A survey on simplicial volume and invariants of foliations and laminations

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Contents

Abstract

We intend to give a not too technical introduction to several recent results of several authors related to hyperbolic volume and non-Hausdorfness of leaf spaces. In particular, we describe results about the normal and transverse Gromov norm of foliations and laminations.

1 Volume and Topology

1.1 3-manifolds

1.1.1 Topological decompositions

All used notions from 3-manifold topology are explained in the glossary in section 1.1.3. Let M be a **closed, orientable 3-manifold**. Then there is the following topological decomposition of M.

Kneser-Milnor: M has a unique decomposition $M = M_1 \sharp \ldots \sharp M_r$ as connected sum, with M_i either irreducible or $S^2 \times S^1$.

Jaco-Shalen-Johannson: If M is irreducible, then there is an (up to isotopy unique) family $T_1, \ldots, T_s$ of incompressible tori such that each connected component C of $M \setminus \cup_{i=1}^{s} T_i$ contains no embedded incompressible torus (except tori homotopic into ∂C).

1.1.2 Geometrization conjecture

Let M be a compact, orientable, irreducible 3-manifold, with boundary a (possibly empty) union of tori. Assume that each embedded incompressible torus can be homotoped into the boundary. Then there are two cases.

Case 1 (**Seifert case**): $\pi_1 M$ contains a (non-peripheral) subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. In this case, M must contain an immersed incompressible torus. By the Torus Theorem (Scott), this implies that M must be a Seifert fibration, i.e. that some finite cover of M is a circle bundle over a surface. (The immersed tori in M arise as projections of embedded tori in this circle bundle.) It is known that each Seifert fibration can be equipped with some locally homogeneous metric (a 'geometric structure') and that there are 7 possible types of geometric structures on Seifert fibrations.

Case 2 (**Atoroidal/Hyperbolic case**): M is atoroidal, that is, $\pi_1 M$ does not contain a (non-peripheral) subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.
Thurston's hyperbolization conjecture states that each compact, irreducible, atoroidal 3-manifold M is hyperbolic. This means the following:
An orientable 3-manifold M is called hyperbolic if there is a faithful representation $\rho : \pi_1 M \to PSL_2\mathbb{C} = Isom^+\left(\mathbb{H}^3\right)$ with discrete, torsionfree image, such that (the interior of) M is diffeomorphic to the quotient of hyperbolic 3-space $\mathbb{H}^3$ under the action of $\rho\left(\pi_1 M\right)$. The assumption that $\rho\left(\pi_1 M\right)$ is discrete and $\pi_1 M$ is torsionfree implies that the projection $\mathbb{H}^3 \to M$ is a covering map and thus (the interior of) M is equipped with

a complete Riemannian metric locally isometric to $\mathbb{H}^3$. In particular, M possesses a Riemannian metric of sectional curvature constant -1.
The conjecture was proved for Haken manifolds, that is, 3-manifolds containing an incompressible, boundary-incompressible surface, by Thurston. At the time of writing, the general hyperbolization conjecture seems to have been proved by Perelman, the proof still being under revision (see the notes of Kleiner-Lott at [18]).

1.1.3 Glossary of notions from the topology of 3-manifolds

A very readable account on the basic notions of 3-dimensional topology is [16].

A 3-manifold M is called **irreducible** if each embedded 2-sphere bounds an embedded 3-ball. By the sphere theorem (cf. [16], Theorem 3.8.), for each orientable 3-manifold M with $\pi_2 M \neq 0$, there exists an embedded 2-sphere representing a nontrivial element in $\pi_2 M$. Therefore, for orientable 3-manifolds M, irreducibility implies $\pi_2 M = 0$. If the Poincare conjecture is true, then also the converse holds.

An immersed surface $F \subset M$ is called **incompressible** if there is no compression disk for F. A compression disk for F is an embedded disk $D \subset M$ with $D \cap F = \partial D$, such that ∂D does not bound a disk in F. By the loop theorem (cf. [16], Corollary 3.3), a two-sided connected surface F in an orientable 3-manifold M is incompressible if and only if the induced homomorphism $\pi_1 F \to \pi_1 M$ is injective.

If M is a compact manifold with nonempty boundary, then an embedded (immersed) surface $F \subset M$ is called properly embedded (immersed) if $\partial F \subset \partial M$.

A properly immersed surface $F \subset M$ is **boundary-incompressible** if there is no boundary compression disk. A boundary compression disk for F is an embedded disk $D \subset M$ with $\partial D = \partial_0 D \cup \partial_1 D, \partial_0 D = \partial D \cap \partial M, \partial_1 D = \partial D \cap F$, such that ∂D does not bound a disk in $F \cup \partial M$. A sufficient condition for boundary-incompressiblity of F is that $\pi_1 (F, \partial F) \to \pi_1 (M, \partial M)$ is injective.

Given a lamination $\mathcal{F}$ of M by properly immersed surface, a leaf F bounding some complimentary region (i.e. a connected component C of $\overline{M - \mathcal{F}}$) is **end-compressible** if there is no end-compressing monogon. An end-compressing monogon for F is a monogon properly embedded in the complimentary region C which is not homotopic (rel. boundary) into ∂C. For example, let $M = \Gamma \backslash \mathbb{H}^3$ be a hyperbolic 3-manifold, and $\mathcal{F} = F$ the projection of a horosphere. Then the projection of the corresponding horoball is an end-compressing monogon for F.

For a 3-manifold M, a subgroup $H \subset \pi_1 M$ is called **non-peripheral** if it is not conjugate in $\pi_1 M$ to a subgroup of $im\,(\pi_1 \partial M \to \pi_1 M)$.

A 3-manifold is called **atoroidal** if each π_1-injective immersion $\mathbb{T}^2 \to M$ is homotopic into the boundary. This is equivalent to the condition that there exists no non-peripheral subgroup of $\pi_1 M$ isomorphic to $\mathbb{Z}^2$.

A 3-manifold M with a given decomposition of ∂M into surfaces with boundary, $\partial M = \partial_0 M \cup \partial_1 M$, is called **pared acylindrical** (with respect to $\partial_1 M$) if each immersion $f : \left(\mathbb{S}^1 \times [0,1], \mathbb{S}^1 \times \{0,1\}\right) \to (M, \partial_1 M)$, which is π_1-injective as a map of pairs, is homotopic into ∂M.
If M is compact, orientable, irreducible, atoroidal, has incompressible boundary ∂M, and is pared acylindrical with respect to some subsurface $\partial_1 M \subset \partial M$, then $DM = M \cup_{\partial_1 M} M$ is hyperbolic according to Thurston's hyperbolization conjecture. (DM contains the in-

compressible, boundary-incompressible surface $\partial_1 M$, therefore one can apply Thurston's hyperbolization theorem for Haken manifolds and does not rely on Perelman's work.) This implies that the original M is hyperbolic with geodesic boundary $\partial_1 M$ and cusps corresponding to $\partial_0 M$.

Guts-terminology. Let N be a compact, oriented, irreducible manifold with incompressible boundary ∂N. The double $DN = N \cup_{\partial N} N$ is obtained by glueing two copies of N (with different orientations) along the common boundary. The double DN is irreducible because N is irreducible and has incompressible boundary. (Each sphere in DN could be homotoped to intersect one copy of N in either a sphere or a compression disk.) Thus we can apply to DN the JSJ-decomposition from section 1.1:

$$DN = S_1 \cup_{T^2} \ldots \cup_{T^2} S_k \cup_{T^2} H_1 \cup_{T^2} \ldots \cup_{T^2} H_l$$

with Seifert fibrations $S_1, \ldots, S_k$ and hyperbolic manifolds $H_1, \ldots, H_l$. For each of these pieces we can consider its intersection with one copy of N. The intersections $S_i \cap N$ are either Seifert fibrations or I-bundles. (If N happens to be atoroidal, then the only possible Seifert fibrations in the decomposition are solid tori.) The intersections $H_j \cap N$ must be atoroidal and pared acylindrical (with respect to $\partial_1 (H_j \cap N) = \partial N \cap H_j$), and thus carry a hyperbolic metric with geodesic boundary $\partial N \cap H_j$ (and cusps corresponding to the intersections with the decomposing tori). The union $\cup_{j=1}^{l} H_j \cap N$ is denoted $Guts\,(N)$.

In particular, if M is a closed, orientable, irreducible 3-manifold and $F \subset M$ an incompressible surface, then we may apply this decomposition to $N := \overline{M - F}$, which is a 3-manifold with boundary consisting of two copies of F. This defines $Guts\left(\overline{M - F}\right)$. (This definition coincides with the definition which we will give for laminations in section 2.2.)

For example, if M is hyperbolic and F is a geodesic surface, then $\overline{M - F}$ is hyperbolic with geodesic boundary and $D\left(\overline{M - F}\right)$ is hyperbolic. Thus $Guts\left(\overline{M - F}\right) = \overline{M - F}$ in this case.

1.2 Hyperbolic volume

In dimensions ≥ 3, hyperbolic metrics (of finite volume) on a given topological manifold are unique up to isometry, by Mostow's rigidity theorem. Therefore, geometric invariants arising from the hyperbolic metric, such as its volume, are topological invariants.

It follows from the Chern-Gauß-Bonnet theorem that in even dimensions (including surfaces) hyperbolic volume is proportional to the Euler characteristic χ. In odd dimensions, χ vanishes by Poincare duality, and one might consider hyperbolic volume as a good replacement.

Of course, there are, especially for hyperbolic 3-manifolds, also plenty of other topological invariants, but according to [31] "one gets a feeling that volume is a very good measure for the complexity" of a 3-manifold, and that the ordinal structure (of the set of hyperbolic volumes as a subset of $\mathbb{R}_+$) "is really inherent in 3-manifolds."

In dimensions $\neq 3$, the set of possible volumes of hyperbolic manifolds is a discrete subset of $\mathbb{R}_+$. In particular, if the dimension is even ($n = 2m$), then all volumes are multiples of π^m. In dimension 3, hyperbolic volumes are sums of dilogarithms of algebraic numbers. An important number-theoretical question is to which extent volumes of hyperbolic 3-manifolds are rationally independent. This is, by work of Goncharov, related to the size of the algebraic K-theory $K_3\left(\overline{\mathbb{Q}}\right)$.

The set of volumes of hyperbolic 3-manifolds is well-ordered, i.e. each subset has a smallest element. In principle, the volume of a hyperbolic 3-manifold (given by Dehn surgery at some link) can be numerically computed by Weeks' program SnapPea. A large number of volumes have been computed by this program, and the smallest closed 3-manifold found so far is the so-called Weeks manifold, whose volume is 0.94... Adams has proved that the smallest nonorientable, noncompact, hyperbolic 3-manifold of finite volume is the Gieseking manifold, whose volume is 1.014... Cao and Meyerhoff have proved that the smallest orientable, noncompact, hyperbolic 3-manifolds of finite volume are the complement of the figure eight knot and its sibling, of volume 2.029... One may naturally ask what are the smallest hyperbolic 3-manifolds with certain topological characteristics, say the smallest fibered manifold, the smallest Haken manifold, the smallest link complement with certain properties of the link, the smallest manifold with a given Betti number,...

Lower bounds. Lower bounds on volumes of hyperbolic 3-manifolds with, for example, specified betti numbers, have been computed by Culler-Shalen and their coworkers in a series of papers (e.g. [8]). It would lead us too far to discuss these results in detail. However, we want to discuss another estimate, Agol's inequality, because it has generalizations to laminations to be discussed in section 3.3. This inequality estimates the volume in terms of the topology of $Guts\left(\overline{M-F}\right)$, for any incompressible surface F. The guts-terminology is explained in section 1.1.3.

The 'original form' of Agol's inequality (which will have a generalization to laminations) is the following, with $V_3 = 1.014..$ the volume of a regular ideal 3-simplex: If M is a hyperbolic 3-manifold containing an incompressible surface F, then $Vol\left(M\right) \geq -2V_3\chi\left(Guts\left(\overline{M-F}\right)\right)$.

In [3], this inequality has been improved as follows, with $V_{oct} = 3.66..$ the volume of a regular ideal octahedron.

Theorem 1 *(***Agol-Storm-Thurston***, [3], Cor.2.2.): If M is a closed hyperbolic 3-manifold containing an incompressible surface F, then*

$$Vol\left(M\right) \geq Vol\left(Guts\left(\overline{M-F}\right)\right) \geq -V_{oct}\chi\left(Guts\left(\overline{M-F}\right)\right).$$

The proof uses analytical methods (Perelman's entropy estimate for the Ricci flow, work of Bray and Miao on the Penrose conjecture) and does not seem to generalize to laminations so far.

The right hand side of the Agol-Storm-Thurston inequality has been computed in a few cases.

If L is an alternating hyperbolic link with a prime, alternating, twist-reduced diagram D of twist number $t\left(D\right)$, then Lackenby ([24]) proved for $M = S^3 - L$ and the two checkerboard surfaces B and W: $\chi\left(Guts\left(\overline{M-B}\right)\right) + \chi\left(Guts\left(\overline{M-W}\right)\right) = t\left(D\right) - 2$. The so obtained inequality $Vol\left(S^3 - L\right) \geq V_{oct}\left(\frac{1}{2}t\left(D\right) - 1\right)$ is sharp: equality holds for the Borromean rings.

If L is a 2-bridge link with its canonical Seifert surface $F \subset M = S^3 - L$, then Agol ([2], section 7) has given an explicit, easily computable expression for the right-hand side in terms of the twists of the Seifert surface.

There are 2-bridge link complements that fiber over the circle, for which this gives a nontrivial lower bound. This suggests that for surface bundles of fiber genus ≥ 2 one may hope to get nontrivial bounds from Agol's inequality. On the other hand, for once-punctured torus bundles M we have proved ([21]) that $Guts\left(\overline{M - F}\right) = \emptyset$ holds for each incompressible surface F.

1.3 Simplicial volume

Hyperbolic volume is a homotopy invariant and one might ask whether it is definable in terms of algebraic topology. Such a homotopy invariant was indeed defined by Gromov for all (compact, orientable) manifolds of arbitrary dimensions.

Let M be a compact, orientable, connected n-manifold, possibly with boundary. Its top integer (singular) homology group $H_n(M, \partial M; \mathbb{Z})$ is cyclic. The image of a generator under the change-of-coefficients homomorphism $H_n(M, \partial M; \mathbb{Z}) \rightarrow H_n(M, \partial M; \mathbb{R})$ is called a fundamental class and is denoted $[M, \partial M]$. If M is not connected, we define $[M, \partial M]$ to be the formal sum of the fundamental classes of its connected components.

The simplicial volume $\| M \|$ is defined as

$$\| M \| = inf \left\{ \sum_{i=1}^{r} | a_i | \right\}$$

where the infimum is taken over all singular chains $\sum_{i=1}^{r} a_i \sigma_i$ (with real coefficients) representing the fundamental class $[M, \partial M]$ in $H_n(M, \partial M; \mathbb{R})$.

Theorem 2 *(***Gromov-Thurston Theorem***, [13], [31]): If $M - \partial M$ carries a complete hyperbolic metric of finite volume $Vol(M)$, then*

$$\| M \| = \frac{1}{V_n} Vol(M)$$

with $V_n = sup \{ Vol(\Delta) : \Delta \subset \mathbb{H}^n \text{ geodesic n-simplex} \}$.

Proof: We outline the proof (for M closed) after [4].

Any simplex in a negatively curved manifold is homotopic (rel. vertices) to a unique geodesic simplex. This can be used to show that each fundamental cycle $\sum_{i=1}^{r} a_i \sigma_i$ can be homotoped such that each σ_i is geodesic and thus satisfies $vol(\sigma_i) < V_n$. This implies $Vol(M) = \sum_{i=1}^{r} a_i vol(\sigma_i) < \sum_{i=1}^{r} | a_i | V_n$, from which the '$\geq$'-part of the theorem follows.

To show the '$\leq$'-part, one needs (for any given $\epsilon > 0$) to find cycles $\sum_{i=1}^{r} a_i \sigma_i$ such that each σ_i has volume $vol(\sigma_i) \geq V_n - \epsilon$. This is done by means of Gromov's smearing construction, which we are going to explain now.

On $Isom^+(\mathbb{H}^n)$ we have the bi-invariant Haar measure dh. For some fixed reflection $r \in Isom^-(\mathbb{H}^n)$, we consider $smr = dh - r^* dh$. This is a signed measure on $Isom(\mathbb{H}^n)$, and does not depend on r. For some fixed regular simplex $\widetilde{\sigma} \subset \mathbb{H}^n$ of volume $vol(\widetilde{\sigma}) = V_n - \epsilon$, we consider the $Isom(\mathbb{H}^n)$-equivariant bijection

$$\Gamma \backslash Isom(\mathbb{H}^n) \rightarrow \{\text{regular n-simplices in } M = \Gamma \backslash \mathbb{H}^n \text{ of volume } V_n - \epsilon\},$$

given by $[g] \rightarrow proj(g\widetilde{\sigma})$, where $proj : \mathbb{H}^n \rightarrow \Gamma \backslash \mathbb{H}^n$ is the projection. This bijection allows us to consider smr as a signed measure on the set of regular n-simplices in M.

There is the so-called measure homology $\mathcal{H}(M)$, which is the homology of the space of signed measures on $map(\Delta^*, M)$ with the obvious boundary operator. smr is a cycle and therefore represents a class in $\mathcal{H}(M)$. One can show that it actually represents $(V_n - \epsilon)[M]$, and it has norm $\| smr \| = Vol(M)$. (This proves the wanted inequality if one were to consider the norm in $\mathcal{H}(M)$.)

To prove the '$\leq$'-part of the theorem, there are then two approaches. The one (see [4] or [26]) is to approximate $\frac{1}{V_n - \epsilon} smr$ by actual singular chains representing the fundamental class in singular homology. The other approach, suggested in Thurston's lecture notes and recently proved in [28], is to give, for any smooth manifold, an isometric isomorphism between singular homology and measure homology.

For the noncompact (cusped) case, the proof can be completed using arguments from Francaviglia ([10], sections 5-6). Here is an alternative proof for the cusped case: Consider the map which pinches all boundary tori to points, and let $(M', \partial M')$ be the quotient, with $\partial M'$ a finite number of points. Using Gromov's theory of multicomplexes one can show that this map induces an isometry of Gromov norms. There is an obvious isometry between $H_*(M', \partial M')$ and the (absolute) homology theory constructed from ideal simplices with all ideal vertices in the cusps of M. It is then fairly easy to check that to the latter homology theory one can apply Gromov's smearing construction, to prove the wanted inequality. (A little care is needed because one can not apply the smearing construction to ideal simplices: one is not allowed to have ideal simplices with vertices not in cusps. This technical point is surmounted in section 2.3. of [23].) $\square$

Uniqueness of Gromov's smearing construction. Jungreis proved in [17] that Gromov's smearing construction is unique in the following sense: if a sequence of fundamental cycles (of a closed hyperbolic manifold of dimension ≥ 3) has l^1-norms converging to $\| M \|$, then the sequence converges to $\pm smr$.

In [19] we generalized this result to noncompact hyperbolic manifolds of dimension ≥ 3 and of finite volume, which are not Gieseking-like. (A hyperbolic 3-manifold $\Gamma \backslash \mathbb{H}^3$ is called Gieseking-like if $\mathbb{Q}(\sqrt{-3}) \cup \{\infty\}$ are parabolic fixed points of elements in Γ. The only known examples are commensurable to the complement of the figure eight knot complement.)

The uniqueness fails for manifolds which are of dimension 2 or Gieseking-like.

Definitions using polyhedral norms. More generally, let P be any polyhedron. Then the invariant $\| M \|_P$ is defined in [2] as follows: we denote by $C_*(M, \partial M; P; \mathbb{R})$ the complex of P-chains (i.e. formal sums of maps $P \rightarrow M$ with real coefficients), and by $H_*(M, \partial M; P; \mathbb{R})$ its homology. There is a canonical chain homomorphism $\psi : C_*(M, \partial M; P; \mathbb{R}) \rightarrow C_*(M, \partial M; \mathbb{R})$, given by some triangulation of P which is to be chosen such that all possible cancellations of boundary faces are preserved. $\| M \|_P$ is defined as the infimum of $\sum_{i=1}^{r} \mid a_i \mid$ over all P-chains $\sum_{i=1}^{r} a_i P_i$ such that $\psi\left(\sum_{i=1}^{r} a_i P_i\right)$ represents the fundamental class $[M, \partial M]$.

In general, one can probably not expect these polyhedral norms to be related to the simplicial volume. However, for the case of hyperbolic manifolds one has the following analogue of the Gromov-Thurston Theorem with $V_P := sup\left\{Vol\left(\Delta\right)\right\}$, where the supremum is taken over all straight P-polyhedra $\Delta \subset \mathbb{H}^n$.

Proposition 1 *(Agol, [2], Prop.4.1.): If $M - \partial M$ admits a hyperbolic metric of finite volume $Vol\left(M\right)$, then*

$$\parallel M \parallel_P = \frac{1}{V_P} Vol\left(M\right).$$

1.4 Properties of the simplicial volume

In spite of its relatively unassuming definition, the simplicial volume is quite hard to calculate. Gromov ([13]) developed the theory of bounded cohomology to prove various vanishing results for the simplicial volume. The bounded cohomology $H_b^*\left(M, \partial M\right)$ is the cohomology of the complex of bounded cochains

$$\left\{f \in C^*\left(M, \partial M; \mathbb{R}\right) : sup\left\{f\left([\sigma]\right) : \sigma \in map\left(\Delta^*, M\right)\right\} < \infty\right\}$$

with the usual coboundary operator. Its relevance for vanishing of the simplicial volume is shown by the following implication, for compact, orientable manifolds M with $n = dim\left(M\right)$:

$$H_b^n\left(M, \partial M\right) = 0 \implies \parallel M \parallel = 0.$$

This was used to prove, for example, that

$$\parallel M \parallel = 0$$

if one of the following assumptions is satisfied:
- $\pi_1 M, \pi_1 \partial M$ are amenable (e.g. virtually solvable) and ∂M is connected ([13], p.57),
- M (closed) admits a covering with n-dimensional nerve by sets with amenable fundamental group ([13], p.40) ,
- M (closed) admits a nontrivial (not necessarily free) S^1-action (Yano, cf. [13], p.41).

On the other hand, [13] proved nontriviality

$$\parallel M \parallel > 0$$

if $int\left(M\right)$ admits a complete metric with $-b^2 \leq$ sectional curvature $\leq -a^2 < 0$ and finite volume. In particular, there is the exact formula for finite-volume hyperbolic manifolds in Theorem 2. Recently, Lafont and Schmidt ([25]) proved nontriviality $\parallel M \parallel > 0$ for closed locally symmetric spaces of noncompact type. The proof uses the barycenter method of Besson-Courteois-Gallot.

We describe an application of the simplicial volume to mapping degrees, taken from [13]. The simplicial volume quantifies the topological complexity of manifolds. Indeed, define a partial order on the set of n-manifolds by: $M_1 \geq M_2$ if there exists a degree 1 map from M_1 to M_2. Then the simplicial volume is an order-preserving map from the set of n-manifolds to $\mathbb{R}_+$. More generally, if there is a degree d map from M_1 to M_2, then $\| M_1 \| \geq d \| M_2 \|$. Thus, nontriviality results for the simplicial volume can be used to get restrictions on the possible mapping degrees for continuous maps between given manifolds.

Additivity properties. Let M be a compact, irreducible 3-manifold and $F \subset M$ a compact, incompressible surface.

If F is a torus, then $\| \overline{M - F} \| = \| M \|$. (This is proved in a more general setting by Gromov in [13], a detailed argument can be found in [20]. In this special case, a proof was also given by Soma ([29]), built on Theorem 6.5.1. from Thurston's lecture notes.)

If F is a geodesic surface in a hyperbolic 3-manifold, then $\| \overline{M - F} \| > \| M \|$. (This follows from Jungreis' result on the uniqueness of the smearing construction in the closed case. In the cusped case, including the case of Gieseking-like manifolds, it is theorem 6.3. in [19].)

On the other hand, if F is a fiber of a fibration $M \to S^1$, then $\| \overline{M - F} \| = \| F \times I \|$ depends only on F, whereas $\| M \|$ can become arbitrarily large.

Soma ([30], Theorem 0.1.) has shown that also in the non-fibered case one can have arbitrarily large $\| M \|$ with given $\| \overline{M - F} \|$.

2 Foliations and Laminations

2.1 Motivation

Since the work of Haken and Waldhausen, compact incompressible surfaces have been a main tool to understand the topology of 3-manifolds. Manifolds containing such a compact, incompressible surface are called Haken manifolds, and they are the first general class of 3-manifolds for which the Geometrization conjecture had been proven. However, with the today knowledge of 3-manifolds topology, it is apparent that most closed 3-manifolds do not contain any closed incompressible surface. (We will illustrate this with an example below.) Therefore one is looking for other topological structures on 3-manifolds which are more frequent and which allow generalizations of methods from the Haken-Waldhausen theory.

The first generalization are taut foliations. A foliation $\mathcal{F}$ of a compact 3-manifold is called *taut* if for every leaf F of $\mathcal{F}$ there exists a circle, transverse to $\mathcal{F}$, which intersects F. Equivalently, there exists a circle, transverse to $\mathcal{F}$, which intersects every leaf of $\mathcal{F}$. (This excludes the existence of Reeb components.)

A common generalization to incompressible surfaces and taut foliations is the concept of essential laminations. Its definition is given in section 2.2.

There is a survey article on essential laminations ([11], section 3), so we only mention that for manifolds containing essential laminations (with possibly noncompact leaves) several theorems from the Haken-Waldhausen theory are still true (each homotopy equivalence is homotopic to a homeomorphism; the diffeomeorphism group is finite; ...) and that for manifolds with essential laminations a weak version of the hyperbolization theorem can be proved. (The latter may now seem obsolete in view of Perelman's work.

However, it is still very interesting and fruitful that one can build a direct connection between topological structures, as essential laminations, and geometric structures, as hyperbolic metrics.)

Essential laminations, in general, need not have any compact leaf, and are thus by far more frequent than compact, incompressible surfaces. It has actually been an open question for long time, whether there exist at all hyperbolic 3-manifolds without essential laminations. The first counterexamples have been found only recently by Fenley ([9]). However, it seems in view of section 3.4. below that for hyperbolic 3-manifolds of small volume it is often harder to get essential laminations.

An illustrating example ([31], section 4).

The hyperbolic knot complement of smallest volume is the complement of the figure eight knot K, its volume is 2.029..

Let us look at all (infinitely many) 3-manifolds M, which are obtained by Dehn-filling (i.e. glueing a solid torus by some $A \in SL(2,\mathbb{Z}) \subset Homeo\left(\mathbb{T}^2\right)$) in the complement of the figure eight knot.

All but 10 of them are hyperbolic. Their volumes are all strictly smaller than 2.029.. (Thurston's hyperbolic Dehn surgery theorem).

From the topological point of view, Thurston has proved that all but 6 of them do not contain a closed, incompressible surface (thus can not be studied by the classical Haken-Waldhausen theory).

Moreover, by Agol's inequality, none of these hyperbolic manifolds can contain a closed geodesic surface.

However, by Hatcher ([15]), it is known that all of these Dehn-fillings of $\mathbb{S}^3 - K$ carry transversely orientable essential laminations.

We remark that, as will be explained in section 3.4, by the generalization of Agol's inequality to laminations, the work of Calegari-Dunfield on tight laminations with empty guts, and a recent paper of Tao Li, hyperbolic 3-manifolds of volume smaller than 2.029.. can not carry (transversely orientable) essential laminations, except possibly if the fundamental groups of these hyperbolic manifolds inject into $Homeo^+\left(S^1\right)$. (The latter condition can in many cases be checked algorithmically, using the methods developed by Calegari-Dunfield).

Thus, Hatcher's construction implies that the fundamental groups of the hyperbolic Dehn-fillings of the figure-eight knot complement inject into $Homeo^+\left(S^1\right)$.

2.2 Structure of laminations on 3-manifolds

We remind that all upcoming notions from 3-manifolds topology are explained in the glossary in section 1.1.3.

Let M be a compact 3-manifold and $\mathcal{F}$ a (codimension one) lamination of M.

By abuse of notation we will denote by $\mathcal{F}$ both the lamination and the laminated subset of M, i.e. the union of leaves. Moreover, we will assume that $\mathcal{F}$ has no isolated leaves (which can always be achieved by blowing up isolated leaves to a product region) and we will denote by $\overline{M-\mathcal{F}}$ the completion (w.r.t. any path metric) of the complement of $\mathcal{F}$. If M has boundary, we will always assume without further mentioning that $\mathcal{F}$ is transverse or tangential to ∂M.

A **lamination** $\mathcal{F}$ of a **3-manifold** M is called **essential** ([12], ch.1) if no leaf is a sphere or a torus bounding a solid torus in M, $\overline{M-\mathcal{F}}$ is irreducible, and $\partial\left(\overline{M-\mathcal{F}}\right)$ is incompressible and end-incompressible in $\overline{M-\mathcal{F}}$. Examples of essential laminations are taut foliations or compact, incompressible, boundary- incompressible surfaces.

Guts of essential laminations. If M is a 3-manifold and $\mathcal{F}$ an essential lamination on M, then $N=\overline{M-\mathcal{F}}$ is, in general, a noncompact manifold. The noncompact ends of N are I-bundles over noncompact subsurfaces of ∂N. After cutting off each of these ends along an annulus $\mathbb{S}^1\times I$, one obtains a compact 3-manifold N^{cut} with boundary. One defines $Guts\,(N)=Guts\,(N^{cut})$, where $Guts\,(N^{cut})$ is defined as in section 1.1.3. Thus $Guts\,(N)$ is compact and it admits a hyperbolic metric with geodesic boundary and cusps. (Be aware that some authors, like [6], include Seifert fibered solid tori into the guts.)

We illustrate this with an example, taken from [6]. Let M be the mapping torus of a surface diffeomorphism $\phi:\Sigma\rightarrow\Sigma$. Assume that $genus\,(\Sigma)\geq 2$ and ϕ is pseudo-Anosov, then M is hyperbolic and there are two ϕ-invariant geodesic laminations $\lambda_\pm$ on Σ. The complement of, say, λ_+ consists of ideal polygons. Let $\mathcal{F}$ be the suspension lamination of λ_+ (i.e. we consider $\lambda_+\times I\subset S\times I$ and project it to $\mathcal{F}\subset M=S\times I/_{(x,0)\sim(\phi(x),1)}$). Then we can decompose $\overline{M-\mathcal{F}}$ into I-bundles over noncompact surfaces (namely neighborhoods of the cusps of the ideal polygons) and one solid torus. Thus, $Guts\left(\overline{M-\mathcal{F}}\right)=\emptyset$ in this case.

Leaf space of essential laminations. To construct the leaf space T of $\mathcal{F}$, one considers the pull-back lamination $\widetilde{\mathcal{F}}$ on the universal covering $\widetilde{M}=\mathbb{H}^3$. The space of leaves T is defined as the quotient of $\widetilde{M}$ under the following equivalence relation $\sim$. Two points $x,y\in\widetilde{M}$ are equivalent if either they belong to the same leaf of $\widetilde{\mathcal{F}}$, or they belong to the same connected component of the completion $\overline{\widetilde{M}-\widetilde{\mathcal{F}}}$. If $\widetilde{\mathcal{F}}$ is an essential lamination, then T is an order tree, with vertices corresponding to the leaves of $\widetilde{\mathcal{F}}$, and segments corresponding to directed, transverse, efficient arcs in $\widetilde{M}$. (See [12], also for the definition of order tree.) Moreover, T is an $\mathbb{R}$-order tree, that is, it is a countable union of segments and each segment is order isomorphic to a closed interval in $\mathbb{R}$. T can be topologized by the order topology on segments (and declaring that a set is closed if the intersection with each segment is closed). The order tree T comes with a fixed-point free action of $\pi_1 M$, because the deck transformations of $\widetilde{M}$ send leaves of $\widetilde{\mathcal{F}}$ to other leaves of $\widetilde{\mathcal{F}}$. (This is proved in [9], Lemma 4.7.) Fenley ([9]) has exhibited hyperbolic 3-manifolds whose fundamental groups do not admit any fixed-point free action on order trees. Thus there are hyperbolic 3-manifolds not carrying any essential lamination.

An essential lamination is called **tight** if its associated $\mathbb{R}$-order tree T is Hausdorff. It is called unbranched if its associated $\mathbb{R}$-order tree T is homeomorphic to $\mathbb{R}$. It is said to have two-sided branching ([5], Def. 2.6.1) if there are leaves $\lambda, \lambda_1, \lambda_2, \mu, \mu_1, \mu_2$ such that the corresponding points in the $\mathbb{R}$-order tree T satisfy $\lambda < \lambda_1, \lambda < \lambda_2, \mu > \mu_1, \mu > \mu_2$ but λ_1, λ_2 are incomparable and μ_1, μ_2 are incomparable. It is said to have one-sided branching if none of the other two cases occurs.

Surfaces in 3-manifolds. Let M be a hyperbolic 3-manifold of finite volume. By the Bonahon-Thurston theorem, each closed, incompressible surface $F \subset M$ is either quasigeodesic or is a virtual fiber. If we consider the essential lamination, obtained by blowing up F to a product region, then it is clearly tight. If F is a virtual fiber, then it is unbranched. If F is quasigeodesic, then it has two-sided branching.

3 Invariants of foliations and laminations

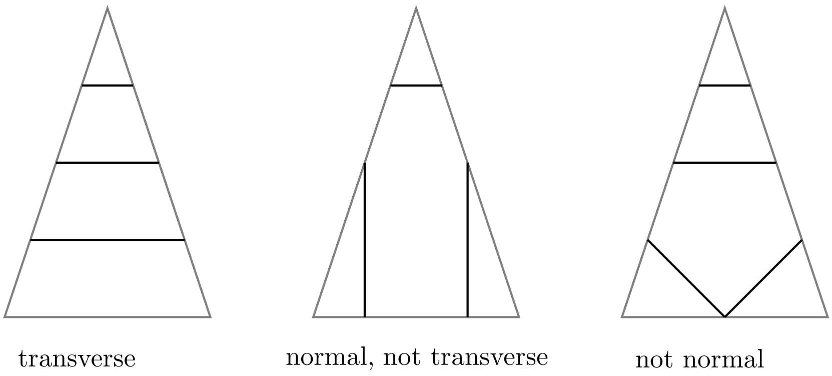
3.1 Definitions

Let M be a manifold and $\mathcal{F}$ a **codimension one lamination** of M. Let Δ^n be the standard simplex in $\mathbb{R}^{n+1}$, and $\sigma : \Delta^n \rightarrow M$ some (continuous, not necessarily differentiable) singular simplex. The lamination $\mathcal{F}$ induces an equivalence relation on Δ^n by: $x \sim y \Longleftrightarrow \sigma(x)$ and $\sigma(y)$ belong to the same connected component of $L \cap \sigma(\Delta^n)$ for some leaf L of $\mathcal{F}$. We say that a singular simplex $\sigma : \Delta^n \rightarrow M$ is laminated if the equivalence relation $\sim$ is induced by a lamination $\mathcal{F}\mid_\sigma$ of Δ^n. We call a lamination $\mathcal{F}$ of Δ^n affine if there is an affine mapping $f : \Delta^n \rightarrow \mathbb{R}$ such that $x, y \in \Delta^n$ belong to the same leaf if and only if $f(x) = f(y)$. We say that a lamination $\mathcal{G}$ of Δ^n is conjugate to an affine foliation if there is a simplicial homeomorphism $H : \Delta^n \rightarrow \Delta^n$ such that $H^*\mathcal{G}$ is an affine foliation.

We say that a singular n-simplex $\sigma : \Delta^n \rightarrow M$, $n \geq 2$, is **transverse** to $\mathcal{F}$ if it is foliated and it is
- either contained in a leaf,
- or $\mathcal{F}\mid_\sigma$ is conjugate to an affine foliation $\mathcal{G}$ of Δ^n.

We say that the simplex $\sigma : \Delta^n \rightarrow M$ is **normal** to $\mathcal{F}$ if, for each leaf F, $\sigma^{-1}(F)$ consists of normal disks, i.e. disks meeting each edge of Δ^n at most once (or being equal to a face of Δ^n).
In the special case of foliations $\mathcal{F}$ it is easy to show that the transversality of σ is equivalent

to the normality of σ.



It was first observed in [5] that a refinement of the simplicial volume gives a meaningful invariant for foliations and laminations.

Definition 1 *: Let M be a compact, oriented manifold, possibly with boundary, and $\mathcal{F}$ a foliation or lamination on M. Then*

$$\parallel M \parallel_{\mathcal{F}} := inf \left\{ \sum_{i=1}^{r} \mid a_i \mid : \psi \left(\sum_{i=1}^{r} a_i \sigma_i \right) \text{ represents } [M, \partial M], \sigma_i : \Delta^n \to M \text{ transverse to } \mathcal{F} \right\}$$

and

$$\parallel M \parallel_{\mathcal{F}}^{normal} := inf \left\{ \sum_{i=1}^{r} \mid a_i \mid : \psi \left(\sum_{i=1}^{r} a_i \sigma_i \right) \text{ represents } [M, \partial M], \sigma_i : \Delta^n \to M \text{ normal to } \mathcal{F} \right\}.$$

There is an obvious inequality

$$\parallel M \parallel \leq \parallel M \parallel_{\mathcal{F}}^{normal} \leq \parallel M \parallel_{\mathcal{F}} .$$

In the case of foliations, equality $\parallel M \parallel_{\mathcal{F}}^{normal} = \parallel M \parallel_{\mathcal{F}}$ holds.

Analogous norms $\parallel M \parallel_{\mathcal{F},P}^{normal}$ can be defined for any polyhedron P instead of the simplex Δ^n.

3.2 Inequalities for the transverse Gromov norm

The transverse Gromov norm seems to measure the branching of foliations or laminations. This is suggested by the following results of Calegari (which are stated in [5] for taut foliations but can straightforwardly be generalized to essential laminations).

Theorem 3 *(Calegari, [5], Theorems 2.2.10., 2.5.9.): Let $\mathcal{F}$ be an essential lamination of a closed 3-manifold M. If $\mathcal{F}$ is either unbranched or branches in only one direction, then $\parallel M \parallel_{\mathcal{F}} = \parallel M \parallel$.*

If M is a hyperbolic 3-manifold, then some standard conjectures would imply that a foliation $\mathcal{F}$ branches in both directions if and only if it is asymptotically separated, i.e. there exists a geodesic half-plane in $\mathbb{H}^3 = \widetilde{M}$ separating some leaf from some other leaf of $\widetilde{\mathcal{F}}$. Thus, as observed by Calegari, the following theorem would imply that, at least for foliations of hyperbolic 3-manifolds, $\parallel M \parallel_{\mathcal{F}}$ decides whether $\mathcal{F}$ has two-sided branching.

Theorem 4 *(Calegari, [5], Theorem 2.4.5.): Let $\mathcal{F}$ be an asymptotically separated lamination of a closed, hyperbolic manifold $M^{n \geq 3}$. Then $\parallel M \parallel_{\mathcal{F}} > \parallel M \parallel$.*

Proof: ([5]) Consider a sequence c_n of fundamental cycles of norms converging to $\parallel M \parallel$. By the uniqueness of Gromov's smearing construction, due to Jungreis (see section 1.3.), the sequence must converge to $\pm smr$. In particular, for large enough n, c_n must involve simplices close to any given regular ideal simplex. However, one finds regular ideal simplices which can not be approximated by simplices transverse to $\mathcal{F}$. $\square$

For example, any lamination containing a closed geodesic leaf is asymptotically separated and thus has nontrivial Gromov norm.

In [19] we have generalised Theorem 4 to noncompact, hyperbolic manifolds of finite volume, under the same assumptions as in section 1.3., i.e. either $dim\left(M\right) \geq 4$ or $dim\left(M\right) = 3$ and M is not Gieseking-like.

3.3 Inequalities for the normal Gromov norm

A straightforward generalization of the proof of Theorem 3 shows the following fact, which suggests that $\parallel M \parallel_{\mathcal{F}}^{normal}$ measures the non-tightness of laminations.

Lemma 1 *: Let $\mathcal{F}$ be an essential lamination of a closed 3-manifold M. If $\mathcal{F}$ is tight, then $\parallel M \parallel_{\mathcal{F}}^{normal} = \parallel M \parallel$.*

However, the analogue of Theorem 4 is clearly not true. There are many tight laminations containing compact, geodesic surfaces, the simplest example being just finite unions of compact, geodesic surfaces.

We will discuss in section 3.5. a general inequality for the normal Gromov norm (Theorem 6), which specialized to the 3-dimensional case yields the following generalization of Agol's inequality. (Actually Theorem 6, which is purely topological, is by a factor 2 weaker than Theorem 5. The improvement by a factor 2 is special to hyperbolic 3-manifolds, as explained at the end of section 3.5. below.)

Theorem 5 *(K.): Let M be a compact hyperbolic 3-manifold and $\mathcal{F}$ an essential lamination of M. Then*

$$\parallel M \parallel_{\mathcal{F}}^{normal} \geq -2\chi\left(Guts\left(\overline{M - \mathcal{F}}\right)\right).$$

If $\mathcal{F}$ consists of one compact, incompressible surface, then $\parallel M \parallel_{\mathcal{F}}^{normal} = \parallel M \parallel$, and the above inequality is exactly the 'original' (weaker) form of Agol's inequality.

3.4 Applications

We discuss the application to non-existence results for laminations on 3-manifolds.

Tao Li ([27]) has proved that the existence of a transversely orientable essential lamination on a given hyperbolic 3-manifold M implies that the same M must also carry a tight lamination. Thus it makes sense to concentrate on the existence question for tight laminations.

If M is hyperbolic and carries a tight lamination with empty guts, then Calegari and Dunfield have shown ([6], Theorem 3.2.) that $\pi_1 M$ acts effectively on the circle, i.e., there is an injective homomorphism $\pi_1 M \rightarrow Homeo\left(S^1\right)$. This implies that the Weeks manifold (the closed hyperbolic manifold of smallest known volume) can not carry a tight lamination with empty guts ([6], Corollary 9.4.). Calegari and Dunfield also observed that the generalization of Agol's inequality to tight laminations (which is stated above in Theorem 5 and proved in [22]) would give obstructions to existence of laminations with nonempty guts, and, in particular, exclude existence of tight laminations on the Weeks manifold. (This was stated as conjecture 9.7 in [6].)
The following corollary applies, for example, to all hyperbolic manifolds M obtained by Dehn-filling the complement of the figure-eight knot in S^3. (It is known that each of these M contains tight laminations. By the following corollary, all these tight laminations have empty guts.)

Corollary 1 *: If M is a closed hyperbolic 3-manifold with $Vol\left(M\right) < 2V_3 = 2.02...$, then M carries no tight lamination with nonempty guts.*

Proof: We use that $\chi\left(Guts\left(\overline{M-\mathcal{F}}\right)\right) = \frac{1}{2}\chi\left(\partial Guts\left(\overline{M-\mathcal{F}}\right)\right) \leq -1$ because the geodesic part of $\partial\overline{M-\mathcal{F}}$ is either closed or contains at least two surfaces with boundary. Hence Corollary 1 folows from Theorem 5. □

Corollary 2: *The Weeks manifold admits no tight lamination $\mathcal{F}$.*

Putting this together with the result of Tao Li, one can even improve this result as follows.

Corollary 3: *The Weeks manifold admits no transversely orientable essential lamination.*

The same argument shows that a hyperbolic 3-manifold M with
- $Vol\left(M\right) < 2V_3$, and
- no injective homomorphism $\pi_1 M \rightarrow Homeo^+\left(S^1\right)$

can not carry a transversely orientable essential lamination. Some methods for excluding the existence of injective homomorphisms $\pi_1 M \rightarrow Homeo^+\left(S^1\right)$ have been developed in [6] (which yielded in particular the nonexistence of such homomorphisms for the Weeks manifold, used in corollary 2), but in general it is still hard to apply this criterion to other hyperbolic 3-manifolds of volume $< 2V_3$.

3.5 Higher dimensions

For keeping the notation not too complicated, we consider in this section the case $\partial M = \emptyset$. (The general statements for $\partial M \neq \emptyset$ can be found in [22].)

For a manifold (with boundary) N^n and a submanifold Q^n we denote $\partial_1 Q = \partial N \cap \partial Q$, $\partial_0 Q = \overline{\partial Q - (\partial N \cap \partial Q)}$. $(Q, \partial_1 Q)$ is pared acylindrical if every π_1-injective map $\left(\mathbb{S}^1 \times [0,1], \mathbb{S}^1 \times \{0,1\}\right) \rightarrow (Q, \partial_1 Q)$ can be homotoped into ∂Q. We say that the decomposition $N = Q \cup \overline{(N-Q)}$ is essential if all inclusions $Q \rightarrow N, \overline{N-Q} \rightarrow N, \partial_0 Q \rightarrow Q, \partial_0 Q \rightarrow \overline{N-Q}$ are π_1-injective for each connected component.

Theorem 6 *(K., [22], Thm.1): Let M be a closed, orientable n-manifold and $\mathcal{F}$ a lamination (of codimension one) of M. Assume that there exists a compact, aspherical, n-dimensional submanifold $Q \subset \overline{M - \mathcal{F}}$ such that, if we let*
$N = \overline{M - \mathcal{F}}, \partial_0 Q = \overline{\partial Q - (\partial N \cap \partial Q)}, \partial_1 Q = \partial N \cap \partial Q$, then
i) each connected component of $\partial_0 Q$ has amenable fundamental group,
ii) $(Q, \partial_1 Q)$ is pared acylindrical,
iii) the decomposition $N = Q \cup \overline{(N-Q)}$ is essential. Then

$$\parallel M \parallel_{\mathcal{F}}^{normal} \geq \frac{1}{n+1} \parallel \partial Q \parallel,$$

$$\parallel M \parallel_{\mathcal{F}} \geq \frac{1}{\left[\frac{n}{2}\right]+1} \parallel \partial Q \parallel .$$

The following corollary gives an explicit bound for the topological complexity of compact, geodesic hypersurfaces in a given compact, negatively curved manifold. Such a bound seems to be new except, of course, in the 3-dimensional case where it is due to Agol ([2]) and (with nonexplicit constants) to Hass ([14]).

Corollary 2 *: Let M be a compact Riemannian n-manifold of negative sectional curvature and finite volume. Let $F \subset M$ be a geodesic $n-1$-dimensional hypersurface of finite volume. Then $\parallel F \parallel \leq \frac{n+1}{2} \parallel M \parallel$.*

Proof: : Consider the lamination given by F. Its complement $N = \overline{M-F}$ is aspherical and (pared) acylindrical. (The latter follows from the fact that $DN = N \cup_{\partial_1 N} N$ is hyperbolic, hence atoroidal.)

If N is compact we can choose $Q = N$, in which case the other assumptions of Theorem 6 are trivially satisfied. In the case that N is not compact we cut off the cuspical ends along submanifolds with amenable fundamental groups to get Q. From Theorem 6 we conclude $\parallel M \parallel_F^{norm} \geq \frac{1}{n+1} \parallel \partial N \parallel$. The boundary of N consists of two copies of F, hence $\parallel \partial N \parallel = 2 \parallel F \parallel$. Moreover $\parallel M \parallel_F^{normal} = \parallel M \parallel$ by Lemma 1. The claim follows. $\square$

Proof: (Sketch of proof of Theorem 6.)

To give a flavor of the argument, we describe it in the simplest case: M is a hyperbolic 3-manifold, $\mathcal{F} = F$ a geodesic surface (i.e. $\partial Guts\left(\overline{M - \mathcal{F}}\right) = 2F$).

Let $\sum a_i\sigma_i$ be a normal cycle representing $[M]$. Then $\sum a_i\left(\sigma_i\cap F\right)$ represents $[F]$ and to get the wanted inequality $\sum \mid a_i \mid\geq \frac{1}{4}\parallel 2F\parallel$ it would suffice to have that each σ_i intersects F in at most 4 simplices.

Of course, there is a priori no reason to have an upper bound on the number of intersections that a normal simplex may have with the geodesic surface.

However, one can easily see that, whenever a 3-simplex intersects the surface more than 4 times, then two of the intersection triangles must have a parallel edge, i.e. cut out a square on one boundary face of the standard 3-simplex.

If σ_i mapped this square to a cylinder (i.e. mapped the opposite edges to the same edge), then one could use the acylindricity of $\overline{M-F}$ to argue that the cylinder degenerates after homotopy, hence can be removed without changing the homology class, and thus the number of intersections can be reduced.

Then the proof consists of defining a straightening which produces the maximally possible number of cylinders. (Some care is needed because the subdivided 1-skeleton can, of course, not be straightened arbitrarily. Even though each 1-simplex can be moved freely, the 2-skeleton imposes homotopy relations between concatenations of 1-simplices, which have to be respected by the straightening.)

Roughly the same argument works whenever $Q=N, P=\emptyset$, i.e. $N=Q$ is acylindrical. We give a short outline of the proof.

The intersection of a normal fundamental cycle $\sum_{i=1}^r a_i\sigma_i$ with ∂Q gives a fundamental cycle for ∂Q.

Since we are interested in proving an upper bound for $\frac{\|\partial Q\|}{\|M\|_{\mathcal{F}}^{normal}}$, we would thus like to bound the intersection of ∂Q with the σ_i's, namely to bound it by the number $n+1$. This is not possible for the, arbitrarily chosen, normal fundamental cycle $\sum_{i=1}^r a_i\sigma_i$, but, using acylindricity, for some fundamental cycle derived from it.

We note that homotoping a cycle, and removing subcycles consisting of degenerate simplices, does not change the homology class and does not increase the norm.

In [22], section 4.1., we define a straightening on Q (i.e. a way of homotoping cycles into some special position). The nontrivial point of this straightening is that, for each pair of connected components of $\partial_1 Q=\overline{\partial Q-P\cap Q}$, we fix among the straight edges one 'special' straight edge connecting them, and that those edges of the new cycle, which are subarcs of edges of the original fundamental cycle of M, are only allowed to be straightened into the 'special' straight edges. Hence, to straighten a simplex it will be necessary to move all edges coming from the original fundamental cycle into the 'special' straight ones, possibly changing the homotopy classes relative to the endpoints. These homotopies extend to a homotopy of the whole new fundamental cycle because we may guarantee that no two edges coming from the original fundamental cycle have a common vertex.

After this straightening one removes all simplices which have become degenerate after straightening.

Using acylindricity, one can show that each simplex, of the fundamental cycle we started with, can (after straightening and removal of degenerate simplices) contribute at most $n+1$ simplices to the fundamental cycle of ∂Q. (This needs a combinatorial exercice: each n-simplex contains at most $n+1$ affine codimension one simplices without parallel edge.) This proves Theorem 6 in the case $Q=N$.

Finally, to handle the general case $Q \neq N$, one would like to define a retraction $r : C_* (N) \to C_* (Q)$. It seems not possible to define r directly, but, using Gromov's work on multicomplexes, one can at least define it up to some 'amenable ambiguity' and use this to prove the general case.

To make a precise statement, such a retraction can be only defined in the following weak sense: there are multicomplexes $K(N)$ and $K(Q)$, with isomorphic bounded cohomology to N resp. Q, and an action of a group G on them, such that there is defined a retraction $r : C_* (K(N), K(\partial N)) \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_* (K(Q), K(\partial Q)) \otimes_{\mathbb{Z}G} \mathbb{Z}$. In other words, there is an ambiguity up to some group action.

If $\partial_0 Q = P \cap Q$ has amenable fundamental group, then the acting group G turns out to be amenable and this allows in some sense to remove the ambiguity (using bounded cohomology), and to prove Theorem 6.

□

Finally we show how Theorem 5 follows from Theorem 6.

Proof: For an essential lamination $\mathcal{F}$, it follows from [12], Theorem 6.1., that the complement $N = \overline{M - \mathcal{F}}$ is aspherical. Thus one can apply Theorem 6 to essential laminations of 3-manifolds.

Let P be the characteristic submanifold and $Q = Guts(\mathcal{F})$ (see section 2.2.). Both are known to be π_1-injective. Moreover, Q admits a hyperbolic metric with geodesic boundary $\partial_1 Q = \overline{\partial Q - P \cap Q}$ and cusps $\partial_0 Q = P \cap Q$, hence must be pared acylindrical ([32],Thm.3). $\partial_0 Q$ consists of tori and annuli, hence has amenable fundamental group. In conclusion, Q satisfies the assumptions of Theorem 6.

From the computation of the simplicial volume for surfaces ([13], section 0.2.) and $\chi(Q) = \frac{1}{2}\chi(\partial Q)$ (which is a consequence of Poincare duality for the closed 3-manifold $Q \cup_{\partial Q} Q$), it follows that

$$-\chi\left(Guts\left(\overline{M - \mathcal{F}}\right)\right) = -\frac{1}{2}\chi\left(\partial Guts\left(\overline{M - \mathcal{F}}\right)\right) = \frac{1}{4} \parallel \partial Guts\left(\overline{M - \mathcal{F}}\right) \parallel .$$

Thus, Theorem 6 for $n = 3$ yields $\parallel M \parallel_{\mathcal{F}}^{normal} \geq -\chi\left(Guts\left(\overline{M - \mathcal{F}}\right)\right)$. It was shown by Agol in [2], end of section 6, that one can use other polyhedral norms, with suitable sequences of polyhedra, to get an improvement by a factor 2 with respect to simplicial norms. This also applies to the normal Gromov norm. (By the way, this is the only argument for Theorem 5 which really uses hyperbolic geometry. The proof of Theorem 6 only uses topological properties of hyperbolic manifolds, especially the acylindricity of $Guts\left(\overline{M - \mathcal{F}}\right)$).

□

References

[1] C. Adams, *The noncompact hyperbolic 3-manifold of minimal volume*, Proc. AMS **100**, pp.601-606 (1987).

[2] I. Agol, *Lower bounds on volumes of hyperbolic Haken 3-manifolds*, Preprint. http://front.math.ucdavis.edu/math.GT/9906182.

[3] I. Agol, P. Storm, W. Thurston, with an appendix by N. Dunfield, *Lower bounds on volumes of hyperbolic Haken 3-manifolds*, Preprint. http://front.math.ucdavis.edu/math.DG/0506338

[4] R. Benedetti, C. Petronio, *Lectures on Hyperbolic Geometry.* Universitext, Springer-Verlag, Berlin (1992).

[5] D. Calegari, *The Gromov norm and foliations.* GAFA **11**, pp.1423-1447 (2001).

[6] D. Calegari, N. Dunfield, *Laminations and groups of homeomorphisms of the circle*, Invent. Math. **152**, pp.149-207 (2003).

[7] C. Cao, R. Meyerhoff, *The orientable cusped hyperbolic 3-manifolds of minimal volume*, Invent. Math. **146**, pp.451-478 (2001).

[8] M. Culler, P. Shalen, *Volumes of hyperbolic Haken manifolds*, Invent. Math. **118**, pp.285-329 (1994).

[9] S. Fenley, *Laminarfree hyperbolic 3-manifolds*, Preprint. http://front.math.ucdavis.edu/math.GT/0210482.

[10] S. Francaviglia, *Hyperbolic volume of representations of fundamental groups of cusped 3-manifolds*, IMRN **9**, 425-459 (2004).

[11] D. Gabai, *3 lectures on foliations and laminations on 3-manifolds*, Contemp. Math. **269**, pp.87-109 (2001).

[12] D. Gabai, U. Oertel, *Essential laminations in 3-manifolds*, Ann. of Math. **130**, 41-73 (1989).

[13] M. Gromov, *Volume and Bounded Cohomology*, Public. Math. IHES **56**, 5-100 (1982).

[14] J. Hass, *Acylindrical surfaces in 3-manifolds*, Michigan Math. J. 42, pp.357-365 (1995).

[15] A. Hatcher, *Some examples of essential laminations in 3-manifolds*, Ann. Inst. Fourier **42**, 313-325 (1992).

[16] A. Hatcher, *Basic Topology of 3-Manifolds*, http://www.math.cornell.edu/*sim*hatcher/3M/3Mdownloads.html

[17] D. Jungreis, *Chains that realize the Gromov invariant of hyperbolic manifolds*, Ergodic Theory and Dynamical Systems **17**, 643-648 (1997).

[18] B. Kleiner, J. Lott, *Notes and commentary on Perelman's Ricci flow papers*, http://www.math.lsa.umich.edu/research/ricciflow/perelman.html

[19] T. Kuessner, *Efficient fundamental cycles for cusped hyperbolic manifolds*, Pac. J. Math. **211**, 283-314 (2003).

[20] T. Kuessner, *Multicomplexes, bounded cohomology and additivity of simplicial volume*, Preprint. http://www.math.uni-siegen.de/~kuessner/preprints/bc.pdf

[21] T. Kuessner, *Guts of surfaces in punctured-torus bundles*, Arch. Math. **86**, 176-184 (2006).

[22] T. Kuessner, *Generalizations of Agol's inequality and nonexistence of tight laminations*, Preprint. http://www.math.uni-siegen.de/~kuessner/preprints/lam.pdf

[23] T. Kuessner, *Proportionality principle for cusped manifolds*, http://www.math.uni-siegen.de/~kuessner/preprints/prop.pdf

[24] M. Lackenby, with an appendix by I. Agol and D. Thurston, *The volume of hyperbolic alternating link complements*, Proc. LMS **88**, pp.204-224 (2004).

[25] J. Lafont, B. Schmidt, *Simplicial volume of closed locally symmetric spaces of non-compact type*, appears in Acta Math.

[26] B. Leeb, *Gromov's proof of Mostow rigidity*, http://poincare.mathematik.uni-tuebingen.de/~leeb/konf/molveno0901/mostow.ps

[27] T. Li, *Compression branched surfaces and tight essential laminations*, Preprint.

[28] C. Löh, *Measure homology and singular homology are isometrically isomorphic*, Math. Z. **253**, 197-218 (2006).

[29] T. Soma, *The Gromov invariant of links*, Invent. Math. **64**, pp.445-454 (1981).

[30] T. Soma, *Volume of hyperbolic 3-manifolds with iterated pseudo-Anosov amalgamations*, Geom. Dedic. **90**, pp.183-200 (2002).

[31] W. Thurston, *The Geometry and Topology of 3-Manifolds* . Lecture Notes, Princeton.

[32] W. Thurston, *Hyperbolic geometry and 3-manifolds.* Low-dimensional topology, Proc.Conf.Bangor, Lond.Math.Soc.Lect.NoteSer.48, pp.9- 25 (1982).

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