Volume cocycle and volume form

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March 15, 2012

Abstract

Let $M = \Gamma\backslash G/K$ be a finite volume locally symmetric space of noncompact type of $\mathbb{R}$-rank 1. In this note we show that the monomorphism $H^*_{c,b}(G) \to H^*_b\left(\overline{M}, \partial\overline{M}\right)$ sends the cohomology class of the volume cocycle to the cohomology class of the volume form. This gives an alternative proof for Gromov's proportionality principle in this case.

1 Introduction

Proportionality principle. In [4] Gromov defined a topological invariant, the simplicial volume $\parallel M \parallel$ and he proved (for closed manifolds M) the proportionality principle $\parallel M \parallel = c_{\widetilde{M}} vol(M)$, where $c_{\widetilde{M}}$ is a constant which depends only on (the geometry of) the universal cover $\widetilde{M}$ with the pull-back metric. In [1, Section 6] Bucher-Karlsson gave a proof of the proportionality principle (for closed manifolds) in terms of bounded cohomology. (See also [3].) Moreover she showed in [1, Section 4] that for closed locally symmetric spaces the proportionality principle is a direct consequence of the (almost obvious) fact that the homomorphism $H^*_c(G) \to H^*_{dR}(M)$ sends the volume class in $H^*_c(G)$ to the cohomological fundamental class, that is the class of the volume form. (This allows for a description of $c_{G/K}$, see Section 4 below.) In this note we will use the constructions from our recent paper [8] to extend the argument to the finite-volume $\mathbb{R}$-rank 1 case.

Volume class. Let Γ be a lattice in a semisimple Lie group G and let $M = \Gamma\backslash G/K$ be the corresponding locally symmetric space. Let $n = dim(M)$. The volume class $v_n \in H^n_c(G)$ is defined as the cohomology class of the volume cocycle v_n given by

$$v_n(g_1, \ldots, g_n) := vol(str(\tilde{x}, g_1\tilde{x}, g_1g_2\tilde{x}, \ldots, g_1g_2\ldots g_n\tilde{x}))$$

for some $\tilde{x} \in G/K$, where $str(\tilde{x}, g_1\tilde{x}, g_1g_2\tilde{x}, \ldots, g_1g_2\ldots g_n\tilde{x})$ denotes the straight simplex with vertices $\tilde{x}, g_1\tilde{x}, g_1g_2\tilde{x}, \ldots, g_1g_2\ldots g_n\tilde{x} \in G/K$. The cohomology class $[v_n]$ does not depend on $\tilde{x}$.

Uniform lattices. If Γ is cocompact, then it is well known (and easy to prove using that $H^*(\Gamma)$ can be computed from the complex of Γ-invariant cochains on G) that the restriction homomorphism $res: H^*_c(G) \to H^*(\Gamma)$ is an injective isometry (a left inverse is the transfer map [5, Lemma 2.1], given by integration over a compact fundamental domain) and that the composition

$$H^*_c(G) \to H^*(\Gamma) \cong H^*_{dR}(M)$$

sends the volume class to the cohomology class of the volume form.

Nonuniform lattices. In this note we will consider non-uniform lattices Γ. Then $res: H_c^*(G) \rightarrow H^*(\Gamma)$ need not be injective (the transfer map is not defined for arbitrary cocycles), however the analogous morphism

$$res: H_{c,b}^*(G) \rightarrow H_b^*(\Gamma)$$

in bounded cohomology is an injective isometry (integration of *bounded* cocycles over the fundamental domain still works, see [12, Section 8.6.2]).

We assume that $M = \Gamma\backslash G/K$ is the interior of a compact n-manifold $\overline{M}$ with boundary $\partial\overline{M}$. Since the locally symmetric metric is defined only on $M = \overline{M} - \partial\overline{M}$ we have to specify what we mean by the cohomology class of the volume form. Let $DCone\left(\partial\overline{M} \rightarrow \overline{M}\right) := DCone\left(\cup_{i\in I}\partial_i\overline{M} \rightarrow \overline{M}\right)$ be the union along $\partial\overline{M}$ of $\overline{M}$ and the (disjoint) cones over the path components of $\partial\overline{M}$. We have a homeomorphism $h: M \cup \{cusps\} \rightarrow DCone\left(\partial\overline{M} \rightarrow \overline{M}\right)$, where the cusps are mapped to the cone points.

The metric on M defines a volume form on M, hence via h a volume form $\overline{dvol}$ on $DCone\left(\partial\overline{M} \rightarrow \overline{M}\right) - \{\text{cone points}\}$, hence with the proof of kkk a simplicial cocycle $\overline{dvol}$ on $DCone\left(\partial\overline{M} \rightarrow \overline{M}\right)$ by integration (declaring the cone points to have measure zero), hence a (bounded) cohomology class $\left[\overline{dvol}\right] \in H_b^*\left(DCone\left(\partial\overline{M} \rightarrow \overline{M}\right)\right)$. **When** G/K **has** $\mathbb{R}$**-rank 1**, then $\partial\overline{M}$ has nilpotent (hence amenable) fundamental group. This implies that $H_b^*\left(DCone\left(\partial\overline{M} \rightarrow \overline{M}\right)\right)$ is isometrically isomorphic to $H_b^*\left(\overline{M}\right)$ and to $H_b^*\left(\overline{M}, \partial\overline{M}\right)$, see Lemma 1 below. Composition with res defines an injective isometry

$$\Xi: H_{c,b}^*(G) \rightarrow H_b^*\left(DCone\left(\partial\overline{M} \rightarrow \overline{M}\right)\right) \cong H_b^*\left(\overline{M}, \partial\overline{M}\right).$$

By [4, Section 1.2] the volume cocycle (and also the volume form) are bounded, i.e. we have $[v_n]_b \in H_{c,b}^n(G)$. The purpose of this note is to show that Ξ maps $[v_n]_b$ to the cohomology class of the volume form.

We mention that a second reason for using bounded cohomology (besides Ξ being an isometry) in the proof is that the isomorphism $H_b^*\left(\overline{M}, \partial\overline{M}\right) \cong H_b^*\left(\overline{M}\right)$ has no counterpart in ordinary cohomology. It seems plausible that the proportionality principle should hold not only for $\mathbb{R}$-rank 1 locally symmetric spaces but in greater generality for noncompact Riemannian manifolds of finite volume. However the argument of this paper does not seem to extend (not even to locally symmetric spaces of $\mathbb{Q}$-rank 1) because the isomorphism $H_b^*\left(\overline{M}, \partial\overline{M}\right) \cong H_b^*\left(\overline{M}\right) \cong H_b^*(\Gamma)$ does not hold in general.

We thank Michelle Bucher-Karlsson for bringing the problem to our attention.

2 Descriptions of Ξ

Let $M = \Gamma\backslash G/K$. Fix some $x \in M, \tilde{x} \in G/K$ such that $\tilde{x}$ projects to x under $\pi: G/K \rightarrow M$. Let $C_*^{str,x}(M) \subset C_*(M)$ be the subcomplex of straight simplices with all vertices in x. There is a chain isomorphism

$$\Phi: C_*(B\Gamma) \rightarrow C_*^{str,x}(M)$$

given by

$$\Phi(\gamma_1, \ldots, \gamma_n) = \pi\left(str\left(\tilde{x}, \ldots, \gamma_1 \ldots \gamma_n \tilde{x}\right)\right).$$

Moreover the inclusion

$$j : C_*^{str,x}(M) \to C_*(M)$$

is a chain homotopy equivalence by [2, Theorem 1a].

We briefly recall some notation from [8].

Let $I = \{1, \ldots, s\}$ be an index set for the path components of $\partial \overline{M} = \partial_1 \overline{M} \cup \ldots \cup \partial_s \overline{M}$. Then, as in [8, Section 4.2.1] we let

$$DCone\left(\cup_{i\in I}\partial_i\overline{M} \to \overline{M}\right)$$

be the union along $\partial\overline{M}$ of $\overline{M}$ and the (disjoint) cones over the path components $\partial_i\overline{M}$, and as in [8, Section 4.2.2] we define

$$B\Gamma^{comp} = DCone\left(\cup_{i\in I} B\Gamma_i \to B\Gamma\right),$$

where Γ_i as in [8, Definition 5] is a fixed subgroup of Γ with $\Gamma_i \cong \pi_1\partial_i\overline{M}$.

In [8] we constructed a chain homotopy equivalence $C_*^{simp}\left(B\Gamma^{comp}\right) \simeq C_*\left(DCone\left(\cup_{i\in I}\partial_i\overline{M} \to \overline{M}\right)\right)$ under the assumption that G/K has $\mathbb{R}$-rank one and Γ is a lattice.

Namely we defined in [8, Definition 11] a simplicial complex $\widehat{C}_*^{str,x}(M)$ containing $C_*^{str,x}(M)$ and chain maps $\widehat{\Phi}, \hat{j}$ such that the following diagram (with i denoting the respective inclusions) commutes.

$$\begin{array}{ccc}
C_*^{simp}(B\Gamma) & \xrightarrow{\ i\ } & C_*^{simp}\left(B\Gamma^{comp}\right) \\
\downarrow \Phi & & \downarrow \widehat{\Phi} \\
C_*^{str,x}(M) & \xrightarrow{\ i\ } & \widehat{C}_*^{str,x}(M) \\
\downarrow j & & \downarrow \hat{j} \\
C_*(M) & \xrightarrow{\ i\ } & C_*(M\cup\{\text{cusps}\}) \\
 & & \cong \downarrow h \\
 & & C_*\left(DCone\left(\cup_{i\in I}\partial_i\overline{M} \to \overline{M}\right)\right)
\end{array}$$

(In the last row we use a homeomorphism $h : M \cup \{\text{cusps}\} \cong DCone\left(\cup_{i\in I}\partial_i\overline{M} \to \overline{M}\right)$.)

$\widehat{\Phi}$ is a chain isomorphism by [8, Lemma 8a]. Moreover, in [8, Lemma 8b] we proved that $\hat{j}$ is a chain homotopy equivalence. Thus $\hat{j}\widehat{\Phi}$ is a chain homotopy equivalence.

We observe that the composition $\left(\left(h\hat{j}\hat{\phi}i\right)^*\right)^{-1} \circ res$:

$$H_{cb}^*(G) \longrightarrow H_b^*(\Gamma) \longrightarrow H_b^*\left(DCone\left(\cup_{i\in I}\partial_i\overline{M} \to \overline{M}\right)\right)$$

agrees with Ξ. Indeed Ξ is the composition of res with the isomorphism $H_b^*(\Gamma) \to H_b^*\left(\overline{M}\right)$, which is given by $\left((j\Phi)^*\right)^{-1}$, and with the isomorphism $H_b^*\left(\overline{M}\right) \to H_b^*\left(DCone\left(\cup_{i\in I}\partial_i\overline{M} \to \overline{M}\right)\right)$, which is given by $\left((hi)^*\right)^{-1}$. Because of $ij\Phi = \hat{j}\widehat{\Phi}i$ this is the same as the composition considered above.

Lemma 1. *Let $\overline{M}$ be a compact manifold with boundary $\partial\overline{M}$ such that all path components of $\partial\overline{M}$ have amenable fundamental group injecting into $\pi_1\overline{M}$. Then*
a) inclusion $\overline{M} \to DCone\left(\partial\overline{M} \to \overline{M}\right)$ induces an isometric isomorphism

$$H_b^*\left(DCone\left(\partial\overline{M} \to \overline{M}\right)\right) \to H_b^*\left(\overline{M}\right),$$

b) the quotient map $C_*\left(\overline{M}\right) \to C_*\left(\overline{M}, \partial\overline{M}\right)$ induces an isometric isomorphism

$$H_b^*\left(\overline{M}, \partial\overline{M}\right) \to H_b^*\left(\overline{M}\right),$$

c) the composed isomorphism $\tau^* : H_b^*\left(DCone\left(\partial\overline{M} \to \overline{M}\right)\right) \to H_b^*\left(\overline{M}, \partial\overline{M}\right)$ is induced by $\tau(z) = z + Cone(\partial z)$ for relative cycles $z \in Z_*\left(\overline{M}, \partial\overline{M}\right)$.

Proof: a) We give an argument using the theory of multicomplexes as developed by Gromov in [4]. We stick to the notation of [9].

For each pair of topological spaces (X, Y) (with $\pi_1 Y \to \pi_1 X$ injective for all path components of Y) one has associated a pair of aspherical minimal multicomplexes $(K(X), K(Y))$. Their 0-skeleton coincides with (X, Y) and their 1-skeleta contains one 1-simplex in each homotopy class rel. $\partial\Delta^1$ of maps $\Delta^1 \to X$. We can assume that the 1-simplices in $K(X)$ are choosen to have minimal number of components of intersection with Y in their homotopy class.

For each subset $A \subset X$ there is an action of a certain group $\Pi_A X$ on $K(X)$, as defined in [9, Section 1.5]. By [9, Lemma 4], the group $\Pi_A X$ is amenable if $\pi_1 A$ is amenable for each path component of A. So in our setting we have the following commutative diagram with $G := \Pi_{\partial\overline{M}}\overline{M}$ and $H := \Pi_{DCone\left(\partial\overline{M}\right)} DCone\left(\partial\overline{M} \to \overline{M}\right)$ both amenable:

$$\begin{array}{ccc}
C_*^{sing}\left(\overline{M}\right) & \xrightarrow{i_1} & C_*^{sing}\left(DCone\left(\partial\overline{M} \to \overline{M}\right)\right) \\
\uparrow j_1 & & \uparrow j_2 \\
C_*^{simp}\left(K\left(\overline{M}\right)\right) & \xrightarrow{i_2} & C_*^{simp}\left(K\left(DCone\left(\partial\overline{M} \to \overline{M}\right)\right)\right) \\
\downarrow k_1 & & \downarrow k_2 \\
C_*^{simp}\left(K\left(\overline{M}\right)\right) \otimes_{\mathbb{Z}G} \mathbb{Z} & \xrightarrow{i_3} & C_*^{simp}\left(K\left(DCone\left(\partial\overline{M} \to \overline{M}\right)\right)\right) \otimes_{\mathbb{Z}H} \mathbb{Z}
\end{array}$$

According to Gromov's results in [4, Section 3.3] the morphisms j_1, j_2, k_1, k_2 induce isometric isomorphisms in bounded cohomology. Thus to prove the claim of the lemma it suffices that i_3 induces an isometric isomorphism in bounded cohomology. We claim that i_3 is actually an isomorphism of chain complexes. To prove this claim we are going to construct an chain homomorphism f_3 inverse to i_3.

First we define f_3 on the 1-skeleton. Let $\tau : [0,1] \to DCone\left(\partial\overline{M} \to \overline{M}\right)$ represent a 1-simplex in $C_1^{simp}\left(K\left(DCone\left(\partial\overline{M} \to \overline{M}\right)\right)\right)$. Upon homotopy we can assume that at most two boundary intervalls $[0,a]$ and $[b,1]$ are mapped to $DCone\left(\partial\overline{M}\right)$. (This follows from $\pi_1\left(DCone\left(\partial\overline{M}\right), \partial\overline{M}\right) = 0$.) Now let $h_1 = \tau\mid_{[0,a]}$ and $h_2 = \tau\mid_{[b,1]}$, and consider[1]

[1] If only $[0,a]$ or only $[b,1]$ or none of them are mapped to $DCone\left(\partial\overline{M}\right)$, then we let $h = \left\{h_1, \overline{h_1}\right\}$ or $h = \left\{h_2, \overline{h_2}\right\}$ or $h = \emptyset$.

$h = \left\{ h_1, \overline{h_1}, h_2, \overline{h_2} \right\} \in H$. Then clearly $\tau = h\sigma$, where $\sigma := \tau \mid_{[a,b]}$ represents a 1-simplex in $C_1^{simp}\left(K\left(\overline{M}\right)\right)$. We define $f_3\left(\tau \otimes 1\right) = \sigma \otimes 1$. Clearly $f_3 i_3 = id$. Since $\tau = h\sigma$ we have $\tau \otimes 1 = \sigma \otimes 1$ and therefore also $i_3 f_3 = id$. This defines $f_3 = i_3^{-1}$ on the 1-skeleton and using asphericity this definition easily extends to the full multicomplex.

b) This is Theorem 1.2 in [7]. (The proof also uses Gromov's theory of multicomplexes.)

c) An analogous commutative diagram as in a) shows that for computing the effect of τ^* on bounded cohomology we can replace $\tau : Z_*\left(\overline{M}, \partial\overline{M}\right) \rightarrow Z_*\left(DCone\left(\partial\overline{M} \rightarrow \overline{M}\right)\right)$ by the simplicial map $Z_*^{simp}\left(K\left(\overline{M}\right), K\left(\partial\overline{M}\right)\right) \otimes_{\mathbb{Z}G} \mathbb{Z} \rightarrow Z_*^{simp}\left(K\left(DCone\left(\partial\overline{M} \rightarrow \overline{M}\right)\right)\right) \otimes_{\mathbb{Z}G} \mathbb{Z}$ which sends $z \otimes 1$ to $\tau\left(z\right) \otimes 1$. On the other hand the proof of [7, Theorem 2.1] and the proof of a) yield that the composed isomorphism is induced by the simplicial map sending $z \otimes 1$ to $z \otimes 1$. But [9, Observation 1] implies for amenable H that $Cone\left(\partial z\right) \otimes 1 = 0$ whenever $\partial z \in C_*^{simp}\left(K\left(\partial\overline{M}\right)\right)$. Thus $z \otimes 1 = \tau\left(z\right) \otimes 1$ which yields the claim.

QED

3 Proof

After these preparations we are ready to prove the main result $\Xi\left(\left[v_n\right]_b\right) = \left[\overline{dvol}\right]_b$.

Theorem 1. *Let $\overline{M}$ be a compact manifold such that $M = \overline{M} - \partial\overline{M}$ is of the form $M = \Gamma \backslash G/K$ with G/K an $\mathbb{R}$-rank one symmetric space of noncompact type and $\Gamma \subset G$ a lattice. Then the injective isometry Ξ defined as the composition*

$$H_{cb}^*\left(G\right) \rightarrow H_b^*\left(\Gamma\right) \cong H_b^*\left(\overline{M}\right) \cong H_b^*\left(DCone\left(\partial\overline{M} \rightarrow \overline{M}\right)\right)$$

sends the cohomology class $\left[v_n\right]_b$ of the volume cocycle to the cohomology class $\left[\overline{dvol}\right]_b$ of the volume form.

Proof: We use the explicit description of the isomorphism $H_b^*\left(\Gamma\right) \cong H_b^*\left(\overline{M}, \partial\overline{M}\right)$ from Section 2.

Let $\left(\gamma_1, \ldots, \gamma_n\right)$ be an n-simplex in $B\Gamma$. Then $\hat{j}\left(\widehat{\Phi}\left(i\left(\gamma_1, \ldots, \gamma_n\right)\right)\right) = j\left(\Phi\left(\gamma_1, \ldots, \gamma_n\right)\right)$ and the latter is, by definition, the image of the straight simplex $str\left(\tilde{x}, \ldots, \gamma_1 \ldots \gamma_n \tilde{x}\right)$ under the projection $\pi : G/K \rightarrow M$. Application of the volume form yields the volume of this simplex, that is $vol\left(\pi\left(str\left(\tilde{x}, \ldots, \gamma_1 \ldots \gamma_n \tilde{x}\right)\right)\right)$. Since π is a local isometry the latter equals

$$vol\left(str\left(\tilde{x}, \ldots, \gamma_1 \ldots \gamma_n \tilde{x}\right)\right)$$

which is, by the very definition of v_n, exactly the value of $< v_n, \left(\gamma_1, \ldots, \gamma_n\right) >$. Hence

$$< h^* \overline{dvol}, \hat{j}\left(\widehat{\Phi}\left(i\left(\gamma_1, \ldots, \gamma_n\right)\right)\right) >=< v_n, \left(\gamma_1, \ldots, \gamma_n\right) >,$$

which implies $v_n = \left(h\hat{j}\widehat{\Phi}i\right)^* \overline{dvol}$. Hence the composition

$$H_{cb}^*\left(G\right) \longrightarrow H_b^*\left(\Gamma\right) \longrightarrow H_b^*\left(DCone\left(\cup_{i \in I} \partial_i \overline{M} \rightarrow \overline{M}\right)\right)$$

maps $[v_n]_b$ to

$$\left(\left(h\hat{j}\hat{\phi}i\right)^*\right)^{-1}([v_n]_b) = \boxed{dvol}_b .$$

QED

4 Applications

If M is a closed, orientable, Riemannian manifold, $Vol\,(M)$ its volume and $\parallel M \parallel$ the topologically defined simplicial volume, then it is a well-known application of the Hahn-Banach Theorem (cf. [1, Section 2]) that

$$\frac{Vol\,(M)}{\parallel M \parallel} = \parallel [dvol] \parallel_\infty .$$

Moreover the Gromov-Thurston proportionality principle states that this quotient $\frac{Vol(M)}{\|M\|}$ depends only on (the geometry of) the universal cover $\widetilde{M}$. This was proved in [4] and with more details in [10].

If $\widetilde{M} = G/K$ is a symmetric space of noncompact type (and again M is closed), then Bucher-Karlsson proved in [1] an explicit (and better, but still hard to compute) description for this quotient, namely

$$\frac{Vol\,(M)}{\parallel M \parallel} = \parallel [v_n] \parallel_\infty ,$$

where $[v_n] \in H_c^n(G)$ is the class of the volume cocycle. The latter depends only on the geometry of the universal cover G/K. (See also [3].) In fact she used that $\Xi : H_c^*(G) \to H^*(M)$ is an injective isometry, which then implies

$$\parallel [dvol] \parallel_\infty = \parallel [v_n] \parallel_\infty ,$$

because in the closed case it is obvious that Ξ sends $[v_n]$ to $[dvol]$.

Corollary 1. *If $M = \Gamma\backslash G/K$ is a finite volume locally symmetric space of noncompact type of $\mathbb{R}$-rank 1, then*

$$\frac{Vol\,(M)}{\parallel \overline{M}, \partial\overline{M} \parallel} = \parallel [v_n] \parallel_\infty .$$

Proof: Let $\beta \in H^{dim(M)}\left(\overline{M}, \partial\overline{M}; \mathbb{R}\right)$ be a cohomological fundamental class. A standard application of the Hahn-Banach Theorem shows $\frac{1}{\|\overline{M},\partial\overline{M}\|} = \parallel \beta \parallel_\infty$.

Let $\tau^* : H_b^*\left(DCone\left(\partial\overline{M} \to \overline{M}\right)\right) \to H_b^*\left(\overline{M}, \partial\overline{M}\right)$ be the isometric isomorphism induced by $\tau(z) = z + DCone\,(\partial z)$ for relative cycles $z \in Z_*\left(\overline{M}, \partial\overline{M}\right)$. We claim that $[dvol] := \tau^* \boxed{dvol}$ represents $Vol\,(M)\,\beta$. Since $H^{dim(M)}\left(\overline{M}, \partial\overline{M}; \mathbb{R}\right)$ is one-dimensional it suffices to check the equality for some representative of $\left[\overline{M}, \partial\overline{M}\right]$. Now, if z is some triangulation of $\left(\overline{M}, \partial\overline{M}\right)$, then $\tau(z) = z + DCone\,(\partial z)$ is a triangulation of $DCone\left(\partial\overline{M} \to \overline{M}\right)$

and we have $<\overline{dvol},\tau(z)>=vol(M)$, hence $<dvol,z>=<\tau^*\overline{dvol},z>=vol(M)$. This proves that $[dvol]$ represents $Vol(M)\beta$ and thus

$$\frac{Vol(M)}{\| \overline{M},\partial\overline{M} \|}=\| [dvol] \|_\infty .$$

Moreover Ξ is an injective isometry and we have proved in Theorem 1 that Ξ sends $[v_n]_b$ to $\boxed{dvol}_b$, thus $\| [v_n] \|_\infty=\| \boxed{dvol} \|_\infty=\| [dvol] \|_\infty$. *QED*

We remark that the proportionality principle in that case (and more generally for finite-volume locally symmetric spaces of $\mathbb{Q}$-rank one) has already been proved in [11], but that proof does not give the explicit constant.

For $G/K=\mathbb{RH}^n$ it follows from the Gromov-Thurston Theorem that $\| [v_n] \|_\infty$ is the maximal volume of an ideal simplex in $\mathbb{RH}^n$. By the Haagerup-Munkholm Theorem the latter equals the volume of a regular ideal simplex in $\mathbb{RH}^n$, e.g. $\| [v_2] \|_\infty=\pi$ and $\| [v_3] \|_\infty=3\Lambda\left(\frac{\pi}{3}\right)$, where Λ is the Lobatschewski function.

For $G/K=\mathbb{CH}^m,\mathbb{HH}^m,Ca\mathbb{H}^2$ it is clear that $\| [v_n] \|_\infty$ is bounded above by the maximal volume of an ideal simplex, but the exact value of $\| [v_n] \|_\infty$ has not been computed so far.

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