Finite-dimensional Torsion Classes

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For a flat family of chain complexes $(E = \oplus E_i, \partial) \to B$, one wants to show that the secondary characteristic classes of E and of the fiberwise homology $H(E)$ agree. Even better, a higher torsion form $T \in \Omega^* (B)$ with $dT = ch^0 (E) - ch^0 (H)$ will be constructed. Finally, an application to fibrewise Morse functions is explained.

The approach will be to consider the formal sum $\partial + \nabla$ as a flat superconnection and to apply secondary characteristic classes for this superconnection to interpolate between the secondary characteristic classes of E and H.

A superconnection of total degree -1 is an odd operator $A = \sum_{j \geq 0} A_j : C^\infty (B, E) \to \oplus_{\geq 0} \Omega^j (B, E)$, such that A_1 is a connection and $A_i \in \Omega^i (B, \mathrm{Hom}(E_j, E_{j-1+i}))$ if $i \neq 1$. It is flat if $A^2 = 0$, i.e. $A_0^2 = 0, [A_0, A_1] = 0, [A_0, A_2] + A_1^2 = 0, \ldots$. In particular, if A is a flat connection and ∂ a parallel differential, then $\partial + \nabla$ is a flat superconnection.

Let g be a hermitian metric, A^* the adjoint of A for g, and $X = \frac{1}{2}(A - A^*) \in \Omega(B, \mathrm{End}E)$. For each (odd) holomorphic function f, one defines a characteristic class (even a differential form) by $\alpha := (2\pi i)^{\frac{1-N}{2}} \; str f(X) \in \Omega(B)$. ($N$ is to be understood as $N = k$ on $\Omega^k(B)$.) We will consider $f(z) = ze^{z^2}$, which gives ch^0.

Lemma 1. *: The cohomology class $\left[ch^0 (E, A, g)\right]$ is independent of the metric.*

Proof: Let $g_t = tg_1 + (1-t) g_0$ and use it to define $X_t = \frac{1}{2}(A - A_t^*)$. One computes $\frac{d}{dt} str f(X_t) = (2\pi i)^{-\frac{N}{2}} d \; str \left(\frac{1}{2}\left(g_t^{-1}\right) \frac{dg_t}{dt} f'(X_t)\right)$, hence $f(X_1)$ and $f(X_0)$ differ by a closed form. □

For $b \in B$ we denote $H(E, \partial)_b$ the homology of $(E, \partial)_b$. By Hodge theory, H is a subspace of E with induced connection and metric.
Rescaling: for $t \in \mathbb{R}$ we define a metric g_t by $g_t = t^N g$, i.e. $g_t = t^i g$ on E_i.

Theorem 1. *(Bismut-Lott):*

$$\lim_{t \to \infty} ch^0 (E, A, g_t) = ch^0 \left(H, \nabla^H, g^H\right),$$

$$\lim_{t \to 0} ch^0 (E, A, g_t) = ch^0 (E, \nabla, g).$$

In view of Lemma 1 this proves that $\left[ch^0 (E, \nabla)\right] = \left[ch^0 \left(H, \nabla^H\right)\right]$. To construct a Stammform for their difference, we consider $\hat{f}(A, g_t) := (2\pi i)^{-\frac{N}{2}} str\left(\frac{N}{2} f'(X_t)\right)$ and note that the proof of Lemma 1, together with $\left(g_t^{-1}\right) \frac{dg_t}{dt} = \frac{N}{t}$, implies that $\frac{1}{t} d\hat{f}(A, g_t) = \frac{d}{dt} str \; f(A, g_t)$. This implies that

$$T := \int_0^\infty \hat{f}(A, g_t) - \lim_{t \to \infty} \hat{f}(A, g_t) - \lim_{t \to 0} \hat{f}(A, g_t) \frac{dt}{t}$$

$$= \int_0^\infty \left(\hat{f}(A, g_t) - \frac{1}{2}\chi'(H) f'(0) - \left(\chi'(E) - \frac{1}{2}\chi'(H)\right) f'\left(\frac{i\sqrt{t}}{2}\right)\right) \frac{dt}{t}$$

satisfies $dT = ch^0\left(H, \nabla^H\right) - ch^0\left(E, \nabla\right)$. (The constant terms are added to assure convergence of the integral.)
If E and H carry parallel metrics, it follows that T is closed and thus defines a cohomology class in $H^*\left(B\right)$.

Fibrewise Morse theory. Let $p: M \rightarrow B$ be a fibration. Let $h: M \rightarrow \mathbb{R}$ be a fibrewise Morse function, and let $C_i \rightarrow B$ be the finite covering by fibrewise critical points of index i. Let F be a flat vector bundle over M and consider the fibrewise Thom-Smale complex (with coefficients in F) $E := \left(p\mid_C\right)_*\left(o\left(W^u x\right) \otimes F\right)$.

If h satisfies the fibrewise Smale condition, then the fibrewise differential ∂ on E is defined and one may consider the flat superconnection $\partial + \nabla$. If h does not satisfy the fibrewise Thom-Smale condition, then Goette still constructs a flat superconnection $A = \nabla + a, a \in \Omega^*\left(B, End^{*-1}E\right)$ of total degree -1 satisfying the axioms of family Thom-Smale complexes.

For such family Thom-Smale complexes, the higher torsion form can be defined as above, but by using the metric $g_t = t^h g$ for $t \leq 1$, $g_t = t^N g$ for $t \geq 1$. (In the case that h satisfies the fibrewise Thom-Smale condition, one can assume that h is fibrewise self-indexing, hence $h = N$ on C.)

References

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[2] S. Goette, *Morse theory and higher torsion invariants I, II*, Preprints.