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Driven Contact Process with Temporal Feedback

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We investigate a directed percolation (DP)-based cyclically coupled model with branching bias, namely, the driven contact process with temporal feedback (DCP-TF). In the DCP-TF, we control not only the temporal feedback by using its memory strength and the power-law tail exponent for the incubation period of the intermediate state but also the preferred direction of branching. The CP-TF in a one-dimensional lattice exhibits continuously varying scaling behaviors, which is attributed to the long-term memory caused by the power-law type feedback. This implies that the CP-TF does not belong to the DP universality class. However, its deviation from the DP exponent is not very big. To clarify this issue, we test the relevance of the external driving to the critical behaviors of CP-TF. If it belongs to the DP universality class, the branching bias is irrelevant to its critical exponents. We find that the branching bias does affect both the region of continuously varying scaling and the value of critical decay exponent. Based on numerical results and intuitive arguments in the DCP-TF, we discuss how the universality class of the CP-TF is related to the universality class of the pair CP with diffusion.

Keywords: Nonquilibrium phase transitions, Contact process, Directed percolation, Long-term memory, Diffusion/branching bias, Universality class

I. introduction

Among many mathematical models of epidemics, the contact process (CP) is widely used as a prototype model to study the critical behaviors of epidemic spreading and absorbing phase transitions (APTs). Under the circumstances that either immunization or recovery (removal) is absent in the system, the CP is the simplest model to belong to the directed percolation (DP) universality class, even in the one-dimensional (1D) lattice.

The central interest in the model study of epidemic spreading is the structure of nonequilibrium stationary states (NESS) and the dynamic pathways to those NESS. It is because the system undergoes an APT from a phase where dynamics is inactive (trapping in an absorbing state) to another phase where dynamics is active as a function of control parameters [1].

There are two distinct well-established universality classes of APTs. The DP universality class [2] has only one or infinitely many absorbing states, whereas the directed Ising (DI) [parity-conserving] universality class [3] (and references therein) has two absorbing states with the Z_2 symmetry. Because of the DP conjecture by Janssen and Grassberger [4], it is an interesting challenge to find new models that belong to another universality class. Among them, the pair CP with diffusion (PCPD) is the most debatable one [5–15].

In this paper, we study a 1D driven CP (DCP)-based cyclically coupled epidemic model with the long-term memory and the branching bias, namely the DCP with temporal feedbacks (DCP-TF). The DCP-TF has the power-law lifetime distribution of the intermediate-exposed (**E**) state between susceptible (**S**) and infected (**I**) states. This can be interpreted as the memory of the incubation (quarantine). The CP-TF was first proposed by H. Park in Chapter 5 of Ph.D. thesis [16], where

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continuously varying scaling behaviors were reported as the memory strength and its power-law decay exponent vary. Such behaviors are similar to those of the generalized pair CP with diffusion (GPCPD) [10], which implies that both the GPCPD and the CP-TF do not belong to either the DP or the DI universality class.

To clarify the universality class of the CP-TF, the branching bias is employed as the external driving because it is well-known that the branching bias is irrelevant to the DP universality class [11] (and references therein). If the branching bias is relevant to the critical behaviors of the CP-TF does not belong to the DP universality class. Our numerical results show that the presence of the branching bias changes both the continuously varying scaling regime and the critical decay exponent value. Based on intuitive arguments, we claim that the CP-TF with the long-term memory belongs to the different universality class from the ordinary CP, not the DP universality class.

This paper is organized as follows. We briefly review the CP and describe the dynamic rule of the CP-TF with the branching bias in Sec. II, where the long-term memory is also discussed as temporal feedbacks. In Sec, III, the critical behaviors of the DCP-TF are numerically compared with those of the CP-TF. We conclude this paper with summary and some remarks in Sec. IV.

II. Model

Consider the ordinary CP in a 1D lattice of L sites, which is a typical single absorbing state model. In the CP, each site can be either empty (**0** for the susceptible state) or occupied by only one active particle (**A** for the infected state). The initial condition is exactly the same for three cases: the CP, the CP-TF, and the DCP-TF: $\rho_{\mathbf{A}}(0) \equiv N_{\mathbf{A}}(0)/L = 1/2$ and $\rho_{\mathbf{B}}(0) \equiv N_{\mathbf{B}}(0)/L = 0$. However, in the CP-TF and the DCP-TF, intermediate exposed states, denoted as **B**, are created as time t elapses. The inactive particle **B** has the lifetime for either the incubation of epidemic disease or the quarantine from epidemic spreading, so that the dynamics of **B** plays a role of the long-term memory in the system as temporal feedbacks.

In Fig. 1, we illustrate how the dynamics of the CP-TF is modified to that of the CP-TF with the branching

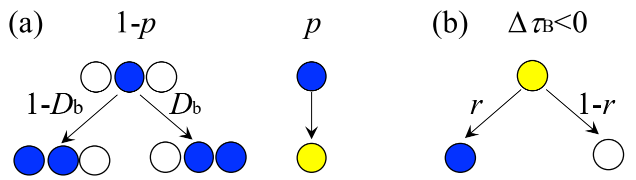

Fig. 1. (Color online) (a) The branching (annihilating or temporarily mutating) dynamics of **A** particle, represented as a blue solid circle, is illustrated: $\mathbf{A}+\mathbf{0}\rightarrow \mathbf{A}+\mathbf{A}$ ($\mathbf{A}\rightarrow\mathbf{B}$) with probability $1-p$ (p), where the branching bias is controlled with probability $D_{\rm b}$ ($1-D_{\rm b}$) to the right (left). (b) The feedback dynamics of **B** particle after the incubation (quarantine) period, represented as a yellow solid circle, is illustrated: $\mathbf{B}\rightarrow\mathbf{A}$ ($\mathbf{B}\rightarrow\mathbf{0}$) with probability r ($1-r$), where $\Delta\tau_{\rm B}$ is the remaining time against its pre-assigned lifetime $\tau_{\rm B}$, $\Delta\tau_{\rm B}\equiv\tau_{\rm B}-t_{\rm elapsed}$.

bias, where we present the most relevant changes in the CP-TF. It is noted that the hard-core repulsion of **A** particles is still preserved for the branching dynamics of **A** particle to the nearest-neighboring sites. However, **B** particle is bosonic, so it does not interact with any particles. Due to the dynamics of **B** particle, the hard-core repulsion of **A** particles is partially broken. As a result, in the CP-TF and the DCP-TF, a site can be occupied by multiple particles, m**A** and n**B**, as t elapses in our model, where m and n are whole numbers.

For the rejection-free algorithm in Monte-Carlo (MC) simulations, we provide the detailed description of the DCP-TF dynamics with step-by-step comments.

1. **Initial setup** – Each site can be empty or occupied by only one **A** at random, so that $\rho_{\mathbf{A}}(0)=1/2$ and $\rho_{\mathbf{B}}(0)=0$. Since we are interested in the temporal decay of the order parameter $\rho_{\mathbf{A}}(t)$ by dynamic simulations near the criticality of the APT, one may think that the fully **A**-occupied system is the best choice as the initial setup. However, due to the hard-core repulsion of **A** particles for the branching dynamics, such an initial condition lets the system undergo the initial intrinsic and transient period.

2. **Updating choice** – To compare with other APTs, we employ the random sequential updating, where the time is continuously updated and the updating site is chosen at random with probability $1/L$. To get rid of the rejection by empty sites, it is useful to make the list of the particles as well as the sites that have at least one particle, which is initially at most the same as the half of the lattice size, $N_{\rm p}(0)=L/2$, from $N_{\rm p}(t)\equiv N_{\mathbf{A}}(t)+N_{\mathbf{B}}(t)$.

3. **Time Increment** – For the rejection-free random sequential updating, the MC simulation time is updated at each dynamics by $\Delta t = 1/N_{\rm p}(t)$, instead of $1/L$.

4. **Dynamics of A particle** – **A** can either *temporally mutate* to **B** with probability p, or *branch* another **A** at one of the nearest-neighbor sites with probability $(1-p)$, provided that the target site does not have **A**. The branching bias is controlled with probability $D_{\rm b}$ $(1-D_{\rm b})$ to the right (left). The new **B** has the lifetime before reactivating for the temporal feedback, which is assigned from the power-law lifetime distribution, $P(\tau_{\rm B}) = c\tau_{\rm B}^{-(\theta+1)}$, such that the survival distribution, $P_s(\tau_B) = c_s\tau_{\rm B}^{-\theta}$ with $P(\tau_{\rm B}) = -\frac{{\rm d}P_s(\tau_{\rm B})}{{\rm d}\tau_{\rm B}}$. We set $c = \theta$.

$$\mathbf{A} \rightarrow \begin{cases} [\text{Temporal Mutation}]\ \mathbf{B} & \text{with } p \\ [\text{Branching}]\ \mathbf{AA} & \text{with } 1-p \\ \begin{cases} \text{to the right} & \text{with } D_{\rm b} \\ \text{to the left} & \text{with } 1-D_{\rm b} \end{cases} \end{cases} \quad (1)$$

5. **Dynamics of B particle** — If **B** is reactive-ready with $\Delta t_{\rm B} \equiv \tau_{\rm B} - t_{\rm elapsed} < 0$, the chosen **B** can either *mutate back* to **A** after incubation of disease with probability r (temporal feedbacks as long-term memory), or *eventually annihilate* with probability $1-r$ after quarantine from disease (just temporal delay, compared to the ordinary CP).

$$\mathbf{B}\ (\text{if } \Delta\tau_{\rm B} < 0) \rightarrow \begin{cases} [\text{Incubation}]\ \mathbf{A} & \text{with } r \\ [\text{Quarantine}]\ \mathbf{0} & \text{with } 1-r \end{cases} \quad (2)$$

In the DCP-TF, the density of **A** particles, $\rho_{\mathbf{A}}(t)$ is still the order parameter because the system is active until when at least one **A** exists, just as the ordinary CP. However, for the case of nonzero r, it is interesting to measure the density of **B** particles, $\rho_{\mathbf{B}}(t)$, and the particle density, $\rho_{\rm A+B}(t) \equiv N_P(t)/L$, for the comparison of $\rho_{\mathbf{A}}(t)$, near the criticality of the APT. We note that the DCP-TF with $D_b = 0.5$ is the CP-TF, and the CP-TF of $r = 0$ corresponds to the temporally delayed CP. In the CP-TF where the annihilation of **A** is modified by the temporal mutation to **B**, the mutated **B** plays a role as the power-law decaying non-order field. It is interesting to find when its feedback becomes relevant to the system (see Fig. 2).

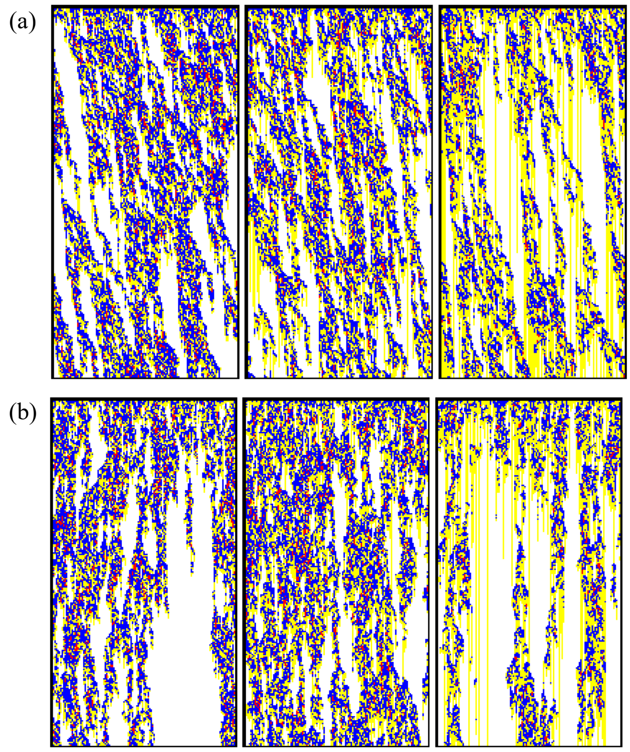

Fig. 2. (Color online) Spatiotemporal patterns of the 1D DCP-TF with (a) the fully-biased branching the right, $D_{\rm b} = 1.0$, and (b) no bias, $D_{\rm b} = 0.5$, are presented for $L = 128$ (horizontally) and $T = 2L$ (vertically) for each panel, where we set $r = 0.5$ and $\theta = \{2.5, 1.5, 1.0\}$ (from left to right) at the corresponding critical threshold, $p = p_c(r,\theta)$. The particles are colored as • for **A**, • for **B** or multiple **B**s, and • for multiple **A**s.

Figure 2 shows the spatiotemporal patterns of the 1D DCP-TF at the criticality with and without the branching bias, respectively (a) $D_{\rm b} = 1$ and (b) $D_{\rm b} = 0.5$, where the relative biased diffusion of **A** is observed in the presence of the branching bias and rapid decay of **A** particles occurs as θ gets smaller. This is similar to the DPCPD [11], where the bias becomes relevant to change the critical scaling behaviors.

III. Numerical Results

In our numerical results, we implement the power-law lifetime (τ) distribution, $P(\tau)$, with the uniform random number (x) distribution, $R(x)$. By definition, $R(x'){\rm d}x' = P(\tau'){\rm d}\tau'$ and $\int_1^\infty P(\tau){\rm d}\tau = 1$, we can find the following simple relation between a random number

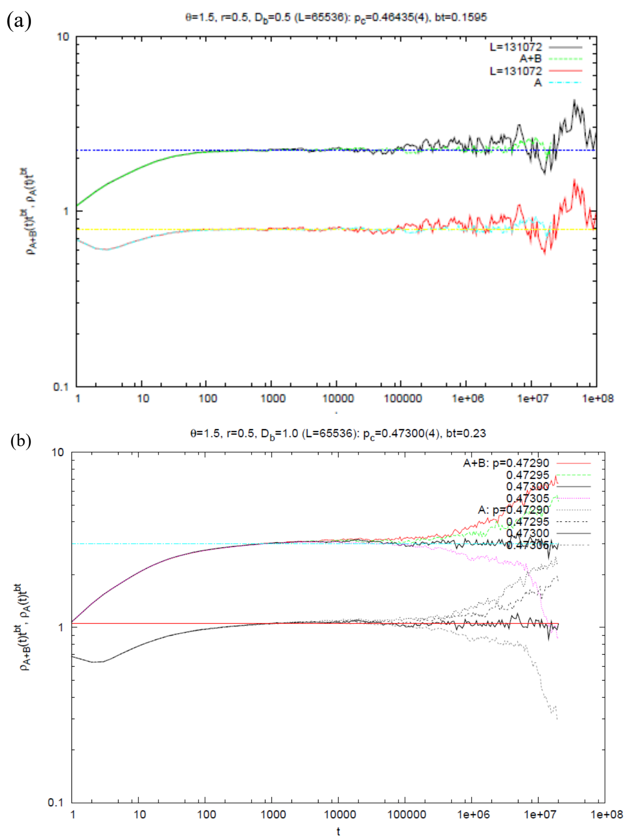

Fig. 3. (Color online) For the case of $\theta = 1.5$ and $r = 0.5$ at $L = 2^{16}$, double logarithmic plots of $\rho_{\rm A}(t)t^{bt}$ and $\rho_{\rm A+B}(t)t^{bt}$ are presented against t to indicate the critical threshold (p_c) and the critical exponent ($\beta/\nu_\parallel$, bt in short), where (a) $D_{\rm b} = 0.5$ (CP-TF) at $p = 0.46435(\approx p_c)$, where both $L = 2^{16}$ (dashed) and 2^{17} (solid) are shown, and (b) $D_{\rm b} = 1.0$ (DCP-TF). Our numerical data are averaged over at least 100 samples.

x of $R(x)$ and a lifetime τ of $P(\tau)$:

$$\int_1^x R(x')\mathrm{d}x' = \int_1^\tau P(\tau')\mathrm{d}\tau' \ \to\ \tau = (1-x)^{-1/\theta}, \quad (3)$$

where $R(x) = 1$ and $P(\tau) = \theta\tau^{-(\theta+1)}$. From $P(\tau_{\rm B})$, we can get the average lifetime $\langle\tau_{\rm B}\rangle$ as follows:

$$\langle\tau_{\rm B}\rangle = \int_1^\infty \tau_{\rm B}P(\tau_{\rm B})\mathrm{d}\tau_{\rm B} \sim \begin{cases} \text{finite} & \text{if } \theta > 1 \\ \infty & \text{if } \theta < 1 \end{cases} \quad (4)$$

Therefore, $\theta = 1$ is a marginal value for the lifetime decay exponent. In the next section, our numerical date is presented to discuss the universality class of the CP-TF as we control $p, r, D_{\rm b},$ and θ.

In order to discuss the scaling behaviors of the order parameter, $\rho_{\rm A}(t)$, and the critical threshold, p_c of the APT [1], we employ the dynamic scaling ansatz , $\rho_{\rm A}(t) \sim t^{-\beta/\nu_\parallel}$ at $p = p_c$ when the system is large enough.

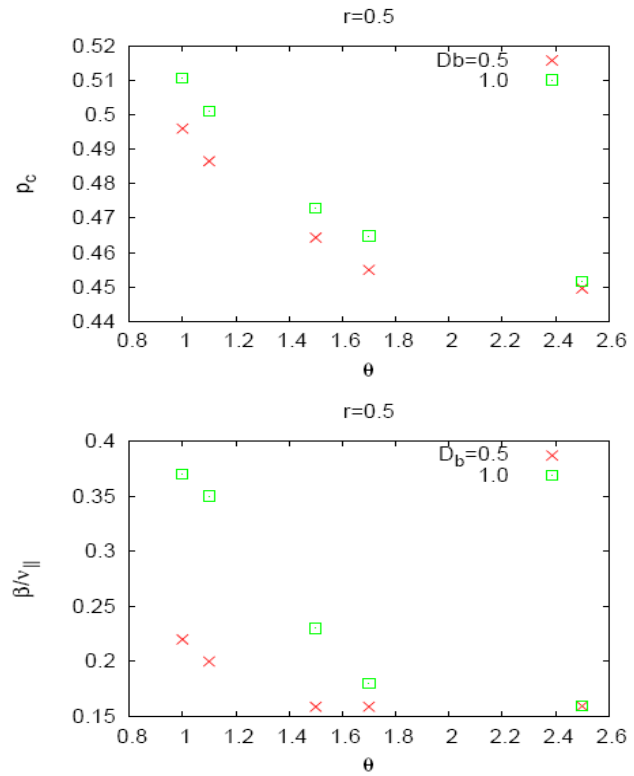

Fig. 4. (Color online) As the power-law decay exponent of the long-term memory, θ, varies, the DCP-TF ($D_{\rm b} = 1$, □) is compared with the CP-TF ($D_{\rm b} = 0.5$, ×) for $L = 10^5$ and $r = 0.5$: (a) the critical threshold, p_c, and (b) the critical exponent, $\beta/\nu_\parallel$.

As shown in Fig. 3, the critical scaling behaviors in the DCP-TF are different from those the CP-TF, in the context of not only the critical threshold but also the decay exponent. For the case of $\theta = 1.5$, the DCP-TF becomes distinct from the DP universality, whereas the CP-TF still follows the DP universality class.

Using the same analyses, we numerically obtain both (a) p_c and (b) $\beta/\nu_\parallel$ as a function of θ at $r = 0.5$, which are plotted in Fig. 4 and summarized in Table 1 (only for $D_{\rm b} = 1.0$). Due to the fact that the branching bias becomes relevant, unlike the ordinary CP (the DP universality class), our results with $r = 0.5$ imply that the CP-TF belongs to the different universality class that has continuously varying exponent.

IV. Summary and discussion

Table 1. The critical threshold and the critical exponent of 1D DCP-TF at $r = 0.5$ and $D_b = 1.0$ are presented with those of the 1D CP-TF. For the comparison, we also show the critical exponents of other classes.

Class		θ	p_c	$\beta/\nu_{\parallel}$
	DCP-TF	2.5	0.45160(4)	0.159(1)
		1.7	0.46483(2)	0.18(1)
		1.5	0.47300(4)	0.23(1)
		1.1	0.50105(4)	0.35(1)
		1.0	0.51065(4)	0.37(1)
	CP-TF [a]	1.5	0.46436(2)	0.159(5)
		1.2	0.47887(4)	0.172(5)
		1.1	0.48654(4)	0.207(5)
		1.0	0.49598(4)	0.237(5)
	GPCPD	1.25(5)[b]		0.197(4)
DP	(from [10])			0.1596(4)
DI	(from [10])			0.286(2)

[a]From Table 5.1 of Chapter 5 in Ref. [16]
[b]This estimate is at $r = 0.5$ and $d = 0.1$ in Ref. [10].

We studied epidemic spreading in the one-dimensional (1D) lattice, in terms of the contact process (CP) with the long-term memory. Owing to the introduction of the intermediate exposed state in the CP with its lifetime, we generated power-law type temporal feedbacks and controlled the memory strength. For the discussion of the universality class of the 1D CP with temporal feedbacks (CP-TF), we tested the relevance of the branching bias. In the 1D driven CP-TF (DCP-TF), the role of the long-term memory caused by the non-ordered field, is similar to single particles in the pair CP with diffusion (PCPD).

As a result, our study reopened the discussion about the universality class of the PCPD. However, we cannot rule out the possibility of the crossover to the directed percolation (DP) universality class, discussed in the recent studies of the PCPD [14,15].

Finally, we suggest that the mean-field behavior of the CP-TF would be an interesting future work for the extended studies on network structures. Some extension related to COVID-19 type epidemic spreading has been under the investigation.

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