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Spatial Correlations and Extended Self-similarity Properties in Sandpiles

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We study scaling behaviors in the metastable states of sandpiles, which exhibit self-organized criticality (SOC). In order to discuss universal spatial correlations and extended self-similarity (ESS) properties, we consider the (1+1)-dimensional non-Abelian stochastic directed sandpile model (SDSM) [D. Hughes and M. Paczuski, Phys. Rev. Lett. **88**, 054302 (2002)]. The non-Abelian SDSM is well known to belong to the same universality class as the Abelian SDSM with respect to the scaling behaviors of avalanches. However, the two models yield totally different structures of metastable states with SOC. Such metastable-state structures are analyzed in terms of the inter-grain distribution functions and the two-point correlation functions by using Monte Carlo simulations. Finally, we show that large-scale networks of grains can be described by using the time-dependent characteristic size of the gap between clusters and find that the ESS properties of the correlations are analogous to those found in fluid turbulence when the avalanche propagation is mapped to 1D interface growth.

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I. INTRODUCTION

Underlying mechanisms and symmetries in complex systems yield spatial and temporal patterns, which may contain some crucial information for universality classes. Lots of examples are ubiquitous in nature. Among them, sandpiles also exhibit the self-organized critical (SOC) metastable patterns. Since Bak, Tang, and Wiesenfeld first introduced the undirected deterministic sandpile model in two dimesions [1], SOC propertites have been studied [2, 3], in particular to universality classes related to various dynamic rules and symmetries.

The basic procedure in sandpiles models consists of the addition of grains, the instant redistribution (toppling) of slowy added grains whenever the site becomes unstable, and the dissipation of toppled grains at boundaries. The main question in sandpiles is how to understand SOC properties and scaling relations. Hence it would be meaningful to focus on the simplest case, such as (1+1)-dimensional directed sandpile models [4,5], which explain the relevance of Abelian symmetry and stochasticity.

Unlike Abelian directed sandpiles, non-Abelian ones present large-scale spatially correlated structures. This was first discussed by Hughes and Paczuski [6], in terms of a simple variant of the Abelian stochastic directed sandpile model (SDSM) [7–9] with a change in the rule of the broken Abelian symmetry for updating unstable sites. Note that sandpile models with Abelian symmetry is irrelevant to the order of topplings.

For (1+1)-dimensional SDSMs, two facts have been known: (i) Both Abelian and non-Abelian cases show the same power-law distributions of avalanche-relates quantities. (ii) Subcritical SOC patterns of non-Abelian metastable states show large-scale structures, consisting of networks of grains. This fact is quite different from those of Abelian case, where such spatial correlations are not presented as shown in Fig. 1.

In this paper, we explore non-Abelian structure of metastable states. Particularly, we discuss the origin of

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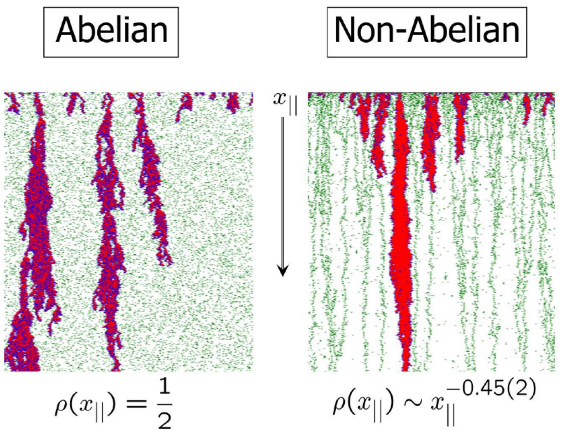

Fig. 1. (Color online) Metastable state patterns in (1+1)-dimensional SDSMs with shaded avalanche scars.

large-scale spatial correlations. Moreover, we test extended self-similarity properties in sandpiles. This paper is organized as follows: In Sec. II, we describe the simplest sandpile model, and briefly reproduce SOC scaling properties. Metastable pattern issues are discussed as well as SOC physical properties in Sec. III, which is numerically checked in Sec.IV. Finally, we conclude the paper in Sec. V with some remarks.

II. MODEL

While (1+1)-dimensional typical sanpile models are illustrared in Fig. 2, we here consider a 45°-rotated two-dimensional square lattice of $(L \times T)$, of which horizontal axis is indiced as $x_\perp = x$ (space) and vertical axis as $x_{||} = t$ (time). As a result, one can describe each site as (x, t), at which the amount of grains is $z(x, t)$. All sites are initially empty and there is the uniform maxium capacity per site, z_c. Once $z(x, t)$ reaches z_c, all grains at the site topples each neighbors based on model dynamics:

1. Randomly choose a site at $t = 0$, and add a grain, $z(x, 0) \to z(x, 0) + 1$. Keep adding a grain at a random site at $t = 0$ until any $z(x, 0)$ reaches z_c.
2. If any x satisfies $z(x, 0) = z_c$, keep toppling as $z(x \pm 1, 1) \to z(x \pm 1, 1) + 1$ until $z(x, 0) = 0$. Keep the similar procedure along to the t direction as t increases by 1. Keep toppling grains until either all sites become stable or the toppled grain reaches at $t = T$.

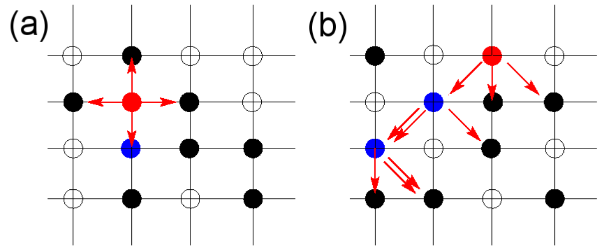

Fig. 2. (Color online) Description of toppling and avalanche in 2D sandiple models: (a) undirected *vs.* (b) directed.

Table 1. The results in (1+1)-dimensional non-Abelian SDSM [6] are reproduced with metastable SOC states. We obtained exponents for $L = L_\perp = 200$ and $T = L_{||} = 2000$, based on linear regression fitting.

Quantity x	s	a	b	m	t	w
Exponent τ_x	1.67(8)	1.74(6)	1.9(1)	1.48(5)	1.80(5)	3.0(1)
[6]				1.43(2)	1.75(1)	

3. If the toppled grain reaches at bottom ($t = T$), it leaves from the system, which is the dissipation. If all sites becomes stable, Stop topplings and go back to the addition of a grain at top ($t = 0$).

III. PHYSICAL QUANTITIES

1. Avalanche Distributions for SOC

Now that model dynamics is well described with three steps, we pose the following question: **What are major quantities to check SOC propertites and metastable patterns in sandpiles?**

The multi-avalanche distribution, $P_{L,T}(s, a, b, m, t, w)$, is measured for each variables, where s is the total number of toppled sites (this is the same as the number of topplings unless multiple toppings occur), a is the area of each avalanche activity (*i.e.*, the total number of sites involved in the activity, including those that didn't topple), b is the total number of untoppled sites, which were involved in the activity, but didn't topple, so $b = a - s$.

For comparison to the results in Ref. 6, we also measure m, the total number of grains involved in the avalanche activity, and t, the duration time of the activity. To optimize the transverse length L, parallel to the direction of avalanche propagation, the maximum width, w, of each avalanches is monitored.

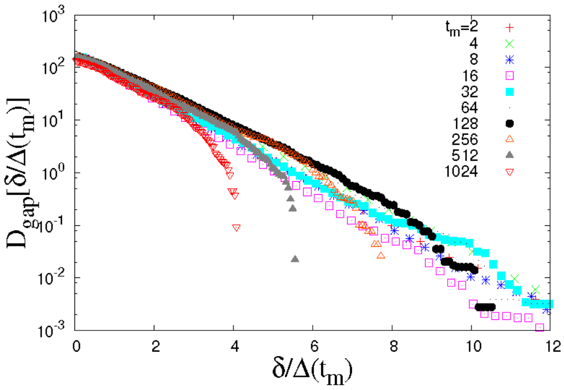

Fig. 3. (Color online) Cumulative gap distributions of metastatble states, chacteristic gap scaling, and data collapse for $t_m = \{2,\ 4,\ 8, ...,\ 1024\}$.

2. Subcritical Patterns of Metastable States

In analyzing the structure of (1+1)-dimensional SDSM metastable states, the following quantities are measured as varying L and T. Note that the bracket is the spatial average, *i.e.*, $\langle A(x,t)\rangle \equiv \frac{1}{L}\sum_{x=0}^{L-1} A(x,t)$.

- The average density of sites occupied with grains at time t and fluctuations:
  $\rho(t) \equiv \langle z(x,t)\rangle$
  $[\Delta\rho(t)]^2 \equiv \langle z(x,t)^2\rangle - \langle z(x,t)\rangle^2$

- Accumulated height up to time t and fluctuations:
  $h(x,t) \equiv \frac{1}{t}\sum_{t'=0}^{t-1} z(x,t')$
  $W^2(t) \equiv \langle h(x,t)^2\rangle - \langle h(x,t)\rangle^2$

- Connected two-point correlation functions:
  $C_z(r,t) \equiv \langle z(x,t)z(x+r,t)\rangle - \langle z(x,t)\rangle\langle z(x+r,t)\rangle$
  $C_h(r,t) \equiv \langle h(x,t)h(x+r,t)\rangle - \langle h(x,t)\rangle\langle h(x+r,t)\rangle$
  Here $C_z(0,t) = [\Delta\rho_T(t)]^2$ and $C_h(0,t) = W^2(t)$, respectively.

- The q-th correlation functions:
  $G_z^{(q)}(r,t) \equiv \langle |z(x,t) - z(x+r,t)|^q\rangle$
  $G_h^{(q)}(r,t) \equiv \langle |h(x,t) - h(x+r,t)|^q\rangle$

- Cumulative gap distribution $D_{\rm gap}$ at time t:
  $D_{\rm gap}(\delta,t) = \int_\delta^\infty P_{\rm gap}(\delta,t)$

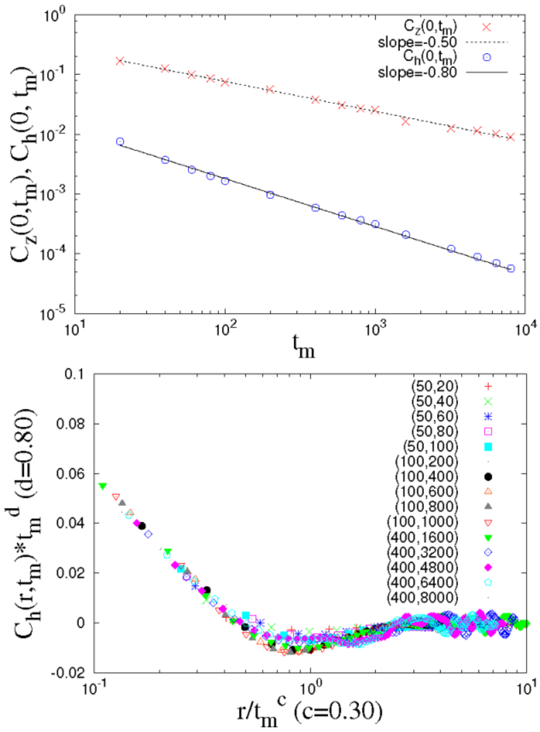

Fig. 4. (Color online) Scaling behaviors in fluctuations of z and h in the upper panel, and two-point height correlation functions C_h in the lower panel.

IV. NUMERICAL RESULTS

Fig. 3 shows that grains are uniformly distributed with a characteristic size of gap between clusters at $t = t_m$, which grows as $t_m^{0.46}$. Due to the fact that $\rho(t_m) \sim t_m^{-\alpha_r}$ with $\alpha_r = 0.45(2)$ and $\rho(t_m) \simeq 1/(1+\Delta(t_m))$, we expect roughly $\Delta(t_m) \sim t_m^{\alpha_r}$, which is consistent with our numerical results.

Density fluctuations and height fluctuations are shown in the upper plots of Fig. 4. Both fluctuations decay in time as $t_m^{-0.50}$ (density) and $t_m^{-0.80}$ (height), respectively. Since connected two-point grain correlation functions, $C_z(r,t_m)$, are too noisy to be analyzed, we only present $C_h(r,t_m)$ in the lower plots of Fig. 4. Anti-correlation dip location r^* scales as t^c with $c = 0.30(5)$. As plotting semi-log scales, data collapse very well.

In Figs. 5, 6 and 7, we check spatial correlations and extended self-similarity properties [10–12] with the q-th moments of correlation functions up to $q = 6$. Although the height-height correlation function $G_h^{(q=2)}$ doesn't show any clear power-law behavior in r, it exhibits dynamic scaling as $G_h^{(q)}(t) \sim t_m^{-dq} f_q(r/t^c)$, where $f_q(x) = [f_1(x)]^q$.

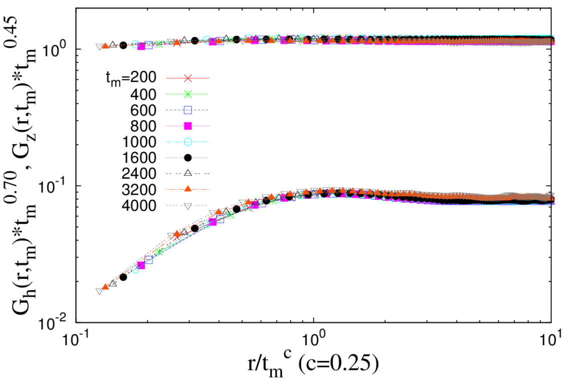

Fig. 5. (Color online) Scaling functions of G_z and G_h, where $q = 2$.

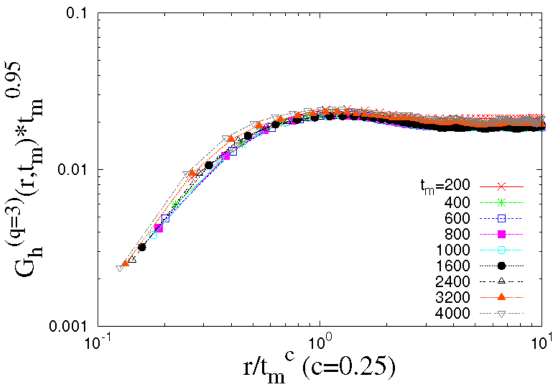

Fig. 6. (Color online) Scaling functions of $G_h^{(q=3)}$.

V. SUMMARY AND REMARKS

In summary, we have investigated spatial correlations and extended self-similarity (ESS) properties of metastable-state SOC patterns for (1+1)-dimensional non-Abelian stochastic directed sandpiles, which can be applicable to (2+1)-dimensional ones. Such patterns can be called as large-scale "scar" networks, which can be quantitively compared to recent observations of space plasma. Moreover, self-organized critical behaviors and universality classes are relevant to stochasticity. Since ESS test indicate whether either multifractal nature or anomalous scaling exist, one can apply it to find some essential scaling properties in real systems.

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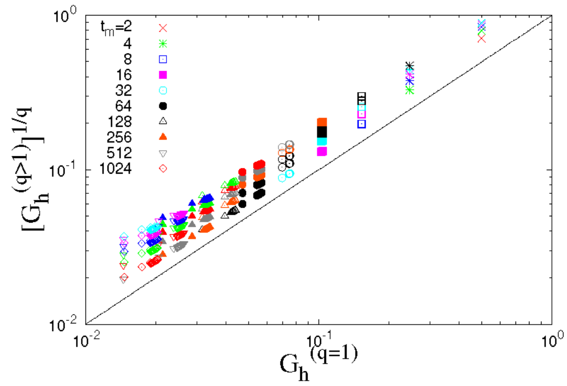

Fig. 7. (Color online) Extended self-similarity test: $G_h^{(q>1)}(G_h^{(q=1)})$ up to $q = 6$.

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