Supplemental Material for
"Role of Hubs in the Synergistic Spread of Behavior"

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CONTENTS

I. GENERATION OF SCALE-FREE NETWORKS

In our simulations of the generalized epidemic process (GEP), we randomly generated the scale-free networks (SFNs) according to the following three-step scheme.

Step 1. Depending on the value of α, fix the maximum degree as

$$k_{\max} = \begin{cases} N-1 & \text{if } \alpha \geq 3, \\ \lfloor \sqrt{N} \rfloor & \text{if } 2 < \alpha < 3. \end{cases} \tag{S1}$$

This ensures that the degrees of adjacent nodes are uncorrelated [S1].

Step 2. Given the degree distribution

$$p_k = \frac{k^{-\alpha}}{\sum_{k'=k_m}^{k_{\max}} k'^{-\alpha}}, \tag{S2}$$

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TABLE S1. Unsigned Stirling numbers of the first kind $\left[{j \atop i}\right]$ for small non-negative integers j and i.

j \ i	0	1	2	3	4	5	6	7
0	1	–	–	–	–	–	–	–
1	0	1	–	–	–	–	–	–
2	0	1	1	–	–	–	–	–
3	0	2	3	1	–	–	–	–
4	0	6	11	6	1	–	–	–
5	0	24	50	35	10	1	–	–
6	0	120	274	225	85	15	1	–
7	0	720	1764	1624	735	175	21	1

generate a degree sequence *deterministically* so that the number of nodes with degree k, denoted by N_k, satisfies

$$\left\lfloor N \sum_{k'>k} p_{k'} \right\rfloor = \sum_{k'>k} N_{k'}, \tag{S3}$$

for every integer $k \in [k_m, k_{\max}]$. This method, used in [S2], reduces the noise stemming from the sample-to-sample fluctuations of the degree sequence at finite N.

Step 3. Randomly connect the nodes according to the given degree sequence, avoiding the creation of self-loops and multiple links between the same pair of nodes.

II. DERIVATION OF EQ. (4)

We first rewrite Eq. (2) of the main text as

$$\begin{aligned} f(q) &= 1 - \sum_{k=k_m}^{\infty} p'_k \left[\sum_{m=0}^{n-1} \binom{k-1}{m} (1-\lambda)^m q^m (1-q)^{k-1-m} + \sum_{m=n}^{k-1} \binom{k-1}{m} (1-\lambda)^{n-1} (1-\mu)^{m-n+1} q^m (1-q)^{k-1-m} \right] \\ &= 1 - \sum_{k=k_m}^{\infty} p'_k \left\{ \left(\frac{1-\lambda}{1-\mu}\right)^{n-1} (1-\mu q)^{k-1} + \sum_{m=0}^{n-2} \binom{k-1}{m} (1-\lambda)^m \left[1 - \left(\frac{1-\lambda}{1-\mu}\right)^{n-m-1} \right] q^m (1-q)^{k-1-m} \right\}, \end{aligned} \tag{S4}$$

whose validity can be easily shown by the binomial expansion of $(1-\mu q)^{k-1}$. Using a notation for the *Lerch transcendent*

$$\Phi_{s,v}(z) \equiv \sum_{i=0}^{\infty} \frac{z^i}{(v+i)^s}, \tag{S5}$$

we can calculate the summations over k in Eq. (S4) to obtain

$$f(q) = 1 - \frac{1}{\zeta_{\alpha-1,k_m}} \left(\frac{1-\lambda}{1-\mu}\right)^{n-1} (1-\mu q)^{k_m-1} \Phi_{\alpha-1,k_m}(1-\mu q) \\ - \frac{1}{\zeta_{\alpha-1,k_m}} \sum_{m=0}^{n-2} \frac{(1-\lambda)^m}{m!} \left[1 - \left(\frac{1-\lambda}{1-\mu}\right)^{n-m-1}\right] (-q)^m \frac{d^m}{dq^m} \left[(1-q)^{k_m-1} \Phi_{\alpha-1,k_m}(1-q)\right]. \tag{S6}$$

In order to expand the rhs of Eq. (S6) with respect to q, we note that the Lerch transcendent has a series expansion [S3]

$$\Phi_{s,v}(z) = z^{-v} \sum_{i=0}^{\infty} \zeta_{s-i,v} \frac{(\ln z)^i}{i!} + z^{-v} \Gamma(1-s)(-\ln z)^{s-1} \tag{S7}$$

for any complex z with $|\ln z| < 2\pi$ and for real numbers s and v satisfying $s \neq 1, 2, 3, \ldots$ and $v \neq 0, -1, -2, \ldots$. Taking advantage of the generating function

$$[\ln(1-x)]^i = (-1)^i \cdot i! \cdot \sum_{j=i}^{\infty} \begin{bmatrix} j \\ i \end{bmatrix} \frac{x^j}{j!} \tag{S8}$$

for the *unsigned Stirling numbers of the first kind* $\begin{bmatrix} j \\ i \end{bmatrix}$ (whose values for small j and i are listed in Table S1), we can derive a useful relation

$$\frac{[\ln(1-x)]^i}{1-x} = -\frac{1}{i+1} \frac{d}{dx} [\ln(1-x)]^{i+1} = (-1)^i \cdot i! \cdot \sum_{j=i+1}^{\infty} \begin{bmatrix} j \\ i+1 \end{bmatrix} \frac{x^{j-1}}{(j-1)!}. \tag{S9}$$

This in turn can be used to rewrite Eq. (S7) in a more convenient form

$$(1-x)^{v-1} \Phi_{s,v}(1-x) = \sum_{j=1}^{\infty} \left\{ \sum_{i=0}^{j-1} (-1)^i \begin{bmatrix} j \\ i+1 \end{bmatrix} \zeta_{s-i,v} \right\} \frac{x^{j-1}}{(j-1)!} + \frac{\Gamma(1-s)}{\zeta_{s,v}} x^{s-1} [1 + O(x)] \\ = \sum_{j=0}^{\infty} \left\{ \sum_{i=1}^{j+1} (-1)^{i+1} \begin{bmatrix} j+1 \\ i \end{bmatrix} \zeta_{s-i+1,v} \right\} \frac{x^j}{j!} + \frac{\Gamma(1-s)}{\zeta_{s,v}} \left[x^{s-1} + O(x^s)\right], \tag{S10}$$

where the second equality is obtained by the change of variables $j \to j+1$ and $i \to i-1$. Using the above expansion in Eq. (S6), a tedious but straightforward calculation yields

$$f(q) = \sum_{j=1}^{\infty} \left\{ \sum_{i=1}^{j+1} \frac{(-1)^{i+1}}{j!} \begin{bmatrix} j+1 \\ i \end{bmatrix} \frac{\zeta_{\alpha-i,k_m}}{\zeta_{\alpha-1,k_m}} \right\} \left(\frac{1-\lambda}{1-\mu}\right)^{n-1} \left\{ \sum_{m=0}^{n-2} \binom{m-1-j}{m} (1-\mu)^m \left[1 - \left(\frac{1-\mu}{1-\lambda}\right)^{n-1-m}\right] - \mu^j \right\} q^j \\ + \frac{\Gamma(2-\alpha)}{\zeta_{\alpha-1,k_m}} \left(\frac{1-\lambda}{1-\mu}\right)^{n-1} \left\{ \sum_{m=0}^{n-2} \binom{m+1-\alpha}{m} (1-\mu)^m \left[1 - \left(\frac{1-\mu}{1-\lambda}\right)^{n-1-m}\right] - \mu^{\alpha-2} \right\} \left[q^{\alpha-2} + O\big(q^{\alpha-1}\big)\right], \tag{S11}$$

where $\binom{m'}{m}$ is a generalized Binomial coefficient defined as

$$\binom{m'}{m} \equiv \frac{m'(m'-1)\cdots(m'-m+1)}{m!} \tag{S12}$$

for any integer m' and a non-negative integer m. The definition implies $\binom{m'}{m} = (-1)^m \binom{|m'|+m-1}{m}$ for any negative m' and $\binom{m'}{m} = 0$ whenever $m > m' \geq 0$. Using these properties and Table S1, the order q component of $f(q)$ is given by

$$\begin{aligned}
&\frac{\zeta_{\alpha-1,k_m} - \zeta_{\alpha-2,k_m}}{\zeta_{\alpha-1,k_m}} \left(\frac{1-\lambda}{1-\mu}\right)^{n-1} \left\{\sum_{m=0}^{n-2} \binom{m-2}{m} (1-\mu)^m \left[1 - \left(\frac{1-\mu}{1-\lambda}\right)^{n-1-m}\right] - \mu\right\} q \\
&= \frac{\zeta_{\alpha-1,k_m} - \zeta_{\alpha-2,k_m}}{\zeta_{\alpha-1,k_m}} \left(\frac{1-\lambda}{1-\mu}\right)^{n-1} \left\{\left[1 - \left(\frac{1-\mu}{1-\lambda}\right)^{n-1}\right] - (1-\mu)\left[1 - \left(\frac{1-\mu}{1-\lambda}\right)^{n-2}\right]\theta_{n-3} - \mu\right\} q \\
&= \frac{\zeta_{\alpha-2,k_m} - \zeta_{\alpha-1,k_m}}{\zeta_{\alpha-1,k_m}} \lambda q,
\end{aligned} \tag{S13}$$

where $\theta_m = 1$ ($\theta_m = 0$) for any integer $m \geq 0$ ($m < 0$). Then Eq. (4) of the main text is obtained by defining $g_{s,n}(\lambda, \mu)$ as in Eq. (5) of the main text.

III. PHASE TRANSITIONS AT INTEGER DEGREE EXPONENTS

If the degree exponent α is an integer, the epidemic outbreaks and their associated critical phenomena are governed by the behavior of $\Phi_{s,v}(z)$ near $z = 1$ for a positive integer s. The relevant series expansion is given by [S3]

$$\Phi_{s,v}(z) \equiv z^{-v} \sum_{n=0}^{\infty} \tilde{\zeta}_{s-n,v} \frac{(\ln z)^n}{n!} + z^{-v} \left[\psi(s) - \psi(v) - \ln(-\ln z)\right] \frac{(\ln z)^{s-1}}{(s-1)!} \tag{S14}$$

for $|\ln z| < 2\pi$ and $v \neq 0, -1, -2, \ldots$, where we have introduced the notations

$$\tilde{\zeta}_{s,v} \equiv \begin{cases} \zeta_{s,v} & \text{if } s \geq 2, \\ 0 & \text{if } s = 1 \end{cases} \tag{S15}$$

and $\psi(s) \equiv \Gamma'(s)/\Gamma(s)$ for the digamma function. Using Eq. (S8), we can recast the above expansion into a more convenient form

$$(1-x)^{v-1}\,\Phi_{s,v}(1-x) = \sum_{j=0}^{\infty} \left\{\sum_{i=1}^{j+1} (-1)^{i+1} \begin{bmatrix} j+1 \\ i \end{bmatrix} \tilde{\zeta}_{s-i+1,v}\right\} \frac{x^j}{j!} - \frac{(-1)^{s-1}}{(s-1)!} \left\{x^{s-1} \ln x + \left[\psi(v) - \psi(s)\right] x^{s-1}\right\} + O(x^s)\,. \tag{S16}$$

Based on this formula, we can expand the rhs of Eq. (S6) as

$$\begin{aligned}
f(q) = &\sum_{j=1}^{\infty} \left\{\sum_{i=1}^{j+1} \frac{(-1)^{i+1}}{j!} \begin{bmatrix} j+1 \\ i \end{bmatrix} \frac{\tilde{\zeta}_{\alpha-i,k_m}}{\zeta_{\alpha-1,k_m}}\right\} g_{j,n}(\lambda,\mu)\, q^j - \frac{(-1)^{\alpha-2}}{\zeta_{\alpha-1,k_m}(\alpha-2)!} g_{\alpha-2,n}(\lambda,\mu)\, q^{\alpha-2} \ln q \\
&- \frac{(-1)^{\alpha-2}}{\zeta_{\alpha-1,k_m}(\alpha-2)!} \Bigg[\left[\psi(k_m) - \psi(\alpha-1)\right] g_{\alpha-2,n}(\lambda,\mu) - \left(\frac{1-\lambda}{1-\mu}\right)^{n-1} \Bigg\{\mu^{\alpha-2} \ln \mu \\
&- \sum_{m=0}^{\min[\alpha,n]-2} (1-\mu)^m \left[1 - \left(\frac{1-\mu}{1-\lambda}\right)^{n-1-m}\right] \binom{m+1-\alpha}{m} \left[\psi(\alpha-1) - \psi(\alpha-1-m)\right]\Bigg\}\Bigg] q^{\alpha-2} + O\big(q^{\alpha-1} \ln q\big)\,,
\end{aligned} \tag{S17}$$

where we have used $g_{s,n}(\lambda, \mu)$ defined in Eq. (5) of the main text. The main difference between Eq. (4) of the main text and Eq. (S17) lies in the presence of $q^{\alpha-2} \ln q$ in the latter, which is always lower-order than $q^{\alpha-2}$. If $\alpha > 5$, the term is simply irrelevant to epidemic outbreaks. If $\alpha \in \{3,4,5\}$, the logarithmic correction has nontrivial effects on the transition behaviors, as discussed case by case below (see Table S2 for a summary).

Case of $\alpha = 5$: the lowest-order terms of Eq. (S17) are given by

$$f(q) = \frac{\zeta_{3,k_m} - \zeta_{4,k_m}}{\zeta_{4,k_m}} \lambda q + \frac{\zeta_{2,k_m} - 3\zeta_{3,k_m} + 2\zeta_{4,k_m}}{2\zeta_{4,k_m}} g_{2,n}(\lambda,\mu)\, q^2 + \frac{g_{3,n}(\lambda,\mu)}{6\zeta_{4,k_m}}\, q^3 \ln q + O\big(q^3\big)\,, \tag{S18}$$

whose form is similar to the corresponding recursive relation for a non-integer $\alpha > 4$. Based on the same arguments described in the main text, the epidemic threshold is obtained as $\lambda_c = \zeta_{4,k_m}/(\zeta_{3,k_m} - \zeta_{4,k_m})$, and the tricritical point (TCP) satisfies $g_{2,n}(\lambda_c, \mu_t) = 0$, which has a physical solution $\mu_t = \lambda_c/(1-\lambda_c) \in (0,1)$ for $n = 2$ and sufficiently large k_m. Near the TCP, we can approximate the above equation as

$$0 \simeq \epsilon_\lambda q + c_{\alpha,k_m} \epsilon_\mu q^2 - c'_{\alpha,k_m} q^3 |\ln q|, \tag{S19}$$

where c_{α,k_m} and c'_{α,k_m} are positive coefficients. Thus the behavior of the outbreak size in this regime satisfies

$$r \sim q \sim \begin{cases} \epsilon_\lambda/|\epsilon_\mu| & \text{if } \epsilon_\mu < 0,\ |\epsilon_\mu| \gg |\epsilon_\lambda \ln \epsilon_\lambda|^{1/2}, \\ |\epsilon_\lambda/\ln \epsilon_\lambda|^{1/2} & \text{if } |\epsilon_\mu| \ll |\epsilon_\lambda \ln \epsilon_\lambda|^{1/2}, \\ \epsilon_\mu/|\ln \epsilon_\mu| & \text{if } \epsilon_\mu > 0,\ |\epsilon_\mu| \gg |\epsilon_\lambda \ln \epsilon_\lambda|^{1/2}. \end{cases} \tag{S20}$$

Case of $\alpha = 4$: the lowest-order terms of Eq. (S17) are obtained as

$$\begin{aligned} f(q) = {} & \frac{\zeta_{2,k_m} - \zeta_{3,k_m}}{\zeta_{3,k_m}} \lambda q - \frac{g_{2,n}(\lambda,\mu)}{2\zeta_{3,k_m}}\, q^2 \ln q \\ & - \frac{1}{2\zeta_{3,k_m}} \Bigg[[3\zeta_{2,k_m} - 2\zeta_{3,k_m} + \psi(k_m) - \psi(3)]\, g_{2,n}(\lambda,\mu) - \left(\frac{1-\lambda}{1-\mu}\right)^{n-1} \Bigg\{ \mu^2 \ln \mu \\ & - \sum_{m=0}^{\min[4,n]-2} (1-\mu)^m \left[1 - \left(\frac{1-\mu}{1-\lambda}\right)^{n-1-m} \right] \binom{m-3}{m} [\psi(3) - \psi(3-m)] \Bigg\} \Bigg] q^2 + O\big(q^3 \ln q\big)\,, \end{aligned} \tag{S21}$$

which implies that the epidemic threshold is at $\lambda_c = \zeta_{3,k_m}/(\zeta_{2,k_m} - \zeta_{3,k_m})$ and that the TCP satisfies $g_{2,n}(\lambda_c, \mu_t) = 0$. As was the case for $\alpha > 4$, the TCP exists only for $n = 2$ and sufficiently large k_m. The near-TCP properties are described by

$$0 \simeq \epsilon_\lambda q + c_{\alpha,k_m} \epsilon_\mu q^2 |\ln q| - c'_{\alpha,k_m} q^2, \tag{S22}$$

for positive coefficients c_{α,k_m} and c'_{α,k_m}. Thus the outbreak size in this regime obeys

$$r \sim q \sim \begin{cases} \epsilon_\lambda/|\epsilon_\mu \ln(\epsilon_\lambda/|\epsilon_\mu|)| & \text{if } \epsilon_\mu < 0,\ |\epsilon_\mu| \gg |\ln \epsilon_\lambda|^{-1}, \\ \epsilon_\lambda & \text{if } |\epsilon_\mu| \ll |\ln \epsilon_\lambda|^{-1}, \\ \mathrm{e}^{-c'_{\alpha,k_m}/(c_{\alpha,k_m}\epsilon_\mu)} & \text{if } \epsilon_\mu > 0,\ |\epsilon_\mu| \gg |\ln \epsilon_\lambda|^{-1}. \end{cases} \tag{S23}$$

TABLE S2. Scaling exponents describing tricritical properties of the GEP (if TCPs exist) on random SFNs for integer degree exponents α.

	$P_\infty \sim \epsilon_\lambda^{\beta_c}$	$r \sim \epsilon_\lambda^{\beta_t}$	$\epsilon_\mu \sim \epsilon_\lambda^{\phi}$
$\alpha = 5$	ϵ_λ	$\lvert\epsilon_\lambda / \ln \epsilon_\lambda\rvert^{1/2}$	$\lvert\epsilon_\lambda \ln \epsilon_\lambda\rvert^{1/2}$
$\alpha = 4$	$\lvert\epsilon_\lambda / \ln \epsilon_\lambda\rvert$	ϵ_λ	$\lvert \ln \epsilon_\lambda\rvert^{-1}$
$\alpha = 3$	$\lambda e^{-c/\lambda}$	λ^0	λ

Case of $\boldsymbol{\alpha = 3}$: the lowest-order terms of Eq. (S17) are given by

$$f(q) = -\frac{1}{\zeta_{2,k_m}} \lambda q \ln q - \frac{1}{\zeta_{2,k_m}} \left[[\zeta_{2,k_m} + \psi(k_m) - \psi(2)]\, \lambda + \left(\frac{1-\lambda}{1-\mu}\right)^{n-1} \left\{ \mu \ln \mu \right.\right.$$
$$\left.\left. - \sum_{m=0}^{\min[3,n]-2} (1-\mu)^m \left[1 - \left(\frac{1-\mu}{1-\lambda}\right)^{n-1-m} \right] \binom{m-2}{m} [\psi(2) - \psi(2-m)] \right\} \right] q + O\big(q^2\big)\,. \tag{S24}$$

At the vanishing epidemic threshold ($\lambda_c = 0$), $q = f(q)$ has (cannot have) a positive root if the sign of the q term on the rhs is positive (negative). Thus μ_t is given by

$$\mu_t \ln \mu_t = \sum_{m=0}^{\min[3,n]-2} (1-\mu_t)^m \left[1 - (1-\mu_t)^{n-1-m} \right] \binom{m-2}{m} [\psi(2) - \psi(2-m)]\,. \tag{S25}$$

We note that μ_t obtained from the above equation is in general not equal to $\lim_{\alpha \downarrow 3} \mu_t$ obtained from Eq. (6) of the main text. If $\mu < \mu_t$, the transition behaviors are described by the approximate formula

$$0 \simeq c_{\alpha,k_m} \lambda q |\ln q| + \left(c'_{\alpha,k_m} \epsilon_\mu - c''_{\alpha,k_m} \lambda \right) q, \tag{S26}$$

where c_{α,k_m}, c'_{α,k_m}, and c''_{α,k_m} are positive coefficients. In this case, the outbreak size satisfies

$$r \sim \lambda q \sim \lambda\, \mathrm{e}^{\left(c'_{\alpha,k_m} \epsilon_\mu - c''_{\alpha,k_m} \lambda\right)/(c_{\alpha,k_m} \lambda)}. \tag{S27}$$

As ϵ_μ approaches zero so that $|\epsilon_\mu| \ll \lambda$ (which can be represented as $\phi = 1$), r abruptly becomes nonzero for an arbitrary positive value of λ. In contrast to the other cases, here r can be already nonzero at $\lambda = \lambda_c$ and $\mu = \mu_t$ in a manner analogous to a discontinuous transition.

IV. ILLUSTRATIONS OF ACTUAL OUTBREAKS

The importance of hubs in the MOTs for $3 < \alpha < 4$ is more directly illustrated in Fig. S1. Using the color scheme described in Fig. S1(a), each circular diagram of Fig. S1(b) shows the final state of the GEP with $n = 3$ at $\lambda = \lambda_c$ and $\mu = 0.5$ on the random SFNs with $N = 360$ nodes and $k_m = 4$. More specifically, each rod on the periphery corresponds to a node, aligned clockwise in the order of decreasing degree (nodes of equal degree are randomly ordered). The seed node (chosen to be the node of the highest degree) is black, the nodes infected in the $\mathbf{S_1}$-state are orange, and those infected in the $\mathbf{S_2}$-state are red. The uninfected nodes are left as vacancies. The links are drawn with grey lines only if they connect two infected neighbors. By comparing these two examples of epidemic outbreaks at $\alpha = 3.5$ and 4.5, it is clear that the $\mathbf{S_2} \to \mathbf{I}$ infections (red nodes) are especially frequent among the high-degree nodes in the case of $\alpha = 3.5$. This reflects the dominant role played by the hubs in the system-wide

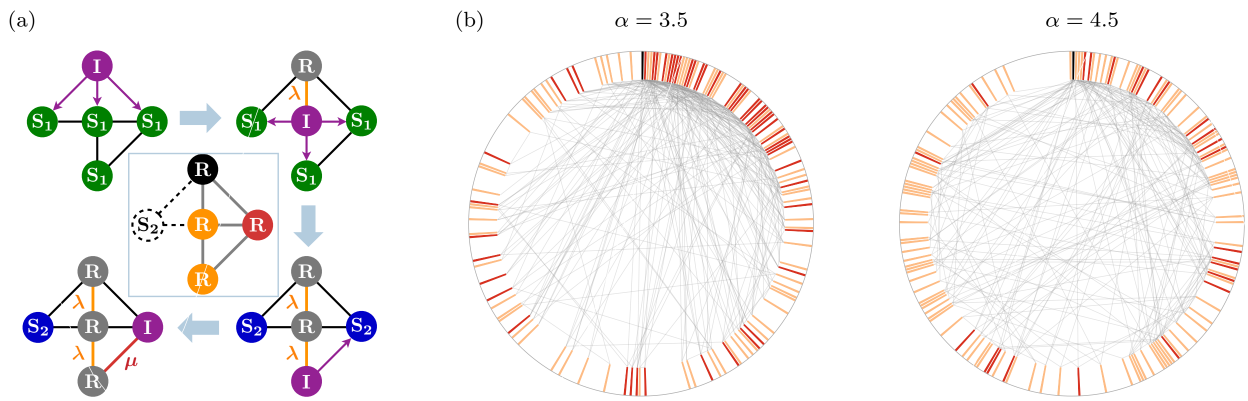

FIG. S1. Examples of the GEP with $n = 3$. (a) Entire dynamics on a five-node network. Each thick arrow represents a time step. Central box: in the final state, the seed is colored black, the nodes infected with probability λ (μ) are colored orange (red), and only the links connecting the infected nodes are shown. (b) Examples of the final state of the GEP on the SFNs with $k_m = 4$ at $\lambda = \lambda_c$, and $\mu = 0.5$. The rods (both colored and white) on the boundary correspond to the nodes, aligned clockwise in the order of decreasing degree. Only the infected nodes and their mutual links are shown according to the color scheme shown in (a). Here the seed is located at the node of the highest degree (the black rod).

avalanche for $3 < \alpha < 4$ (note that $\mu = 0.5 > \mu_t \approx 0.371$ in this case). In contrast, for $\alpha = 4.5$, the high cooperation threshold $n = 3$ and the dominance of two-neighbor effects reduce the significance of cooperative infections among the hubs at the transition, which is bound to be purely continuous. Consequently, the nodes infected by the cooperative mechanism are more evenly distributed among different degrees in the latter case.

V. NEAR-TCP CROSSOVER FOR $\alpha = 5.5$

In Fig. S2, we show the near-TCP crossover behaviors for the GEP with $n = 2$ on the SFNs with $\alpha = 5.5$ and $k_m = 4$, supplementing Fig. 2 of the main text.

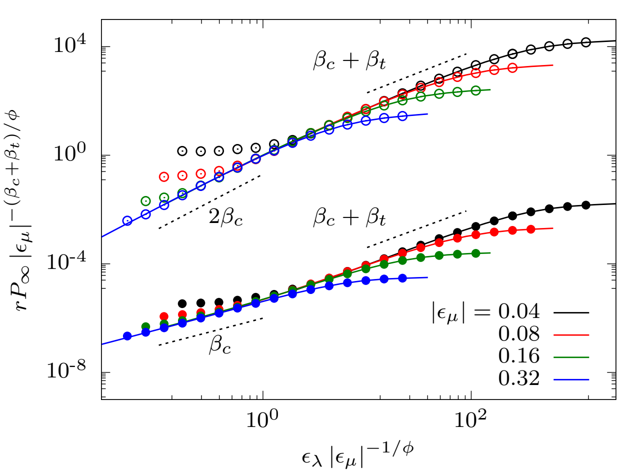

FIG. S2. The near-TCP crossover behaviors for $\alpha = 5.5$ and $n = 2$ described by Eq. (8) of the main text. The lines are obtained from the roots of Eq. (4) of the main text, and the symbols are simulation results obtained using 10^5 SFNs with $N = 10^7$ and $k_m = 4$. The upper (lower) data correspond to the $\epsilon_\mu < 0$ ($\epsilon_\mu > 0$) regime. To remove overlaps, all data for $\epsilon_\mu < 0$ have been divided by 10^6.

VI. COMPARISON BETWEEN THEORY AND NUMERICS

In Fig. S3, we show that deviations of the numerical data from the theoretical predictions of $\langle R \rangle$ converge to zero as the network size N increases to infinity.

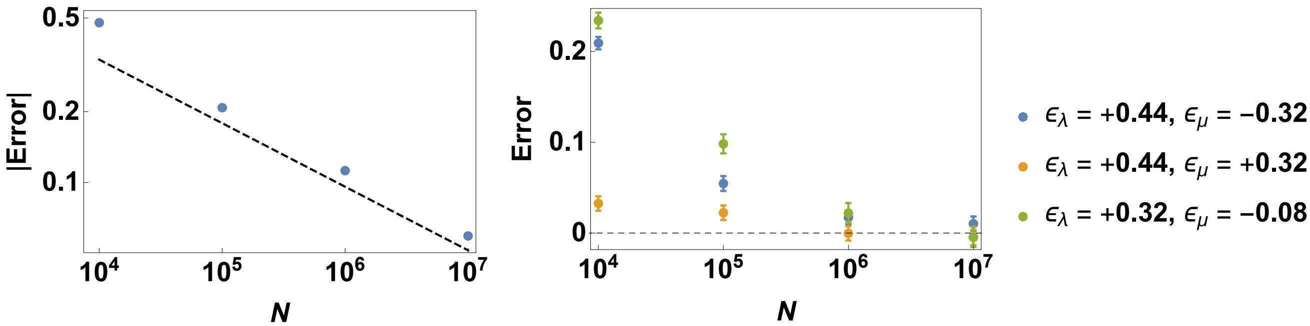

FIG. S3. (Left) Error ratio of $\langle R \rangle$ (i.e. $\frac{\text{numerics}}{\text{prediction}} - 1$) for scale-free networks with $\alpha = 2.5$ and $k_m = 4$ at $\lambda = 0.11$ and $\mu = 0.08$. The dashed line indicates a power-law decay $N^{-0.27}$. (Right) Error ratio of $\langle R \rangle$ for scale-free networks with $\alpha = 3.5$ and $k_m = 4$. The error bars indicate the range of sampling error.

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