Toppling rules of non-Abelian deterministic DSMs on a (1 + 1)-dimensional tilted lattice: (a) aND, (b) bND, and (c) cND. The gray colored grains and arrow represent the state before toppling and the black colored ones represent the state after toppling, respectively.

We give intuitive arguments on the interplay between avalanche flow and metastable patterns or scars in DSMs. Let us define $N(t)$ as the number of grains transferred from the layer t to the next layer $t+1$ within an avalanche, scaling as $N(t) \sim w(t)h(t) \sim t^{D_w + D_h}$. The evolution of $N(t)$, avalanche flow, can be written as

$$\frac{dN(t)}{dt} \approx N(t) - N(t-1) = \sum_{i\in w(t)} n_i(t), \tag{2}$$

where $n_i(t)$ denotes the amount of the avalanche flow at each site (i, t), and the summation is over the sites between avalanche boundaries belonging to w.

We begin with the Abelian case, for which the metastable patterns are fully uncorrelated. In the AD, it is well known that $D_h = 0$ by definition and $D_w = \frac{1}{2}$ by mapping avalanche boundaries onto the random walks [5]. The avalanche flow of the AD can be written as $\frac{dN}{dt} \approx \eta$. An uncorrelated noise η of zero mean and unit variance denotes the fluctuation of grain density. In the AS with the same η, the bulk contribution to avalanche flow plays a crucial role, so that we get $\frac{dN}{dt} \approx \sqrt{w}\eta$ [6,7]. Here $\sqrt{w}$ represents the fluctuation of the number of toppled sites in w when the topplings are uncorrelated. The lack of correlation in metastable patterns again leads to $D_w = \frac{1}{2}$, so $D_h = \frac{1-D_w}{2} = \frac{1}{4}$. In relation to the scar density, we get $\alpha_{sc} = D_w = \frac{1}{2}$ but $\alpha = 0$ for the Abelian case. It is found that the measurement of α_{sc} is more efficient than that of D_w. From now on, we suggest to measure α_{sc} as one independent exponent in DSMs.

For the non-Abelian case with the spatial correlation of metastable patterns or scars, the fluctuation of grain density is numerically found to scale as $\rho(t)$. Furthermore, ρ_{sc} plays the same role as ρ, i.e., $\alpha_{sc} = \alpha$. In the NS [8], keeping $\sqrt{w}$ due to the stochastic nature of toppling just as in the AS, we can write $\frac{dN}{dt} \approx \sqrt{w}\rho$ and get the new scaling relation, $D_h = 1 - \frac{3}{2}\alpha$. In contrast to the NS, in the ND, all sites of w contribute to avalanche flow so that $\frac{dN}{dt} \approx w\rho$. We get the different relation $D_h = 1 - \alpha$. Therefore, all avalanche exponents in the non-Abelian DSMs can be obtained from the grain density exponent α (or α_{sc}) as shown in Table I.

The new scaling relations with α enable us to clarify the effect of metastable pattern with $\alpha \approx 0.45$ in the NS on its avalanche dynamics. Interestingly, if $\alpha = \frac{1}{2}$ is assumed with systemic errors and/or possible logarithmic corrections, the NS has exactly the same avalanche exponents as

TABLE I. Avalanche exponents $\{\tau_x, D_x\}$, grain density exponent α, and scar exponent α_{sc} in (1 + 1)-dimensional DSMs with our conjecture. Note that $\alpha_{sc} = \alpha$ for the non-Abelian case. We check the value of $K_x \equiv D_x(\tau_x - 1)$, which is a universal constant for any x [10], and show its averaged value over x excluding K_w and K_h due to their poor statistics.

Model	τ_s, D_s	τ_t, D_t	τ_a, D_a	τ_w, D_w	τ_h, D_h	$\langle K_x \rangle$	α, α_{sc}
Mean field ($d_u = 2$)	$\frac{3}{2}$, 2	2, 1	$\frac{3}{2}$, 2	3, $\frac{1}{2}$	∞, 0	1	
Abelian							
Deterministic	$\frac{4}{3}$, $\frac{3}{2}$	$\frac{3}{2}$, 1	$\frac{4}{3}$, $\frac{3}{2}$	2, $\frac{1}{2}$	∞, 0	$\frac{1}{2}$	0, $\frac{1}{2}$
Stochastic	$\frac{10}{7}$, $\frac{7}{4}$	$\frac{7}{4}$, 1	$\frac{3}{2}$, $\frac{3}{2}$	$\frac{5}{2}$, $\frac{1}{2}$	4, $\frac{1}{4}$	$\frac{3}{4}$	0, $\frac{1}{2}$
Non-Abelian							
Stochastic	$\frac{2(3-\alpha)}{4-\alpha}$, $2 - \frac{\alpha}{2}$	$2 - \frac{\alpha}{2}$, 1	$\frac{4+\alpha}{2(1+\alpha)}$, $1 + \alpha$	$\frac{2+\alpha}{2\alpha}$, α	$\frac{4(1-\alpha)}{2-3\alpha}$, $1 - \frac{3\alpha}{2}$	$1 - \frac{\alpha}{2}$	α
-numerics	1.43(1), 1.77(2)	1.78(1), 1.00(1)	1.53(1), 1.46(2)	2.74(2), 0.44(1)	3.18(2), 0.31(3)	0.77(1)	0.45(3)
Deterministic	$\frac{3}{2}$, 2	2, 1	$\frac{2+\alpha}{1+\alpha}$, $1 + \alpha$	$\frac{1+\alpha}{\alpha}$, α	$\frac{2-\alpha}{1-\alpha}$, $1 - \alpha$	1	α
-aND	1.49(1), 1.97(3)	1.94(1), 1.00(1)	1.52(1), 1.88(2)	2.10(1), 0.87(1)	6.64(3), 0.07(1)	0.96(1)	0.86(3)
-bND	1.43(1), 1.82(4)	1.79(1), 1.00(1)	1.48(1), 1.76(7)	2.10(1), 0.76(6)	5.90(9), 0.06(1)	0.81(2)	0.69(5)
-cND	1.52(3), 1.99(4)	2.04(3), 1.00(1)	1.51(1), 1.95(3)	2.03(1), 0.86(2)	9.26(7), 0.06(4)	0.99(2)	0.91(11)

those of AS: $\tau_s = \frac{2(3-\alpha)}{(4-\alpha)} = \frac{10}{7}$ and $\tau_t = 2 - \frac{\alpha}{2} = \frac{7}{4}$. Another scenario for $\alpha = \frac{1}{2}$ can be found by mapping metastable patterns onto the space-time configuration of $2A \to A$ coagulation-diffusion model defined in $d = 1$, where the particle density decays as $t^{-1/2}$ [11]. One can say that the NS belongs to the same universality class as the AS in the following sense: For Abelian cases the flowing avalanche can sweep and lose many grains at the same time due to the uniform grain density. On the other hand, for non-Abelian case the flowing avalanche can sweep only a few grains due to the power-law decaying grain density and leave few grains behind by taking all grains at the toppling sites. In other words, the scaling property of $N(t)$ is apparently unaffected by the grain density as long as the toppling rule is stochastic.

From $D_h = 1 - \alpha$ in the ND, we find that the avalanche exponents for mass and duration have the mean-field (MF) values, independent of α, i.e., $\tau_s = \frac{3}{2}$ and $\tau_t = 2$, whereas other exponents depend on α. We point out that one should consider other avalanche exponents as well as those of mass and duration in order to discuss the universality class of the ND. If $\alpha = D_w = 1$ is assumed from the linear behavior of avalanche boundaries (scars) as shown in Fig. 2, all avalanche exponents turn to the MF values, except for the case of width [12]. This may also correspond to the MF behavior of coagulation-diffusion model, where the particle density decays as t^{-1} in $d \geq d_u = 2$ [11]. This peculiar "MF" behavior of three ND versions appeared in the low dimensional system can be understood by considering the shape of $N(t)$ with its width and height. We now focus on how avalanche boundaries behave linearly, which implies $D_w = 1$. The ND toppling rules we considered suppress the fluctuations of height profile of the flowing avalanche more than the stochastic one does, which leads to spreading grains wider and making the avalanche boundaries grow faster, almost ballistically. This positive feedback enables $D_w = 1$ to be larger than $\frac{1}{2}$ for all other DSMs. Moreover, we like to note that the resultant $D_h = 0$ indicates the MF behavior for the non-Abelian case, whereas $D_h = 0$ for any dimension in the AD.

We performed extensive numerical simulations for all DSMs to confirm our conjecture about the avalanche exponents in terms of the scar exponent, $\alpha_{sc} = \alpha$, up to $T = 2^{13}$ and $L = T/2$ ($T = 2^{15}$ or $L = T$ in some cases). We measure the avalanche exponent set $\{\tau_x, D_x\}$ of all x, α, and α_{sc} using the moment analysis and the conventional successive slope techniques of avalanche distributions, for about 10^9 avalanches at the steady state after the transient period. The spatially correlated scars are observed in all DSMs with the nonzero values of α_{sc}, while the spatial correlations of metastable patterns are only observed in non-Abelian DSMs with the nonzero values of α. To

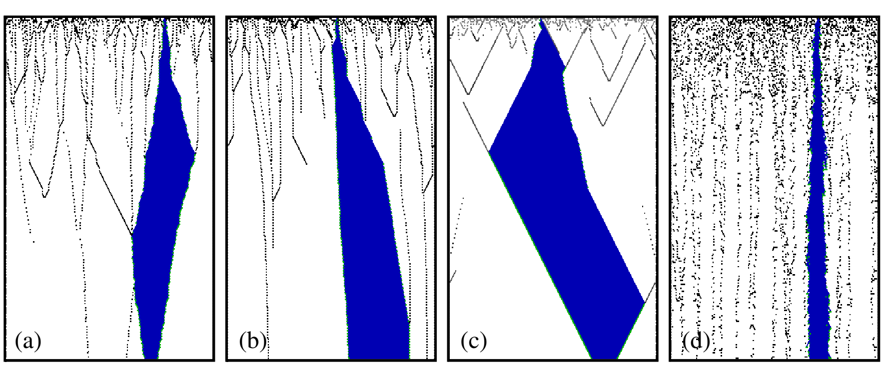

FIG. 2 (color online). Typical metastable patterns for non-Abelian case: The occupied sites are shown as black dots and the typical shapes of dissipative avalanches consisting of the toppled sites as blue (or gray)-shaded areas on a lattice with $L = 150$ and $T = 250$. Here (a) aND, (b) bND, (c) cND, and (d) NS, respectively.

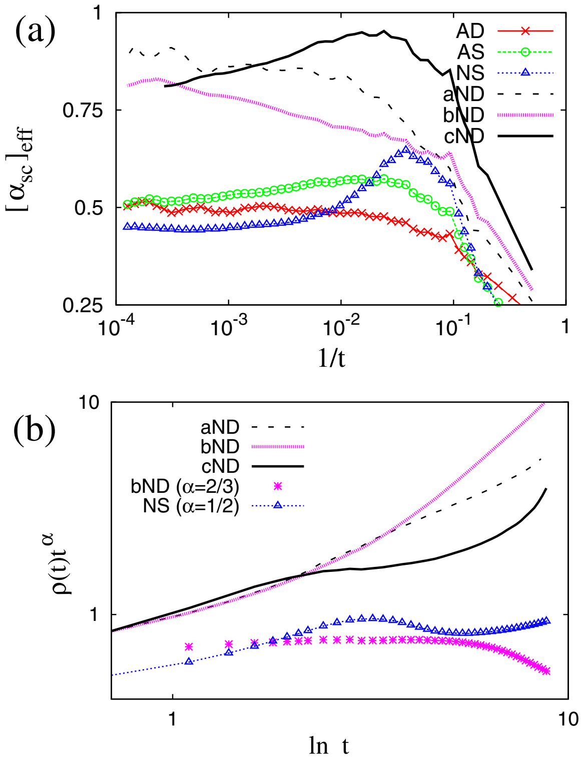

FIG. 3 (color online). (a) The effective scar exponent $[\alpha_{sc}]_{eff}$ as a function of $\frac{1}{t}$, and (b) double-logarithmic plots of $\rho(t)t^{\alpha}$ versus $\ln t$ for logarithmic correction checks, where $\alpha = 1$ unless noted. For the bND, we cannot exclude the possibility of $\alpha = \frac{2}{3}$ either. Here the data of each model were obtained from 10^8 or more avalanches on a lattice of $L = 2^{14}$ and $T = L$, except for the cND where $T = 2^{13}$.

validate the stability of the scar exponent, we plotted the effective scar exponent $[\alpha_{sc}]_{eff}$ as a function of $\frac{1}{t}$ for all DSMs. As shown in Fig. 3(a), there seems to be two asymptotic values, $\frac{1}{2}$ and 1 in the large t limit, with quite long or unusual initial transient behaviors and finite-size corrections. For the non-Abelian case, we also checked the possibility of logarithmic corrections to scaling in ρ_{sc} and ρ, both of which behave qualitatively the same. Thus, we only show ρ in Fig. 3(b) as $\rho(t) \sim t^{-\alpha}(\ln t)^{\phi}$, where the existence of linear parts in curves represents logarithmic corrections.

We finally discuss the relevance of Abelian symmetry in DSMs. Based on our results, Abelian symmetry turns out to be irrelevant to the stochastic version only when $\alpha = \frac{1}{2}$ in the NS. The breaking of Abelian symmetry in the deterministic version yields the MF behavior of avalanche dynamics even in a $(1 + 1)$-dimensional setup. In all NDs, we also confirm that the values of D_h are quite close to 0, which can be the sign of the MF behavior as we argued. Furthermore, it turns out the NDs do not show any criticality in a $(0 + 1)$-dimensional setup. All numerical results are listed in Table I with our conjecture. Only the results of the bND seem to be inconsistent with the values we conjectured. Such discrepancy may be attributed to the relevant effect of bias in toppling rule or relatively large logarithmic corrections. The validity of our conjecture for other possible values of α or α_{sc} is under investigation, with the role of toppling bias in DSMs [13,14].

In summary, we have explored the role of Abelian symmetry and stochasticity in directed sandpiles, and conjectured the new scaling relations for critical avalanche dynamics entangled with the underlying structure of metastable patterns and scars. Our conjecture provides clear guidelines on discussing the universality class in directed sandpile models. Moreover, our results provide essential information on analyzing the self-organized criticality in real systems as well as answering how ubiquitous long-range spatial correlations in nature can be developed and affect real avalanche dynamics.

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*Corresponding author.
msha@kaist.ac.kr

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